SOME EXPLICIT RESULTS FOR THE DISTRIBUTION PROBLEM OF STOCHASTIC LINEAR PROGRAMMING

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This paper presents the “wait and see” problem of stochastic linear programming. We are interested in finding the distribution of \( \max z(x) \)

\[
\begin{align*}
\text{Max } z(x) &= c^T x \\
\text{s.t.: } (A,l)x &= b \\
x &\geq 0
\end{align*}
\]
Early works on the distribution problem can be found in:


For additional references:


most analyses rely on approximation techniques or simulation:


Contribution:

In this paper, we obtain the first explicit results for stochastic c and the first results for non-exponential distributions, namely, uniform, gamma, and triangle.
Basic Theory

A complete theoretical discussion of this problem can be found in


Bereanu discovered that under certain assumptions, the sample space of the random coefficients can be partitioned into non-overlapping sets.

The ith basis is optimal:

\[
\begin{align*}
     c_B^i B_i^{-1} a_j - c_j & \geq 0 \\
     B_i^{-1} b & \geq 0
\end{align*}
\]
The case of stochastic $b$

Given the $m$-tuple $b = (b_1, ..., b_m)$

$$s_{1i} = \{b | (B_i^{-1}b)_j \geq 0 \text{ for all } j = 1, ..., m \}$$

$$P(s_{1i} \cap s_{1j}) = 0 \quad i \neq j$$

The **CDF of max $z$** may be found by evaluating:

$$\sum \text{Pr}\{[z(x) \leq \emptyset] \cap S_{2i}\} = \sum \int \ldots \ldots \int f(b) \prod_{i=1}^{n} db_i$$

The limits of integral will be linear equations representing the space enclosed by the $m$ or $n$ hyperplanes.
Ewbank, Foote and Kumin (EFK) Theory

The feasibility condition is: \[ B_i^{-1}b \geq 0 \]

And by substituting we have:

\[ B(B^{-1}b) = b = Br \]

The probability that a basis remains feasible is:

\[ P = \int \ldots \int f(b) \prod_{i=1}^{n} db_i \]

\[ s = \{ b \mid B_i^{-1}b \geq 0 \} \]

And by substituting we have:

\[ P = \int \ldots \int f(Br|J_r) \prod_{i=1}^{m} dr_i \]

\[ r \geq 0 \]
Where the Jacobian is defined by:

\[ J_r = \det \left( \frac{\partial b_k}{\partial r_i} \right) = \det(B_{ki}) = \det(B) \]

\[ b_k = (B r)_k = \sum_i^m B_{kj} \, r_j \]

\[ \frac{\partial b_k}{\partial r_i} = B_{ki} \]

\[ J_r = \det(B_{ki}) = \det(B) \]
Case of stochastic c

\[ \sum_H \Pr\{[z(x) \leq \emptyset] \cap S_{2i}\} = \int \cdots \int f'(c) \prod_{i=1}^{n} dc_i \]

\[ S_{2i} = \{c | c_i^i B_i^{-1} a_j - c_j \geq 0\} \text{ for all } j = 1, ..., m \]

The limits of integral will be linear equations representing the space enclosed by the hyperplanes.

\[ J_r = (-1)^{n-h} \det(B^T) = (-1)^{n-h} \det(B) \]
Problem Solving/ Structure

The steps in Bereanu's algorithm are:

**Step 1:** Create all the necessary parameters

**Step 2:** Generate all the possible bases. They are the candidates which will be checked for optimality, Find the total number of candidates

**Step 3:** Check the optimality condition (deterministic) for the first selected Candidate

**Step 4:** Calculate the region formed by the intersection of $m$ or $n$ hyperplanes where the objective function is less than $\phi$

**Step 5:** Calculate the integral of the joint density intersected with the region of the basis which satisfies the optimality condition

**Step 6:** Add all of the values of integrals in order to find the cumulative distribution of $\phi$
Problem Solving/ Structure

The steps in EFK algorithm are:

**Step 1:** Create all of the necessary parameters

**Step 2:** Generate all the possible bases. They are the candidates which will be checked for optimality, Find the total number of candidates

**Step 3:** Same as Bereanu's algorithm, verify the optimality condition (deterministic)

**Step 4:** Apply the transformation \( b = Br \)

**Step 6:** Calculate the region of the integration which is intersection of \( m \) or \( n \) hyperplanes where the objective function is less than \( \phi \)

**Step 7:** Calculate the integral of the joint density intersected with the region of the basis which satisfies the optimality condition

**Step 8:** Add all of the integrals to find the cumulative distribution of \( \phi \)
Problem Solving/ Tools

• Software : Wolfram Mathematica software 8.0

• For Case II : The Vertex enumeration package to enumerate the feasible vertices of the convex polytope given by the constraints.


