Generalized quantiles as risk measures

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Quantiles as minimizers

It is well known that the quantiles $x^*_\alpha(X)$ of a random variable $X$ can be defined as the minimizers of an asymmetric piecewise linear loss function:

$$x^*_\alpha(X) = \arg \min_{x \in \mathbb{R}} \pi_\alpha(X, x)$$

with

$$\pi_\alpha(X, x) := E[\alpha(X - x)^+ + (1 - \alpha)(X - x)^-],$$

for $\alpha \in (0, 1)$.

This fundamental property lies at the heart of quantile regression (see for example Koenker, 2005) and has been exploited by Rockafellar and Uryasev (2002) for their approach to the computation of the Conditional Value at Risk (CVaR).
Generalized quantiles

In the statistical literature, several generalizations of the notion of quantile have been suggested, corresponding to more general loss functions:

- expectiles (Newey and Powell, 1987)
- $L^p$-quantiles (Chen, 1996)
- $M$-quantiles (Breckling and Chambers, 1988)

We will consider general asymmetric loss functions of the type

$$
\pi_\alpha (X, x) := \alpha E \left[ \Phi_1 (X - x)^+ \right] + (1 - \alpha) E \left[ \Phi_2 (X - x)^- \right],
$$

with $\Phi_1, \Phi_2 : [0, +\infty) \to [0, +\infty)$ convex and strictly increasing satisfying $\Phi_i(0) = 0, \Phi_i(1) = 1$ and $\lim_{x \to +\infty} \Phi_i(x) = +\infty.$
Functionals that are defined as the minimizers of a suitable expected loss are called *elicitable* in statistical decision theory (see Gneiting, 2011 and the references therein). It has been recently suggested (see for example Embrechts and Hofert, 2013) that elicitability is a very important property of a risk measure, since it provides a natural methodology to perform backtesting. As an example, it is very well known how to perform backtesting of VaR, while backtesting of CVaR (that is not elicitable) is not equally straightforward. The connections between elicitability and coherence are also investigated in Ziegel (2013); elicitable functionals in a dynamic setting have been considered by Bignozzi (2013).
Luxemburg norm

We recall that the Orlicz space $L^\Phi$ and the Orlicz heart $M^\Phi$ associated to a convex Young function $\Phi : [0, +\infty) \to [0, +\infty)$ satisfying $\Phi(0) = 0$, $\Phi(1) = 1$, $\lim_{x \to +\infty} \Phi(x) = +\infty$ are defined as

$L^\Phi : = \left\{ X : E \left[ \Phi \left( \frac{|X|}{a} \right) \right] < +\infty, \right.$ for some $a > 0 \right\}$,

$M^\Phi : = \left\{ X : E \left[ \Phi \left( \frac{|X|}{a} \right) \right] < +\infty, \right.$ for every $a > 0 \right\}$.

The Luxemburg norm is defined as

$\| Y \|_\Phi := \inf \left\{ a > 0 : E \left[ \Phi \left( \frac{|Y|}{a} \right) \right] \leq 1 \right\}$.

and makes both $L^\Phi$ and $M^\Phi$ Banach spaces. The case $\Phi(x) = x^p$ corresponds to the usual $L^p$ spaces.
First order condition

**Theorem**

Let \( X \in M^{\Phi_1} \cap M^{\Phi_2}, \alpha \in (0, 1) \). Then \( x^*_\alpha \in \arg \min \pi_\alpha(X,x) \) if and only it satisfies the following first order conditions:

\[
-\alpha E \left[ 1_{\{x \geq x^*\}} \Phi_1^+(X - x^*)^+ \right] + (1 - \alpha) E \left[ 1_{\{x < x^*\}} \Phi_2^-(X - x^*)^- \right] \leq 0
\]

\[
-\alpha E \left[ 1_{\{x > x^*\}} \Phi_1^-(X - x^*)^- \right] + (1 - \alpha) E \left[ 1_{\{x \leq x^*\}} \Phi_2^+(X - x^*)^+ \right] \geq 0
\]

where \( \Phi_i^- \) and \( \Phi_i^+ \) are the left and right derivatives of \( \Phi_i \).

