Long range dependence have been observed in high frequency stock return time series [17].

Fractional Brownian Motion [13] and Fractional Lévy process model [16], [14] and [9] have been studied to capture the long range dependence.

Option pricing with Fractional Brownian motion have been studied in many literature including [10, 7, 1, 18, 6].

The option pricing model based on the fractional Brownian motion allows ARBITRAGE. Rogers (See [15]) suggested an adjusted fractional process, which is semimartingale, using a new kernel.

In this paper, we construct a new fractional Lévy process model using the Rogers’ kernel and apply it to option pricing theory.

Empirical illustration will present to see the performance of the new model.
Let \((B^+(t))_{t \geq 0}\) and \((B^-(t))_{t \geq 0}\) be two independent copies of Brownian motion, and let

\[
B(t) = \begin{cases} 
B^+(t) & \text{if } t \geq 0 \\
B^-(t) & \text{if } t < 0.
\end{cases}
\]

Fractional Brownian Motion \((B_H(t))_{t \geq 0}\):

\[
B_H(t) = c(H) \int_{-\infty}^{t} (\varphi(t - s, H) - \varphi(-s, H)) dB(s)
\]

where \((B(t))_{t \geq 0}\) is Brownian Motion, \(\varphi(x, H) = (x)^{H-\frac{1}{2}} 1_{x \geq 0}\) and

\[
c(H) = \left( \int_{0}^{\infty} ((1 + x)^{H-\frac{1}{2}} - x^{H-\frac{1}{2}})^2 dx + \frac{1}{2H} \right)^{-\frac{1}{2}}.
\]

Fractional Brownian Motion can capture continuous time long range dependence.

The option pricing model based on the fractional Brownian motion allows ARBITRAGE (See [15])!!
Let $(Z^+(t))_{t \geq 0}$ and $(Z^-(t))_{t \geq 0}$ be two independent copies of a Lévy process, and let

$$Z(t) = \begin{cases} 
Z^+(t) & \text{if } t \geq 0 \\
Z^-(t) & \text{if } t < 0 
\end{cases}.$$ 

Fractional Lévy Process $(X_H(t))_{t \geq 0}$:

$$X_H(t) = c(H) \int_{-\infty}^{t} (\varphi(t - s, H) - \varphi(-s, H))dZ(t) = \int_{-\infty}^{t} f(s, t)dZ(s)$$

where $f(s, t) = c(H)(\varphi(t - s, H) - \varphi(-s, H))$. 
Fractional RDTS Process

If $Z$ is RDTS (See [12]), then we obtain the fractional RDTS process.

- The characteristic function of fractional RDTS $X_H(t)$ is
  
  $$E[\exp(iuX_H(t))] = \exp \left( C \int_{-\infty}^{t} F(iuf(t, s); \alpha, \lambda_+, \lambda_-) ds \right)$$

- $E[X_H(t)] = 0$

- $\text{var}(X_H(t)) = C\Gamma \left( 1 - \frac{\alpha}{2} \right) (\lambda_+^{\alpha-2} - \lambda_-^{\alpha-2}) t^{2H}$

- $\text{cov}(X_H(s), X_H(t)) = C\Gamma \left( 1 - \frac{\alpha}{2} \right) (\lambda_+^{\alpha-2} - \lambda_-^{\alpha-2})(t^{2H} + s^{2H} - |t - s|^{2H})$

- $(X_H(t))_{t \geq 0}$ has the long-range dependence if $H \in (\frac{1}{2}, 1)$ and short-range dependence if $H \in (0, \frac{1}{2})$.

- The process $(X_H(t))_{t \geq 0}$ is NOT a semimartingale.
Because \((X_H(t))_{t \geq 0}\) is not a semimartingale, the fractional Lévy process including the fractional RDTS process allows for arbitrage opportunities. As a consequence, the fractional RDTS process cannot be used directly for option pricing.

The \(f_R\)-RDTS process which is a semimartingale approximation of the fractional RDTS process

\[
X(t) = \int_{-\infty}^{t} f_R(t, s; H, \epsilon) dZ(s),
\]

where \((Z(t))_{t \geq 0} \sim RDTS(\alpha, C, \lambda_+, \lambda_-)\) and \(f_R\) is Rogers’ kernel.