Larger size of the problem → more memory capacity

appropriate alternative is OU supercomputing center for education and research
OU Supercomputer:

Among all of the (HPC) systems, Boomer(I) was selected because of the performance and memory capacity.

**CPUs:** All of Boomer's compute nodes have dual Intel Xeon E5-2650 "Sandy Bridge" oct core 2.0 GHz CPUs; there is also one "fat node" with quad Intel Xeon E7-4830 "Westmere" oct core 2.13 GHz CPUs.

**RAM:** Most of Boomer's compute nodes have 32 GB of 1333 MHz RAM and 23 with 64 GB of 1333 MHz RAM; the one "fat node" has 1 TB of 1066 MHz RAM, which is called large memory.

**Accelerators:** Boomer also has 18 NVIDIA Tesla M2075 cards, for an aggregate of an additional approximately 9 TFLOPs double precision.
## Explicit results

The problem was examined for 4 different types of distributions as shown below. The coefficients were randomly generated in the interval [1,3]

<table>
<thead>
<tr>
<th>Distributions</th>
<th>Defined equation</th>
<th>Parameters</th>
<th>pdf</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Exponential</strong></td>
<td>( f(x) = \lambda e^{-\lambda x} )</td>
<td>( \lambda = 1 )</td>
<td>( f(x) = e^{-x} )</td>
</tr>
<tr>
<td><strong>Uniform</strong></td>
<td>( f(x) = \frac{1}{a-b} )</td>
<td>( b = 8 ) ( a = 0 )</td>
<td>( f(x) = \frac{1}{8} )</td>
</tr>
<tr>
<td><strong>Gamma</strong></td>
<td>( f(x) = \frac{(x-\gamma)^{\alpha-1} \exp[-\frac{(x-\gamma)}{\beta}]}{\beta^\alpha \Gamma(\alpha)} )</td>
<td>( \beta = 1 ) ( \alpha = 1 ) ( \gamma = 1 )</td>
<td>( f(x) = xe^{-x} )</td>
</tr>
<tr>
<td><strong>Triangular</strong></td>
<td>( f(x</td>
<td>a, b, c) = \begin{cases} 0 &amp; x &lt; a, b &lt; x \ \frac{2(x-a)}{(b-a)(c-a)} &amp; a \leq x \leq c \ \frac{2(b-a)}{(b-a)(b-c)} &amp; c &lt; x \leq b \end{cases} )</td>
<td>( a = 0 ) ( c = 0.5 ) ( b = 1 )</td>
</tr>
<tr>
<td>Case</td>
<td>Case I, EFK Model</td>
<td></td>
<td></td>
</tr>
<tr>
<td>------</td>
<td>-------------------</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>$\begin{bmatrix} 1 &amp; 1 \ 1 &amp; 2 \end{bmatrix}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>$b_i \sim \text{Gamma}(\alpha = 2, \beta = 1), i = 1,2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>${2,3}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>CDF</td>
<td>$\left{ \begin{array}{ll} \frac{1}{648}e^{-\phi}(-648 + 648e^{\phi} - 648\phi - 207\phi^2 - 26\phi^3) &amp; \phi &gt; 0 \ 0 &amp; \phi &lt; 0 \end{array} \right.$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>CDF Plot</td>
<td><img src="image" alt="CDF Plot" /></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Time</td>
<td>1.685 sec</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Case</td>
<td>Case II, EFK Model</td>
<td></td>
<td></td>
</tr>
<tr>
<td>------</td>
<td>-------------------</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
| A    | \[
\begin{bmatrix}
2 & 5 \\
2 & 4 \\
\end{bmatrix}
\] |
| B    | \[c_i \sim c_1 \times c_2 e^{-(c_1 + c_2)}, i = 1, 2\] |
| C    | \{3, 1\} |

