Convex approximations for risk-averse stochastic integer programming

Ward Romeijnders
Ruben van Beesten and Maarten H. van der Vlerk†

University of Groningen

June 1, 2017
Outline of the presentation

- Recourse models and convex approximations
- Known results for risk-neutral integer recourse
- New convex approximation for two-stage risk-averse mixed-integer recourse models
- Asymptotic error bound
- Error bound for simple integer recourse
- Discussion and future research
Risk-averse mixed-integer recourse models

Stochastic Programming

- Decision making under uncertainty
- Only probabilistic information available

Uncertainty

- Random goal constraints \( T x = \omega \)
- \( x \in \mathbb{R}^{n_1}_+ \) are decision variables
- \( \omega \in \mathbb{R}^m \) is a random vector with known cdf \( F \)

Integrality

- Advantage: Powerful modeling tool
- Disadvantage: More difficult to solve
Risk-averse mixed-integer recourse actions

When goal constraints are violated ($Tx \neq \omega$)

- Recourse actions $y$ can be undertaken: $Wy = \omega - Tx$
- High unit costs $q$

Consequence for decision making

- Tradeoff between costs incurred now and recourse costs in the future

Risk-aversion

- Conditional Value at Risk: $\text{CVaR}_\beta$
- $\text{CVaR}_\beta(\omega) = "\text{Expectation of worst } 100(1 - \beta)\% \text{ values of } \omega"$
- Typically, $\beta = 0.95$ or $0.99$
Two-stage mixed-integer mean-risk models

Model

\[
\min_{x} \{ cx + Q(x) : Ax = b, \ x \in \mathbb{R}^{n_1}_+ \}
\]

where

\[
Q(x) = \rho_1 \mathbb{E}_\omega [v(\omega - Tx)] + \rho_2 \text{CVaR}_\beta (v(\omega - Tx))
\]

and

\[
v(s) = \min_{y} \{ qy : Wy = s, \ y \in \mathbb{Z}^{n_2}_+ \times \mathbb{R}^{n_3}_+ \}
\]

Recourse actions \( y = y(x, \omega) \)

- To satisfy random goal constraints: \( Wy = \omega - Tx \)
- Risk neutral case: \( \rho_1 = 1 \) and \( \rho_2 = 0 \)
Assumptions

Second-stage value function $v$

1. The recourse is complete: $v(s) < +\infty$
2. The recourse structure is sufficiently expensive: $v(s) > -\infty$

Recourse function $Q$

- $Q(x)$ is finite for all $x \in \mathbb{R}^{n_1}$
Difficulties of solving recourse models

Difficulty of mixed-integer value function $\nu$

- Second-stage $\nu$ is non-convex

Solution approach: convex approximations

- Easier to solve
- Need error bound to guarantee solution quality
Convex approximation for pure integer recourse

Expected value function $Q(x)$

- $Q(x) = \mathbb{E}_\omega \left[ \min_y \{ qy : Wy \geq \omega - Tx, \ y \in \mathbb{Z}^{n_2}_+ \} \right]
\quad \text{with } \omega \text{ a continuous random vector}$

Approximation of VdV (2004): for $\alpha \in \mathbb{R}^m$

- $\hat{Q}_\alpha(x) = \mathbb{E}_\xi \left[ \min_y \{ qy : Wy \geq \xi - Tx, \ y \in \mathbb{R}^{n_2}_+ \} \right]
\quad \text{with } \xi = \left\lceil \omega - \alpha \right\rceil + \alpha \text{ a discrete random vector}$
For $W$ totally unimodular

- Tender variables $z = Tx$


- $Q(z) = \hat{Q}_\alpha(z)$ for $z \in \alpha + \mathbb{Z}^m$
- $\omega_i$ independent and uniform $\Rightarrow \hat{Q}_\alpha$ is convex hull of $Q$
- Uniform error bound for $\|Q - \hat{Q}_\alpha\|_\infty$

$$\|Q - \hat{Q}_\alpha\|_\infty \leq G(|\Delta|_{f_i}, \ldots)$$

Definition total variation

- $|\Delta|_f$ denotes the total variation of the pdf $f$ of $\omega$
- $|\Delta|_f := \sup_U \sum_{i=1}^{N} |f(x_{i+1}) - f(x_i)|$

with partitions $U$

$|\Delta|_f = 2h$
Known results

Special case $W = I$

- Error bound for simple integer recourse (SIR)

$$\|Q - \hat{Q}_\alpha\|_\infty \leq \sum_{i=1}^{m} q_i \frac{|\Delta| f_i}{8}$$

Generalization for two-stage mixed-integer recourse

- Approximate $v$ by a convex function $\hat{v}$:

$$\|Q - \hat{Q}_\alpha\|_\infty \leq G(|\Delta| f_i, \ldots)$$

- with $G(|\Delta| f_i, \ldots) \to 0$ as $|\Delta| f_i \to 0$. 
Main contribution of this presentation

Goal

- Construct convex approximations $\hat{Q}$ in the risk-averse case
- Derive corresponding error bound $\|Q - \hat{Q}\|_{\infty} \leq G(|\Delta|f_i, \ldots)$
- $G(|\Delta|f_i, \ldots) \rightarrow 0$ as $|\Delta|f_i \rightarrow 0$

Approach

- Use same convex approximation $\hat{v}$ for $v$
- Use same techniques as in the risk-neutral case
- Specialized analysis for risk-averse SIR
Definition

Recourse function

\[ Q(x) = \rho_1 \mathbb{E}_\omega [v(\omega - Tx)] + \rho_2 \text{CVaR}_\beta(v(\omega - Tx)) \]

- Assume \( \rho_1 = 0 \) and \( \rho_2 = 1 \)