Rogers’ kernel [15] equals

\[
f_R(t, s; H, \epsilon) = c(H)(\varphi_R(t - s, H, \epsilon) - \varphi_R(-s, H, \epsilon))
\]

where

\[
\varphi_R(x, H, \epsilon) = (x^2 + \epsilon)^{2H-1/4} 1_{x \geq 0}.
\]
\( H = 0.3 \)  
\( H = 0.75 \)

\( \epsilon = 0 \) : Pure Mandelbrot Kernel  
\( \epsilon > 0 \) : Rogers Kernel
$\alpha = 0.7$, $\lambda_+ = 10$, $\lambda_- = 5$, $\varepsilon = 0.01$;
Consider the $f_R$-RDTS process $(X(t))_{t \geq 0}$. Suppose that the market stock price process $(S(t))_{t \geq 0}$ is given by

$$S(t) = \frac{S(0) \exp \left( \int_0^t \mu(s) ds + X(t) \right)}{E_P[\exp(X(t))]},$$

(2)
on the market measure $\mathbb{P}$, and the risk-neutral stock price process is given by

$$S(t) = \frac{S(0) \exp \left( \int_0^t (r(s) - d(s)) ds + X(t) \right)}{E_Q[\exp(X(t))]},$$

(3)
on the risk neutral measure $\mathbb{Q}$, where

$$\int_0^t \mu(s) ds - \log E_P[\exp(X(t))] = \int_0^t (r(s) - d(s)) ds - \log E_Q[\exp(X(t))].$$

(4)

Then, the discounted price process $(\tilde{S}(t))_{t \geq 0}$,

$$\tilde{S}(t) = \exp \left( - \int_0^t r(s) ds \right) S(t),$$

is a MARTINGALE on the risk-neutral measure $\mathbb{Q}$. 

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Change of measures

As a result, we obtain the following conditions of the equivalent martingale measure $Q$ with respect to the market measure $P$ by (4) and Theorem ??:

1. $\alpha = \tilde{\alpha}$
2. $C = \tilde{C}$
3. $\tilde{\lambda}_+^{\alpha-1} - \tilde{\lambda}_-^{\alpha-1} = \lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}$
4. For all $t \geq 0$, $\lambda_+, \lambda_-, \tilde{\lambda}_+, \tilde{\lambda}_-$, $r(t)$ and $\mu(t)$ must satisfy

$$\int_0^t \mu(s)ds - C \int_{-\infty}^t F \left( uf_R(t, s; H, \epsilon); \alpha, \lambda_+, \lambda_- \right) ds$$

$$= \int_0^t (r(s) - d(s))ds - C \int_{-\infty}^t F \left( uf_R(t, s; H, \epsilon); \alpha, \tilde{\lambda}_+, \tilde{\lambda}_- \right) ds.$$
Model Calibration, S&P 500 Call, Aug. 6, 2008

Aug 06, 2008

- Market Call Price
- CGMY Model Price
- CIR–CGMY Model Price
- $R_f$–RDTS Model Price

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H time series

09/29/2008: 9.22% Drop for Bankruptcy of Lehman Brothers
05/20/2010: 3.29% Drop for Greece's Debt Crisis

Daily Log-return of S&P 500 Index
First Passage Time for Tempered Stable Process and Its Application to Perpetual American Option and Barrier Option Pricing

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The perpetual American option and barrier option pricing also not only studied on BS model but also studied on Lévy models.

[8] discussed the perpetual American option pricing formula on the BS model with the martingale approach,

found perpetual American option pricing formula using the Wiener-Hopf factorization on the Lévy model.

The barrier option price formula under the BS model is provided in literature including [11].

[3] presented the barrier option pricing method on Lévy model.

The partial integro-differential equation method has been very popularly used for barrier option pricing (See [4] and [5]) on Lévy model.

Let $X = (X(t))_{t \geq 0}$ be a Lévy process. Suppose that $\phi_X(t)$ is the characteristic function (ch.F) of $X(t)$ and $\psi_X$ is the Lévy symbol of $X$ that is given by $\psi_X(u) = \log \phi_X(1)(u)$ so that $\phi_X(t)(u) = e^{t\psi_X(u)}$. Let $l \in \mathbb{R}$ be a level. We define a first passage time $\tau(l)$ for the Lévy process $X$ to touch the level $l$ as follows:

$$
\tau(l) = \begin{cases} 
\inf\{t \geq 0 | X(t) \leq l\} & \text{if } l < 0 \\
\inf\{t \geq 0 | X(t) \geq l\} & \text{if } l > 0.
\end{cases}
$$

(5)