**CDF**

\[
e^{-6\phi}(13078406e^{6\phi} - 4108797e^{\frac{16\phi}{3}} (3 + 5\phi) + 343(1483 + 5874\phi + 8712\phi^2)) + \\
3e^{\frac{11\phi}{3}} (3688569 + 8952405\phi + 4209590\phi^2))
\]

\[
\begin{align*}
\text{CDF Plot} & \\
\text{Time} & = 11.544 \text{ sec}
\end{align*}
\]
<table>
<thead>
<tr>
<th>Case</th>
<th>Case I. Stochastic Recourse Vector. Bereanu’s model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>[2 2 3 1 2 2 2 3 2]</td>
</tr>
<tr>
<td></td>
<td>[1 1 1 1 2 2 2 3 1]</td>
</tr>
<tr>
<td></td>
<td>[2 3 3 2 1 1 1 3 1]</td>
</tr>
<tr>
<td></td>
<td>[2 2 1 3 3 2 3 1 3]</td>
</tr>
<tr>
<td></td>
<td>[3 1 3 3 1 3 1 2]</td>
</tr>
<tr>
<td></td>
<td>[3 3 3 1 1 1 3 1 2]</td>
</tr>
<tr>
<td></td>
<td>[1 1 1 3 1 2 1 1 3]</td>
</tr>
<tr>
<td></td>
<td>[1 1 1 3 1 2 1 1 3]</td>
</tr>
<tr>
<td></td>
<td>[3 1 1 2 2 1 3 2 2]</td>
</tr>
</tbody>
</table>

- B: $b_i \sim \text{Gamma} (\alpha = 2, \beta = 1), i = 1,2,3,4,5,6,7,8,9$
- C: $\{3, 5, 3, 4, 5, 4, 4, 2, 4\}$
- CDF: Insolvable, The integrals got killed after 48 hours
<table>
<thead>
<tr>
<th>Case</th>
<th>Case I, EFK Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>[3 3 1&lt;br&gt;1 3 1&lt;br&gt;2 3 3]</td>
</tr>
<tr>
<td>B</td>
<td>{b_1, b_2, b_3} \sim \text{Uniform}(0,8)</td>
</tr>
<tr>
<td>C</td>
<td>{2,2,2}</td>
</tr>
</tbody>
</table>

**CDF**

\[
\begin{align*}
\text{CDF} &= \begin{cases} 
\frac{43}{24} & \phi > 8 \\
\frac{\phi(2688-204\phi+5\phi^2)}{6144} & 0 < \phi \leq 8 \\
0 & \phi < 0
\end{cases}
\end{align*}
\]

**CDF Plot**

**Time**

13.29 sec
<table>
<thead>
<tr>
<th>Case</th>
<th>Case II, EFK Model</th>
</tr>
</thead>
</table>
| A    | \[
|      | \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}\] |
| B    | \(\{c_1, c_2\} \sim \text{Uniform}(0,8)\) |
| C    | \(\{3,5\}\) |

\[
\begin{align*}
\text{CDF} &= \frac{2}{3} e^{-\phi_{15}}(-35 + 20e^{150} + 15e^{90} + 24e^{60} - 9e^{45} + \\
& 40e^{30} + 40e^{300} - 15e^{45} - 15e^{150} + 15e^{15}) \\
& 0 \quad \phi > 0 \\
& \quad \phi < 0
\end{align*}
\]

CDF Plot

Time

2.6 sec
<table>
<thead>
<tr>
<th>Case</th>
<th>Case I, EFK Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$\begin{bmatrix} 1 &amp; 1 \ 1 &amp; 2 \end{bmatrix}$</td>
</tr>
<tr>
<td>B</td>
<td>$b_i \sim \text{Exponential}(\lambda = 1), i = 1, 2$</td>
</tr>
<tr>
<td>C</td>
<td>${2, 3}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Case</th>
<th>CDF</th>
</tr>
</thead>
</table>
| CDF | $\begin{cases} 
\frac{1}{6} e^{-\phi (-6 + 6e^{\phi} - \phi)} & \phi > 0 \\
0 & \phi < 0 
\end{cases}$ |