Definition CVaR

- Use definition of Rockafellar and Uryasev (2002)

\[ Q(x) = \text{CVaR}_\beta(v(\omega - Tx)) \]

\[ = \min_{\alpha} \left\{ \alpha + \frac{1}{1 - \beta} \mathbb{E}_\omega \left[ (v(\omega - Tx) - \alpha)^+ \right] \right\} \]
Results

New objective function

- Interpret $\alpha$ as first-stage variable

$$\min_{x, \alpha} \left\{ cx + \alpha + \frac{1}{1 - \beta} \mathbb{E}_\omega \left[ \left( \nu(\omega - Tx) - \alpha \right)^+ \right] \right\}$$

Redefine second-stage value function $\nu$

- Define $\bar{\nu}_\beta(\omega, x, \alpha) = \frac{1}{1 - \beta} (\nu(\omega - Tx) - \alpha)^+$
- Then, $\min_{x, \alpha} \left\{ cx + \alpha + \mathbb{E}_\omega [\bar{\nu}_\beta(\omega, x, \alpha)] \right\}$
- Similar as in risk-neutral setting

Asymptotic error bound

- $\| Q - \hat{Q} \|_\infty \leq \frac{1}{1 - \beta} \bar{G}(|\Delta| f_i, \ldots)$
- with $\frac{1}{1 - \beta} \bar{G}(|\Delta| f_i, \ldots) \to 0$ as $|\Delta| f_i \to 0$
Special case: SIR

Definition value function \( v \)

\[
\nu(s) = \min_y \{ y : y \geq s, \ y \in \mathbb{Z}_+ \}
= \lceil s \rceil^+
\]

Definition recourse function \( Q_\beta \)

\[
Q_\beta(x) = \text{CVaR}_\beta(\nu(\omega - Tx))
= \text{CVaR}_\beta(\lceil \omega - Tx \rceil^+)
= \mathbb{E}_\omega[\lceil \omega - Tx \rceil^+ | \omega \geq q_\beta]
\]

\( \triangleright \) with \( q_\beta = F^{-1}(\beta) \)

\( \triangleright \) Expectation of worst 100(1 - \( \beta \))% values of \( \lceil \omega - Tx \rceil^+ \)
Convex approximation

Original function

\[ Q_\beta(x) = \mathbb{E}_{\omega}[(\omega - T x)^+ | \omega \geq q_\beta] \]

Convex approximation

\[ \hat{Q}_\beta(x) = \mathbb{E}_{\omega}[(\omega + 1/2 - T x)^+ | \omega \geq q_\beta] \]

Expected difference function

\[ Q_\beta(x) - \hat{Q}_\beta(x) = \mathbb{E}_{\omega}[(\omega - T x)^+ - (\omega + 1/2 - T x)^+ | \omega \geq q_\beta] \]
Total variation error bound

Expected difference function

\[ Q_\beta(x) - \hat{Q}_\beta(x) = \mathbb{E}_{\omega} \left[ (\omega - Tx)^+ - (\omega + 1/2 - Tx)^+ \right| \omega \geq q_\beta \]

Use conditional density \( g_\beta \)

\[ g_\beta = f \mid \omega \geq q_\beta \]

\[ Q_\beta(x) - \hat{Q}_\beta(x) = \mathbb{E}_{g_\beta} \left[ (\omega - Tx)^+ - (\omega + 1/2 - Tx)^+ \right] \]

Expected difference of risk neutral SIR

Total variation bound

\[ \| Q_\beta - \hat{Q}_\beta \|_\infty \leq \frac{|\Delta| g_\beta}{16} \]
Interpretation error bound

Expression for $g_\beta$

$$g_\beta(x) = \begin{cases} 
0, & x < q_\beta, \\
\frac{f(x)}{1 - \beta}, & x \geq q_\beta.
\end{cases}$$

Computing $|\Delta|g_\beta$

$$|\Delta|g_\beta = \frac{f(q_\beta) + |\Delta|f([q_\beta, \infty))}{1 - \beta}$$

Error bound in terms of $f$

$$\|Q_\beta - \hat{Q}_\beta\| \leq \frac{1}{16} \times \frac{f(q_\beta) + |\Delta|f([q_\beta, \infty))}{1 - \beta}$$
Interpretation error bound

Error bound

\[ \| Q_\beta - \hat{Q}_\beta \| \leq \frac{1}{16} \times \frac{f(q_\beta) + |\Delta| f([q_\beta, \infty))}{1 - \beta} \]

Monotone decreasing tails

- Assume \( f \) is decreasing on \([q_\beta, \infty)\) for sufficiently large \( \beta \)
- Consequence error bound:

\[ \| Q_\beta - \hat{Q}_\beta \| \leq \frac{1}{8} \times \frac{f(q_\beta)}{1 - F(q_\beta)} \]

Interpretation

- Hazard rate of \( \omega \) at \( q_\beta \)
- Increasing failure rate vs decreasing failure rate
Error bound as a function of $\beta$

Error bound

\[ \| Q_\beta - \hat{Q}_\beta \| \leq \frac{1}{8} \times \frac{f(q_\beta)}{1 - F(q_\beta)} \]

Example Weibull distribution

\[ f(x) = \frac{k}{\lambda} \left( \frac{x}{\lambda} \right)^{k-1} \exp\left\{-\left(\frac{x}{\lambda}\right)^k\right\} \]

- $k = 1$: exponential distribution

Interpretation

- Convex approximation is good for fat-tailed distributions
Conclusion

Main result

- Convex approximation for risk-averse mixed-integer recourse
- Asymptotic error bound in general
- Tight bound for simple integer recourse with CVaR

Future research

- More general models than simple integer recourse
- Other risk measures than CVaR
- Multistage models