Lemma

For all $u \in \mathbb{R}$, if there exist a complex function $\eta(u)$ such that $\Re(-ln(\eta(u))) < 0$, $\phi_X(1)(-i\eta(u))$ is well defined, and

$$
iu + \psi_X(-i\eta(u)) = 0
$$

(6)

then the ch.F of $\tau(l)$ equals to

$$
\phi_{\tau(l)}(u) = E\left[e^{iu\tau(l)}\right] = e^{-ln(\eta(u))}.
$$

(7)
Let $X = (X(t))_{t \geq 0} = (\mu t + \sigma B(t))_{t \geq 0}$ where $(B(t))_{t \geq 0}$ is Brownian motion, $\mu \in \mathbb{R}$, and $\sigma > 0$. The ch.F of the first passage time $\tau(l)$ for $X$ is

$$
\phi_{\tau(l)}(u) = \begin{cases} 
\exp \left( \frac{l \mu - l \sqrt{\mu^2 - 2\sigma^2 u i}}{\sigma^2} \right) & \text{if } l > 0 \\
\exp \left( \frac{l \mu + l \sqrt{\mu^2 - 2\sigma^2 u i}}{\sigma^2} \right) & \text{if } l < 0 
\end{cases}
$$
If $X \sim \text{NIG}(\theta, \beta, \gamma, \mu)$, the ch.F of the first passage time $\tau(l)$ for NIG process as

$$
\phi_{\tau(l)}(u) = \begin{cases} 
\exp(-l\eta^+(u)) & \text{if } l > 0 \\
\exp(-l\eta^-(u)) & \text{if } l < 0 
\end{cases}
$$

where

$$
\eta^+(u) = \frac{-(2\mu\theta + (\mu - \beta)iu) + \sqrt{(2\mu\theta + (\mu - \beta)iu)^2 + ((\mu - \beta)^2 + 2\theta\gamma^2)(u^2 - 4\theta iu)}}{\mu - \beta)^2 + 2\theta\gamma^2},
$$

and

$$
\eta^-(u) = \frac{-(2\mu\theta + (\mu - \beta)iu) - \sqrt{(2\mu\theta + (\mu - \beta)iu)^2 + ((\mu - \beta)^2 + 2\theta\gamma^2)(u^2 - 4\theta iu)}}{\mu - \beta)^2 + 2\theta\gamma^2}.
$$
Figure: Pdf’s of standard NIG distributions (left) and the first passage time of standard NIG processes (right).
In the case of $X \sim \text{NTS}(\alpha, \theta, \beta, \gamma, \mu)$, we find $\eta(u)$ satisfying (6) that is

$$0 = iu + (\mu - \beta)\eta(u) - \frac{2\theta^{1-\frac{\alpha}{2}}}{\alpha} \left( \left( \theta - \beta \eta(u) - \frac{\gamma^2 \eta(u)^2}{2} \right)^{\frac{\alpha}{2}} - \theta^{\frac{\alpha}{2}} \right),$$

which has no explicit solution, but we can find the solution numerically.

In the case of $X \sim \text{CGMY}(\alpha, C, \lambda_+, \lambda_-, \mu)$, we find $\eta(u)$ satisfying (6) that is

$$0 = iu + (\mu - C \Gamma(1-\alpha)(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}))\eta(u) - C \Gamma(-\alpha) \left( (\lambda_+ - \eta(u))^\alpha - \lambda_+^\alpha + (\lambda_- + \eta(u))^\alpha - \lambda_-^\alpha \right)$$

which has no explicit solution.
Figure: Function $\eta(u)$'s and ch.F's for first passage time of standard CGMY processes. The upper left is the function $\eta(u)$ for stdCGMY(0.75, 3, 1) and the upper right is for stdCGMY(0.75, 1, 3). The bottom left is the characteristic function for the stdCGMY(0.75, 3, 1) and the center is for stdCGMY(0.75, 1, 3). Pdf's of standard CGMY distributions are on the right left. The pdfs of the first passage time of standard CGMY processes are on the bottom right.
Let $X = (X(t))_{t \geq 0}$ be a Lévy process (Arithmetic BM, NTS, CGMY, and so on), and $\tau(l)$ is the first hitting time given by (5). Let $r$ and $d$ be the risk free rate of return and the continuous dividend rate of a given underlying asset, respectively. The underlying asset price process $(S(t))_{t \geq 0}$ is assumed as

$$S(t) = S(0)e^{X(t)}.$$ 

All of the market models in this paper are based on the risk-neutral world which has no-arbitrage. So we assume that the discount price process $(\tilde{S}(t))_{t \geq 0}$ with $\tilde{S}(t) = e^{-(r-d)t}S(t)$ is martingale. In this case, we referred to the risk-neutral price model as Lévy market model.
We consider a perpetual American call and put options with strike price $K$. The perpetual call price is equal to

$$C_{\text{perpetual}} = \begin{cases} 
\frac{K}{\eta^+(ir) - 1} \left( \frac{S(0)(\eta^+(ir) - 1)}{K\eta^+(ir)} \right)^{\eta^+(ir)} & \text{if } S(0) \leq L^+ \\
\frac{S(0) - K}{S(0) - K} & \text{if } S(0) > L^+ 
\end{cases},$$