**CDF Plot**

<p>| Time | 2.8 sec |</p>
<table>
<thead>
<tr>
<th>Case</th>
<th>Case II, EFK Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$\begin{bmatrix} 1 &amp; 2 \ 2 &amp; 1 \end{bmatrix}$</td>
</tr>
<tr>
<td>B</td>
<td>$c_i \sim \frac{1}{100} e^{-\frac{1}{10^i(c_1+c_2)}}, i = 1,2$</td>
</tr>
<tr>
<td>C</td>
<td>${10,10}$</td>
</tr>
</tbody>
</table>

CDF

$$\begin{cases} 1 - 2e^{-\frac{\phi}{50}} + e^{-\frac{30}{100}} & \phi > 0 \\ 0 & \phi < 0 \end{cases}$$

Time

1.747 sec
<table>
<thead>
<tr>
<th>Case</th>
<th>Case I, EFK Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$\begin{bmatrix} 3 &amp; 3 \ 5 &amp; 1 \end{bmatrix}$</td>
</tr>
<tr>
<td>B</td>
<td>$b_i \sim \text{Triangular}(x</td>
</tr>
<tr>
<td>C</td>
<td>${3,1}$</td>
</tr>
</tbody>
</table>

**CDF**

$$
\begin{align*}
0 & \quad \varnothing \leq 0 \\
1 & \quad \varnothing > 2 \\
0.83 + 0.6\varnothing - 1.05\varnothing^2 + 1.97\varnothing^3 - 0.85\varnothing^4 & \quad 0.75 \leq \varnothing \leq 1 \\
-6.3 + 8\varnothing + 14.6\varnothing - 11\varnothing^2 - 3.6\varnothing^3 + 0.46\varnothing^4 & \quad 1.5 \leq \varnothing \leq 2 \\
0 + 0.72\varnothing^2 - 0.063\varnothing^4 & \quad 0 \leq \varnothing \leq 0.75 \\
0 & \quad \varnothing < 0
\end{align*}
$$

**CDF Plot**

Time | 22.153 sec
<table>
<thead>
<tr>
<th>Case</th>
<th>Case II, EFK Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$\begin{bmatrix} 3 &amp; 3 \ 5 &amp; 1 \end{bmatrix}$</td>
</tr>
<tr>
<td>B</td>
<td>$b_i \sim \text{Triangular}(x</td>
</tr>
<tr>
<td>C</td>
<td>${3,1}$</td>
</tr>
</tbody>
</table>

**CDF**

$$
\begin{align*}
0 & \quad \varphi \leq 0 \\
3.5 & \quad \varphi > 1 \\
0 - 6\varphi^2 + 120\varphi^3 - 298\varphi^4 & \quad 0.1 \leq \varphi \leq 0.2 \\
-2.5 + 8\varphi + 4 - 8\varphi^3 + 2\varphi^4 & \quad 0.5 \leq \varphi \leq 1 \\
0 + 6\varphi^2 - 2\varphi^4 & \quad 0.2 \leq \varphi \leq 0.5 \\
0.302\varphi^4 & \quad \varphi < 0
\end{align*}
$$

**CDF Plot**

<table>
<thead>
<tr>
<th>Time</th>
<th>39.593 sec</th>
</tr>
</thead>
</table>
Comparing the run times of the different sizes of matrix $A$ problems of size 2, 3, 6 and 9 problems of size 9 were tested using the Jacobian transformation method when the matrix $(b)$ includes random variables.

Case I

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Size: 2 $\times$ 2</th>
<th>Size: 3 $\times$ 3</th>
<th>Size: 6 $\times$ 6</th>
<th>Size: 9 $\times$ 9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exponential</td>
<td>1.89</td>
<td>5.37</td>
<td>754.05</td>
<td>20907.31</td>
</tr>
<tr>
<td>Gamma</td>
<td>5.73</td>
<td>36.14</td>
<td>2743.74</td>
<td>56149</td>
</tr>
<tr>
<td>Uniform</td>
<td>0.69</td>
<td>9.46</td>
<td>8563.63</td>
<td>61625.48</td>
</tr>
<tr>
<td>Triangular</td>
<td>19.26</td>
<td>395.32</td>
<td>38814</td>
<td>491</td>
</tr>
</tbody>
</table>

