Factor Models for Scenario Construction in Long Term Asset Allocation

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Outline

Scenario Generation

Multi-factor Models Overview
  Statistical Factors
  Economic/Financial Factors

Case Study
  Independent Component Analysis: Results
  Economic Models: Results

Conclusion

Bibliography
Scheme for Optimization Problem

**Asset Model:** \( r_i = g \left( \{ F_j \}^M_{j=1} ; \beta ; \epsilon_i \right) \)

**Est. quant.:** \( \hat{\beta}, \{ \hat{F}_j \}^M_{j=1}, \hat{\epsilon}_i \)

**Discr. Sample Space:**
- Monte Carlo Sim.
- Discr. True joint Distr.

**Gen. Port. Meas.:** i.e. Wealth \( R = \sum_{i=1}^{N} x_i r_i \)

**Def. Opt. Problem:** \( \min / \max_{x_i} f(R) \)
  s.t. constraints

**Allocation Decision**
Scenario Generating Models

Linear Models

Non Linear Models
Linear vs Non Linear Models

Non Linear Models:

\[ r_{t,i} = g_i (f_1, ..., f_N) + e_t \]

Example: *Black & Scholes model.*

Linear Models:

\[ r_i = \sum_{j=1}^{N} \beta_{i,j} f_j + e_t \]

Example *CAPM model.*
Scenario Generating Models

- Linear Models
  - Static Models
  - Dynamic Models

- Non Linear Models
Static Linear Model.

The relations between asset returns and their factors do not change through time

\[ r_t = \beta \ast F_t + \epsilon_t \]

where \( \beta \in \mathbb{R}^n \) is the vector of unknown parameters. \( F_t \in \mathbb{R}^n \) is the vector of the available Factors.
Dynamic Models.

The factor exposures $\beta$ change through time. The problem can be modelled as a State Space one:

$$r_t = \beta_t \ast F_t + \epsilon_t$$
$$\beta_t = A \ast \beta_t + u_t$$

A Filter to extrapolate the trajectories of the unknown process $\beta_t$ is necessary. If $\beta_t$ is an AR process, the Kalman Filter can be used.
Scenario Generating Models

Linear Models

Static Models

Economic Factors

Statistical Factors

Dynamic Models

Non Linear Models
Static Linear Model

The general Static Linear Model has the following representation

\[ r_i = \sum_{j=1}^{n} \beta_{i,j} F_j + \epsilon_i \]

where \( r_i \) is the return of asset \( i \).
\( \beta_{i,j} \) is the contribution of the factor \( F_j \) to the return of asset \( i \).

The factors \( F_j \) can be:

- Prespecified, that is, chosen using an Economic or Financial View. The Cause-Effects are known. \( F_j \) are named **Economic/Financial Factors**.

- **Statistical Factors**: The Cause-Effect relations are unknown. Therefore the factors \( F_j \) can be extrapolated using statistical procedures.
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The observable financial/economic variables can be seen as dependent ones of fewer unobservable factors. Then a statistical method for extrapolating such latent variables is necessary.

This procedure is helpful in obtaining the following aims:

- Reduction of the dimensionality.
- Reproduction of the distributional Features observed in the asset returns.
The classic statistical methods, such as PCA or FA, could be inappropriate in the financial context, since the latters are coherent with a Gaussian world.

We investigate the ability of the Independent Component Analysis to extrapolate unobservable Factors.
Independent Component Analysis: Definition 1

Given a set of observable r.v.’s $X_1(t), ..., X_n(t)$, we assume that:

$$
\begin{pmatrix}
X_1(t) \\
... \\
X_i(t) \\
... \\
X_n(t)
\end{pmatrix}
= A
\begin{pmatrix}
S_1(t) \\
... \\
S_i(t) \\
... \\
S_n(t)
\end{pmatrix}
$$

where $A$ is an unknown mixing matrix. $S_i(t)$ contains unobservable independent signals.
ICA estimation procedures

There are different Statistical Methodologies for obtaining the ICs classified into:

- Maximum Likelihood Methods: we need to assume a prior distribution for the components.
- Minimization of Mutual Information. That means: minimize the dependence between random varaibles.
- Maximization of non-Gaussianity.

We obtain the Independent components using a Maximization of non-gaussianity method using the FastICA Algorithm.
Definition

The Variance Gamma r.v. is a Normal Variance Mean Mixture where the mixing random variable is Gamma distributed, i.e.

\[ Y = \mu_0 + \theta V + \sigma \sqrt{V} Z \]

where \( V \sim \Gamma(a, 1) \).
Advantages of the choice:

- The mathematical tractability of the VG.
- The VG is approximated by a finite mixture of normal and its parameters are estimated through EM algorithm.
- The VG is able to capture skewness and kurtosis in financial time series.
- Estimated on daily data, the annual return distribution is computed by VG summation property.
Scenario Generation

Observed Returns $R_j(t)$

FastICA

Ind. Comp. $S_j$
Mixing Matrix A

EM algorithm.

$S_j \sim VG_{\hat{\theta}}$
$\hat{\theta}$ vector param.

