Simulation and Validation of Models for Interest Rate Risk

Johan Hagenbjörk, Jörgen Blomvall
The Interest Rate Market

- Equity Market: 57,000 BUSD.
- Bond Market: 49,000 BUSD.
- Interest Rate Swap Market: 514,000 BUSD (notional).
Contents

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Background

• Our hedging study using a rather primitive interest rate risk model proved successful.
  – Can we improve this further?
• Important to be able to simulate risk factors in finance (CVA, VaR).
• Improve our theoretical understanding of interest rate risk.
Interest Rate Risk

- Principal Component Analysis (PCA) 
  Litterman and Scheinkman (1991) 
  – Day to day changes.
- Eigenvectors become our risk factors.
- Principal component time series describe daily shocks to these risk factors.
Risk Factors for Cubic Splines

• Cubic splines are standard methodology in practical finance.

• Spot rates are often used for PCA.
Yield Curve Estimation using Blomvall (2017)

\[
\min_f h(f) + \frac{1}{2} z^T E z
\]

s.t.
\[
ge_e(f) + F z_e = p_{market}
\]

\[
h(f) = \int_{T_0}^{T_n} a_0(t) \left(f(t) - \bar{f}(t)\right)^2 dt + \sum_{k=1}^{K} \int_{T_0}^{T_n} a_k(t) \left(f^{(k)}(t) - \bar{f}^{(k)}(t)\right)^2 dt
\]
Risk Factors for Blomvall Forward Curves

Eigenvectors for Blomvall_10-2-2

Principal Components

Maturity [Days]

Modelling

• *How to simulate the distribution of our risk factors?*
Monte Carlo Simulation

- Using a model for the volatility together with our risk factors we can simulate scenarios.
- Since our factors are approximately independent we use univariate models for each factor.
  - Constant Volatility:
    \[ \sigma_t = c \]
  - GARCH
    \[ \sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{o} \gamma_i \epsilon_{t-j}^2 I_{\epsilon_{t-j}<0} + \sum_{k=1}^{q} \beta_k \sigma_{t-k}^2 \]
Monte Carlo Simulation

- We can also use different models for the mean while simulating principal component values $y_t$:
  - Zero
    \[ y_t = \epsilon_t \]
  - Constant
    \[ y_t = \mu + \epsilon_t \]
  - Autoregressive
    \[ y_t = \mu + \sum_{i=1}^{p} \Phi_{Li} y_{t-L_i} + \epsilon_t \]
Monte Carlo Simulation

• The residuals are drawn from either $N(0, \sigma)$ or Student-t$(0, \sigma, \nu)$ using our volatility models.

• The Akaike information criterion (AIC) is used to select the appropriate distribution and order of the GARCH-models.

• Using the inversion principle, uniformly distributed random variables are used to generate scenarios

$$X \sim U(0,1)$$
$$\epsilon = F^{-1}(X)$$
Out-of-Sample QQ-plots for Blomvall_10-2-2 using analytic Univariate Models consisting of ZeroMeanModel and GARCHModel with indepCopula

Zero Mean Model Shift $p=1 \ a=0 \ q=1 \ t(6.5)$

Zero Mean Model Twist $p=1 \ a=0 \ q=2 \ t(5.1)$

Zero Mean Model Butterfly $p=1 \ a=1 \ q=2 \ t(2.9)$

Zero Mean Model 4th $p=1 \ a=1 \ q=2 \ t(3)$
Term Structure Scenarios

- Using risk factors together with random variables from our models we can generate term structure scenarios.
Evaluation

• *How do we evaluate our simulation methods?*
Evaluation

• The realized term structure could be compared to the scenarios
  – Very high dimension!

• In practice we are interested in using the generated scenarios for pricing.
  – We construct a portfolio to valuate using all scenarios and the realized term structure.
  – Dimension is now one!
Illustration of Scenario Generation

- One problem remains: not a continuous distribution!
Estimation of a Continuous Distribution

- Use a kernel estimator as described in Kristensen, Shin (2012)
  - Place a Gaussian distribution centered in each observation and normalize.
  - Use \( \left( \frac{4s^5}{3n} \right)^{\frac{1}{5}} \) as variance, where \( s \) is the sample standard deviation and \( n \) is the number of samples.
A Statistical Test

Diebold, Gunter and Tay (1998)

If realized outcomes transformed by the cdf are uniformly distributed, then the cdf will be preferred independently of the loss function.
A Statistical Test

- Giacomini and White (2006)
  - Wald type test statistic to compare the log likelihood ratio between two models using $\chi^2(1)$. 
Simulation

• How is the simulation carried out?
SWAP Portfolio Generation

• Randomize the Notional amount USD $100x$, $x \sim N(0,1)$ (paying/receiving).