For all distributions Minitab shows that the increase is not linear and the run time more likely follows a Weibull or Gamma distribution (with p-value > 0.250).
Eectiveness of Ewbank, Foote and Kumin transformation method

100 problems were solved for size \( m = n = 3 \) using both Bereanu's method and the EFK transformation method to compare the two.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Bereanu method</th>
<th>EFK method</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu )</td>
<td>36.45</td>
<td>4.94</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>30.31</td>
<td>3.64</td>
</tr>
<tr>
<td>min</td>
<td>8.3</td>
<td>1.22</td>
</tr>
<tr>
<td>max</td>
<td>145.03</td>
<td>17.1</td>
</tr>
<tr>
<td>Median</td>
<td>28.65</td>
<td>3.931</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Time Reduction(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu )</td>
<td>86.4%</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>88.0%</td>
</tr>
<tr>
<td>min</td>
<td>85.3%</td>
</tr>
<tr>
<td>max</td>
<td>88.2%</td>
</tr>
<tr>
<td>Median</td>
<td>86.3%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Bereanu method</th>
<th>EFK method</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu )</td>
<td>2.730430358</td>
<td>0.20075758</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>2.384626268</td>
<td>0.1760174</td>
</tr>
<tr>
<td>min</td>
<td>0.3620647</td>
<td>0.016</td>
</tr>
<tr>
<td>max</td>
<td>10.7089013</td>
<td>0.749</td>
</tr>
<tr>
<td>Median</td>
<td>2.2457554</td>
<td>0.14</td>
</tr>
</tbody>
</table>
Impact of Distributions

measure the impact of EFK method on different types of distributions

\[
\text{Time reduction} = \frac{\text{Run time for Bereanu's method} - \text{Run time for EFK method}}{\text{Run time for Bereanu's method}} \times 100\%
\]

\[
\text{Standard deviation reduction} = \frac{\text{SD for Bereanu's method} - \text{SD for EFK method}}{\text{SD for Bereanu's method}} \times 100\%
\]

indicate how the EFK method reduces the variability of time elapsed for solving the problems.
Reducing the computational time

Uniform, exponential, gamma, triangular

Standard deviation reduction

Uniform, exponential, gamma, triangular

Work faster by using the transformation method

Case I
Exponential, gamma, uniform, triangular

Case II
Exponential, uniform, gamma, triangular
Conclusion:

- new explicit results for four different distributions including exponential, gamma, uniform and triangular were obtained (stochastic b and stochastic c).

- The program was verified by comparing the solved example in the Ewbank, Foote and Kumin paper.

- Comparing the run times of both algorithms indicates that the Jacobian transformation introduced by Ewbank, et al. significantly reduces the computational time.
Future Research:

1. Comparing the results to approximation methods.

2. Considering the problem when each of the objective function coefficients or right-hand side follows a separate distribution.

3. Considering the problem in the case which b and c are random at the same time.

4. Comparing the run time for the same distributions but with different parameters.

5. Considering the method for additional types of distribution.

6. Considering the method for larger problems especially for random c.

7. Applying the method to real and practical problems.
Thank you
End of Presentation

contact: afrooz.ansaripour@ou.edu
Back up slides
Problem Solving / My structure

Case I Bereanu's algorithm

Case I EFK algorithm
Problem Solving/ My structure

Case II Bereanu’s algorithm

Case II EFK algorithm
Bereanu’s Theory - Stochastic b

The ith basis is said to satisfy the optimality condition when:

\[ c_B^i B_i^{-1} a_j - c_j \geq 0 \]  
Optimality criterion

\[ B_i^{-1} b \geq 0 \]  
Feasibility criterion

If only the b vector is random, the CDF of max z may be found by evaluating:

\[ Pr\{[z(x) \leq \phi] \cap S_{2i}\} = \int ... \int f(b) \prod_{i=1}^{n} db_i \]

\[ S_{2i} = \{b \mid (B_i^{-1}b)_j \geq 0 \text{ for all } j = 1, ..., m\}, \]

\[ V_i = \{b \mid b \in S_{2i}\}c \]

The limits of integral will be linear equations representing the space enclosed by the hyperplanes.
Bereanu’s Theory - Stochastic c

If only the c vector is random, the **CDF of max z** may be found by evaluating:

An arbitrary finite number

\[
Pr\{[z(x) \leq \phi] \cap S_{1i}\} = \int \cdots \int f'(c) \prod_{i=1}^{n} dc_i
\]

\[
U_i = \{c | c \in S_{1i}\}c
\]

\[
S_{1i} = \{c | c_B B_i^{-1} a_j - c_j \geq 0 \text{ for all } j \text{ such that } x_j^i \in W^i\}
\]

The limits of integral will be linear equations representing the space enclosed by the hyperplanes.
Ewbank, Foote and Kumin (EFK) Theory

For any basis $G$:

$$B(B^{-1}b) = b = Br$$

Feasibility condition for basis $G$:

$$P = \int \int \int_{r \geq 0} f(Br \mid J_r) \prod_{i=1}^{m} dr_i$$

$$J_r = \det(\partial b_k / \partial r_i) = \det(B_{ki}) = \det(B)$$

Adding optimality condition:

$$P_G = \int \int \int_{r \geq 0} f(Br \mid \alpha_G \mid \det(B) \mid \prod_{i=1}^{m} dr_i$$

$$\alpha_G = \begin{cases} 
1 & \text{The basic } G \text{ satisfies the optimality condition} \\
0 & \text{otherwise}
\end{cases}$$
Ewbank, Foote and Kumin (EFK) Theory

\[ P_G(\phi) = Pr\{z(b) \leq \phi \text{ and } G \text{ is an optimal basis}\} \]

\[ P_G(\phi) = \alpha_G \int \int \ldots \int f(b) \prod_{i=1}^{m} db_i \]

\( S_\emptyset = \{b|B^{-1}b = r \geq 0 \text{ and } z(b) = c_Br \leq \emptyset\} \)

And CDF of “max Z” = \( \frac{P_G(\emptyset)}{1-P} \)

\[ F(\phi) = \sum_H \frac{P_G(\phi)}{(1-P^{-H})} \]

\[ \sum_H P_G = 1 - \bar{P} \]

\( Pr\{\text{no } G \text{ is feasible}\} \)

Is the set of all possible bases
Ewbank, Foote and Kumin (EFK) Theory

The final integral becomes:

\[ P_G = \int \cdots \int f'(\hat{\bar{\mathbf{B}}}, \hat{\bar{\mathbf{A}}}, \hat{s}) \alpha_G | J_c | \prod_{i=1}^{h} dt_i \prod_{i=h+1}^{h} ds_i \]

When:

\[
\hat{\mathbf{c}}_B = (\hat{c}_1, \ldots, \hat{c}_h), \quad \hat{\mathbf{t}} = (t_1, \ldots, t_h) \\
\mathbf{s} = c_B \mathbf{B}^{-1} \mathbf{A} - \mathbf{c}^* = t \mathbf{A} - \mathbf{c}^* \\
\mathbf{c}^* = \{c_1, \ldots, c_n\}, \quad \hat{\mathbf{c}}_B = \hat{\mathbf{t}} \mathbf{B} \\
\bar{s} = (s_{h+1}, \ldots, s_n) \\
c_i = (\hat{\mathbf{t}} \mathbf{B})_i \quad i = 1, 2, \ldots, h \\
c_i = (\hat{\mathbf{t}} \mathbf{A})_i - s_i \quad i = h + 1, \ldots, n
\]

\[\bar{\mathbf{A}} = \text{matrix including the last } n - h \text{ columns and the first rows } h \text{ of } \mathbf{A} \]

\[\hat{\mathbf{B}} = \text{The first } h \text{ rows and columns of } \mathbf{B} \]

\[ J_c = (-1)^{n-h} \det(\mathbf{B}^T) = (-1)^{n-h} \det(\hat{\mathbf{B}}) \]
Adding optimality condition:

\[ J_c = (-1)^{n-h} \det(B^T) = (-1)^{n-h} \det(\hat{B}) \]

Investigating the distribution of \( z(c) \)

\[ z(c) = c_B B^{-1} b = tb = \sum_{i=1}^{m} t_i b_i = \sum_{i=1}^{h} t_i b_i = \hat{t}b \]

\[ P_G(\phi) = \Pr\{ z(c) \leq \phi \text{ and } G \text{ is feasible} \} = \]

\[ \alpha_G \int \int \ldots \int_{R_\phi} f'(\hat{t}B, \hat{t}A - \bar{s}) \alpha_G | \det(\hat{B}) | \prod_{i=1}^{h} dt_i \prod_{i=h+1}^{h} ds_i \]

\[ R_\phi = \{ t, s | t \geq 0, s \geq 0 \text{ and } z(c) = \hat{t}b \leq \phi \} \]
Ewbank, Foote and Kumin (EFK) Theory