Sampling

From $\hat{S}_j(t)$ we get a scenario for $R_j(t)$

$\hat{R}_j = A \ast \hat{S}_j$

From $VG_{\hat{\theta}}$ We simulate a scenario $\hat{S}_j(t)$
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Economic/Financial Factor Models

Financial Economic Factors:
\[ F = \begin{cases} \text{Inflation} \\ \text{Interest Rates} \\ \text{GDP} \\ \text{Currency Ex.} \\ \text{etc} \end{cases} \]

Asset Behavior:
\[ r = \begin{cases} \text{Bonds} \\ \text{Equities} \\ \text{Commodities} \\ \text{Real Est.} \\ \text{Renewables} \\ \text{Infrastruct.} \\ \text{etc} \end{cases} \]

Questions:
1. How to estimate \( \beta \)?
2. How to model the Economic Factors?
How to Model $\beta$? OLS

Under the assumptions:

- **Strict exogeneity**: the errors in the regression have conditional zero mean $E(\epsilon|F) = 0$.
- **The regressors must be linear independent**: The rank of $F$ is equal to the number of factors.
- **Homoskedastic and no-autocorrelated Errors**, $Var(\epsilon|F) = \sigma I$

The $\beta$ is estimated using the OLS:

$$\beta = (FF')^{-1}Fr'$$
How to model the Financial/Economic Factors?

A good model must capture the behaviour of the marginal distribution of each economic/financial factor, as well as the dependence structure. The key problem is to identify an appropriate joint distribution.

In a multivariate context, both Parametric and Non-Parametric estimation of joint distributions is a challenging task.

In order to reduce the complexity, we apply the ICA methodology to the Economic/Financial Factors.
Scenario Generating

- **Observed Factors** $F_j(t)$
- **Observed Ret** $R_j(t)$
- **FastICA**
- **ICs.** $S_j$
- **Matrix A**
- **EM algorithm.**
- **$S_j \sim VG_{\hat{\theta}}$**
- **Sampling**
- **We simulate the scenario** $\hat{S}_j(t)$.

\[ \hat{R}_j(t) = \hat{\beta} \ast \hat{F}_t \]

\[ \hat{F}_j(t) = A \ast \hat{S}_j(t) \]
The dataset is composed by the following Assets and Factors:

<table>
<thead>
<tr>
<th>Assets</th>
<th>Economic Factors</th>
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<tbody>
<tr>
<td>EU Commodity Index</td>
<td>ROCOM.NA</td>
</tr>
<tr>
<td>EU Renewbels Index</td>
<td>Beuenrg.Ind</td>
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<tr>
<td>EU Private Equity</td>
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<td>MXEUOIN.Ind</td>
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<tr>
<td>EU Defense Infrastruct.</td>
<td>MXEUOAD</td>
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</table>

Data ranging from 01-06-2010 to 21-01-2013
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Independent Components with Variance Gamma distributions

![Graphs showing independent components with VG distributions](image)
Return Distributions using Statistical Factors

Return Distributions with VG Statistical Factors

- EU Commodity Index
- EU Renewables Index
- EU Private Equity Index
- EU Real Estate Index
- EU Infrastructure Index
- EU Defense Infrastructure Index

Example Tree using Statistical Factors

Tree of EU Private Equity

\[ \omega_1 = 0.0545, \quad p_{\omega_1} = 0.4 \]
\[ \omega_2 = -0.0681, \quad p_{\omega_2} = 0.6 \]
\[ \omega_3 = 0.0735, \quad p_{\omega_3} = 0.136 \]
\[ \omega_4 = -0.0943, \quad p_{\omega_4} = 0.264 \]
\[ \omega_5 = 0.0954, \quad p_{\omega_5} = 0.204 \]
\[ \omega_6 = -0.072, \quad p_{\omega_6} = 0.396 \]
\[ \omega_7 = 0.0845, \quad p_{\omega_7} = 0.0571 \]
\[ \omega_8 = -0.0826, \quad p_{\omega_8} = 0.0652 \]
\[ \omega_9 = 0.0773, \quad p_{\omega_9} = 0.1267 \]
\[ \omega_{10} = -0.0835, \quad p_{\omega_{10}} = 0.1372 \]
\[ \omega_{11} = 0.0632, \quad p_{\omega_{11}} = 0.1060 \]
\[ \omega_{12} = -0.0843, \quad p_{\omega_{12}} = 0.09792 \]
\[ \omega_{13} = 0.0694, \quad p_{\omega_{13}} = 0.18216 \]
\[ \omega_{14} = -0.0714, \quad p_{\omega_{14}} = 0.2138 \]
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Independent Components of Economic Factors

ICs obtained From Economic Factors with VG distributions

ICs Dens. Of Econ.  VG Dens.
Return distribution with Economic Factors

Return Distributions with VG Economic Factors

- EU Commodity Index
- EU Renewables Index
- EU Private Equity Index
- EU Real Estate Index
- EU Infrastructure Index
- EU Defense Infrastructure Index
### Statistical Versus Economic Factors: Empirical Results

\[
\sqrt{MSE_f} = \sqrt{\frac{1}{N} \sum_{i=1}^{N} [f_{i,\text{emp}} - f_{i,\text{VG}}]^2}
\]

\[
\sqrt{MSE_F} = \sqrt{\frac{1}{N} \sum_{i=1}^{N} [F_{i,\text{emp}} - F_{i,\text{VG}}]^2}
\]

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</table>
Conclusion

Choosing the right distribution for factors is important since we get better estimates for asset returns. These latter quantities enter in the Stochastic Optimization Problem and influence the investment decision.
Economic/Financial and Statistical Factor Models.
Scenario Tree Generation

Variance Gamma

Independent Component Analysis