• 2001-2002
  – Buy 1-10Y each day

• 2002-
  – Buy 10Y each day.

• This results in approximately one cash flow each future day with around 2500 active instruments.
SWAP simulation

- In sample estimation 1996-2002
- Out-of-sample backtest 2002-2017 (3755 days)
- Simulate one day ahead.
- 1000 scenarios each day.
- Recalibration of models to newest data every year.

- Using a customized version of QuantLib (C++) we can simulate one day in 1.5 seconds per model.
Results

• *How well did our models capture the interest rate risk?*
Blomvall – GARCH vs Constant Vol

QQ-plot for Blomvall_10-2-2 using analytic Univariate Models consisting of ZeroMeanModel and GARCHModel with indepCopula

Histogram for Blomvall_10-2-2 using analytic Univariate Models consisting of ZeroMeanModel and GARCHModel with indepCopula

T-stat
102,285

P-value
4.80e-24
Cubic Splines – Forward vs Spot Rates

QQ-plot for CubicSplinesNaturalForward using analytic Univariate Models consisting of ZeroMeanModel and GARCHModel with indepCopula

Histogram for CubicSplinesNaturalForward using analytic Univariate Models consisting of ZeroMeanModel and GARCHModel with indepCopula

QQ-plot for CubicSplinesNaturalSpot using analytic Univariate Models consisting of ZeroMeanModel and GARCHModel with indepCopula

Histogram for CubicSplinesNaturalSpot using analytic Univariate Models consisting of ZeroMeanModel and GARCHModel with indepCopula

T-stat 0.04067
P-value 0.84
Forward Rates – Blomvall vs Cubic Splines

QQ-plot for Blomvall 10-2-2 using analytic Univariate Models consisting of ZeroMeanModel and GARCHModel with indepCopula

Histogram for Blomvall 10-2-2 using analytic Univariate Models consisting of ZeroMeanModel and GARCHModel with indepCopula

QQ-plot for CubicSplinesNaturalForward using analytic Univariate Models consisting of ZeroMeanModel and GARCHModel with indepCopula

Histogram for CubicSplinesNaturalForward using analytic Univariate Models consisting of ZeroMeanModel and GARCHModel with indepCopula

T-stat 57.57
P-value 3.257e-14
Simulations for longer horizons

- Two possibilities to simulate our models
  - Analytical forecast: set shocks to 0
    - Mean reverting.
  - Simulation forecast: simulate shocks
    - Path dependent trajectories.
    - Computationally and memory intensive!
- Term structure paths must be simulated in order to set appropriate fixing rates for all scenarios.
- Statistical test requires independent observations!
Weekly – Blomvall vs Cubic Splines

QQ-plot for Blomvall_10-2-2 using simulation Univariate Models consisting of ZeroMeanModel and GARCHModel with indepCopula

Histogram for Blomvall_10-2-2 using simulation Univariate Models consisting of ZeroMeanModel and GARCHModel with indepCopula

QQ-plot for CubicSplinesNaturalForward using simulation Univariate Models consisting of ZeroMeanModel and GARCHModel with indepCopula

Histogram for CubicSplinesNaturalForward using simulation Univariate Models consisting of ZeroMeanModel and GARCHModel with indepCopula

T-stat: 12.256
P-value: 0.0805207
Forward Rate Agreements (FRA)

• Forward Rate Agreements are more sensitive to the forward rate than SWAPs.

• Use the same technique but build portfolio using FRA.
  - Faster simulation due to shorter maturity.
FRA – Blomvall vs Cubic Spline

QQ-plot for Blomvall_10-2-2 using analytic Univariate Models consisting of ZeroMeanModel and GARCHModel with indepCopula

Histogram for Blomvall_10-2-2 using analytic Univariate Models consisting of ZeroMeanModel and GARCHModel with indepCopula

QQ-plot for CubicSplinesNaturalForward using analytic Univariate Models consisting of ZeroMeanModel and GARCHModel with indepCopula

Histogram for CubicSplinesNaturalForward using analytic Univariate Models consisting of ZeroMeanModel and GARCHModel with indepCopula

T-stat 2067

P-value 0
Conclusions

• *What have we learnt so far?*
Conclusions

• Huge market where risks needs to be both measured and handled!

• Yield curve estimation is important!

• With high statistical significance, we have proved that cubic splines give an inappropriate distribution for interest rate risk.
  – PCA on spot rate changes does not help.
References

Johan Hagenbjörk

www.liu.se