The distribution of $z$ given that $G$ is an optimal basis is

$$F(\phi) = \sum_H \frac{P_G(\phi)}{1 - P_\infty}$$

$$\sum_H P_G = 1 - \bar{P}$$

$Pr\{\text{Unbounded solution accurate}\}$

Is the set of all possible bases
### Case II

**Bereanu Algorithm**

<table>
<thead>
<tr>
<th>Case</th>
<th>Case I. Stochastic Recourse Vector. Bereanu's model</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td><img src="image1" alt="Matrix" /></td>
</tr>
<tr>
<td>B</td>
<td>( b_i \sim \text{Gamma} (\alpha = 2, \beta = 1), i = 1,2,3,4,5,6,7,8,9 )</td>
</tr>
<tr>
<td>C</td>
<td>{3, 5, 3, 4, 5, 4, 4, 2, 4}</td>
</tr>
<tr>
<td>CDF</td>
<td>Insolvable, The integrals got killed after 48 hours</td>
</tr>
</tbody>
</table>

**Time**

- 105.144 Seconds
- 4471.77 Seconds
### TRIANGULAR:

#### Case I

<table>
<thead>
<tr>
<th>Case</th>
<th>Case I. Stochastic Recourse Vector. Ewbank's model</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>3 3 3 1 3 2 2 2 2 2 3 3 1 3 1 3 1 1 1 1</td>
</tr>
<tr>
<td>B</td>
<td>( b_1, b_2, b_3, b_4, b_5 ) ( \text{Triangular}(x</td>
</tr>
<tr>
<td>C</td>
<td>( {c_1, c_2} \sim \text{Triangular}(x</td>
</tr>
</tbody>
</table>

#### Case II

<table>
<thead>
<tr>
<th>Case</th>
<th>Case I. Stochastic Recourse Vector. Ewbank's model</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>3 3 3 1 3 2 2 2 2 2 3 3 1 3 1 3 1 1 1 1</td>
</tr>
</tbody>
</table>
| B    | {c_1, c_2} \sim \text{Triangular}(x|0,1,0.5) = \)
| C    | {3,1} \}

#### CDF

<table>
<thead>
<tr>
<th>Time</th>
<th>11036 Seconds</th>
</tr>
</thead>
</table>

#### CDF Plot

- **CDF Plot**
- **Time** 39.593 Seconds
### UNIFORM:

#### Case II

<table>
<thead>
<tr>
<th>Case</th>
<th>Case I. Stochastic Recourse Vector, Ewbank's model</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td><img src="image" alt="Matrix" /></td>
</tr>
<tr>
<td>B</td>
<td>${c_1, c_2, c_3}\sim \text{Uniform}(0, 8)$</td>
</tr>
<tr>
<td>C</td>
<td>${2, 2, 1}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CDF</th>
<th><img src="image" alt="CDF Chart" /></th>
</tr>
</thead>
<tbody>
<tr>
<td>CDF Plot</td>
<td><img src="image" alt="CDF Plot" /></td>
</tr>
<tr>
<td>Time</td>
<td><img src="image" alt="Time Chart" /></td>
</tr>
</tbody>
</table>

| Time | 105.144 Seconds |
Exponential:

**Case II**

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$[1 \quad 1]_{2}$</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>${b_1, b_2} \sim \text{Uniform } (0,8)$</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>$[2, 3]$</td>
<td></td>
</tr>
</tbody>
</table>

**CDF**

\[
\begin{align*}
\frac{1}{4608} (480 - 11.0 \phi) & \quad \phi > 16 \\
\frac{1}{312} (-256 + 96 \phi - 3 \phi^2) & \quad 12 < \phi \leq 16 \\
0 & \quad \text{True}
\end{align*}
\]

**CDF Plot**

1.0

0.8

0.6

0.4

0.2

0

Time

1.76 Seconds

1.747 SECONDS