where

$$L^+ = \frac{\eta^+(ir)K}{\eta^+(ir) - 1},$$

and $\eta^-(ir)$ is the value satisfying (6) and (7) for $l > 0$ and $u = ir$. 
The perpetual put price is equal to

\[
P_{\text{perpetual}} = \begin{cases} 
\frac{K}{1 - \eta^{-}(ir)} \left( \frac{S(0)(\eta^{-}(ir) - 1)}{K\eta^{-}(ir)} \right)^{\eta^{-}(ir)} & \text{if } S(0) \geq L^{-}, \\
K - S(0) & \text{if } S(0) < L^{-}
\end{cases}
\]

where

\[
L^{-} = \frac{\eta^{-}(ir)K}{\eta^{-}(ir) - 1},
\]

and \(\eta^{-}(ir)\) is the value satisfying (6) and (7) for \(l < 0\) and \(u = ir\).
Figure: Perpetual call (left) and put (right) prices under the CGMY market model (Solid Curve) and BS model (Dash-dot Curve).
Barrier Option

Let $B$ be the barrier level, $K$ be the strike price of call and put, and $T$ be the time to maturity. Suppose that the current underlying asset price is $S(0)$, and let $l = \log(B/S(0))$. Then we have the down-and-in and down-and-out options if $l < 0$ and the up-and-in and up-and-out options if $l > 0$. Let $r$ and $d$ be the risk free rate of return and the continuous dividend rate of a given underlying asset, respectively.

The down-and-in call ($c_{di}$), up-and-in put ($p_{ui}$) are priced by

$$c_{di} = \frac{e^{-rT} K^{1+\rho}}{(2\pi)^2 B^\rho} \int_{-\infty}^{\infty} \left( \frac{B}{K} \right)^{iu} \left( \frac{H(u)}{(\rho - iu)(1 + \rho - iu)} \right) du, \quad \rho < -1$$

and

$$p_{ui} = \frac{e^{-rT} K^{1+\rho}}{(2\pi)^2 B^\rho} \int_{-\infty}^{\infty} \left( \frac{B}{K} \right)^{iu} \left( \frac{H(u)}{(\rho - iu)(1 + \rho - iu)} \right) du, \quad \rho > 0,$$
and the up-and-in call ($c_{ui}$) and down-and-in put ($p_{di}$) are priced by

$$c_{ui} = \begin{cases} 
    e^{-rT} \frac{K^{1+\rho}}{(2\pi)^2 B\rho} \int_{-\infty}^{\infty} \left( \frac{B}{K} \right)^{iu} \left( \frac{H(u)}{(\rho-iu)(1+\rho-iu)} \right) du & \text{if } K \leq B, \\
    e^{-rT} \frac{K^{1+\rho}}{2\pi S(0)^\rho} \int_{-\infty}^{\infty} \left( \frac{S(0)}{K} \right)^{iu} \left( \frac{\phi_X(T-t)(u+i\rho)}{(\rho-iu)(1+\rho-iu)} \right) du & \text{if } K > B, \quad \rho < -1.
\end{cases}$$

and

$$p_{di} = \begin{cases} 
    e^{-rT} \frac{K^{1+\rho}}{(2\pi)^2 B\rho} \int_{-\infty}^{\infty} \left( \frac{B}{K} \right)^{iu} \left( \frac{H(u)}{(\rho-iu)(1+\rho-iu)} \right) du & \text{if } K \geq B, \\
    e^{-rT} \frac{K^{1+\rho}}{2\pi S(0)^\rho} \int_{-\infty}^{\infty} \left( \frac{S(0)}{K} \right)^{iu} \left( \frac{\phi_X(T-t)(u+i\rho)}{(\rho-iu)(1+\rho-iu)} \right) du & \text{if } K < B, \quad \rho > 0.
\end{cases}$$

where

$$H(u) = \int_{-\infty}^{\infty} \frac{e^{T\psi_X(u+i\rho)} - e^{-ivT}}{\psi_X(u + i\rho) + iv} \phi_T(v)dv.$$
Figure: Down-and-in call (left) and put (right) prices where the barrier level is $B = 1850$, current underlying index price $S(0) = 1968.89$, and time to maturity is $T = 6$ months (0.5 year fraction). The solid curves are call/put prices of CGMY model, and the dashed curves are of BS model.
Figure: Up-and-in call (left) and put (right) prices where the barrier level is $B = 2020$, current underlying index price $S(0) = 1968.89$, and time to maturity is $T = 6$ months (0.5 year fraction). The solid curves are call/put prices of CGMY model, and the dashed curves are of BS model.
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