If \( \Phi_1 \) and \( \Phi_2 \) are differentiable and \( \Phi_1^+(0) = \Phi_2^+(0) = 0 \), then the f.o.c. becomes simply

\[
\alpha E \left[ \Phi_1^+(X - x^*)^+ \right] = (1 - \alpha) E \left[ \Phi_2^-(X - x^*)^- \right].
\]
Generalized quantiles as shortfall risk measures

When the f.o.c. is given by an equation, generalized quantiles may also be defined as the unique solutions of the equation

\[ E[\psi(X - x^*_\alpha)] = 0, \]

where

\[ \psi(t) := \begin{cases} 
-(1 - \alpha)\Phi'_2(-t) & t < 0 \\
\alpha\Phi'_1(t) & t \geq 0 
\end{cases} \]

is nondecreasing with \( \psi(0) = 0 \).

This shows that generalized quantiles can be seen as special cases of zero utility premium principles, also known as shortfall risk measures or u-mean certainty equivalents (see Deprez and Gerber, 1985, Föllmer and Schied, 2002, Ben-Tal and Teboulle, 2007).
Properties of generalized quantiles

The following properties are elementary:

- \( \pi_\alpha(X, x) \) is law-invariant, non-negative, convex, and satisfies
  \[
  \lim_{x \to -\infty} \pi_\alpha(X, x) = \lim_{x \to +\infty} \pi_\alpha(X, x) = +\infty
  \]

- the infimum of \( \pi_\alpha(X, x) \) is always attained; indeed
  \[
  \arg \min \pi_\alpha(X, x) := [x_{\alpha}^-, x_{\alpha}^+]
  \]

- if \( X \) is essentially bounded, then
  \[
  [x_{\alpha}^-, x_{\alpha}^+] \subset [\text{ess inf } (X), \text{ess sup } (X)]
  \]

- if \( \alpha \leq \beta \), then \( x_{\alpha}^- \leq x_{\beta}^- \) and \( x_{\alpha}^+ \leq x_{\beta}^+ \)

- if \( \Phi_1 \) and \( \Phi_2 \) are strictly convex, then \( x_{\alpha}^- = x_{\alpha}^+ \).
Properties of generalized quantiles /2

Let $\Phi_1, \Phi_2 : [0, +\infty) \rightarrow [0, +\infty)$ be strictly convex and differentiable with $\Phi'_1(0) = \Phi'_2(0) = 0$. Then:

- $x^*_\alpha(X + h) = x^*_\alpha(X) + h$, for each $h \in \mathbb{R}$
- $x^*_\alpha(X)$ is positively homogeneous if and only if $\Phi_1(x) = \Phi_2(x) = x^\beta$, with $\beta > 1$.
- if $X \geq Y$ a.s. or if $X \geq_{st} Y$, then $x^*_\alpha(X) \geq x^*_\alpha(Y)$
- $x^*_\alpha(X)$ is convex if and only if the function $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is convex; it is concave if and only if $\psi$ is concave.
- as a consequence, $x^*_\alpha(X)$ is coherent if and only if $\Phi_1(x) = \Phi_2(x) = x^2$ and $\alpha \geq \frac{1}{2}$.

Thus expectiles with $\alpha \geq \frac{1}{2}$ are the only generalized quantiles that are a coherent risk measure.
Alternative formulations

Before focusing on expectiles, we mention that other choices of the loss function have been considered in the literature. In connection with the so called Haezendonck-Goovaerts risk measures, Bellini and Rosazza Gianin (2012) introduced the so called Orlicz quantiles, defined as the minimizers of

$$\pi_\alpha (X, x) := \alpha \| (X - x)^+ \|_{\Phi_1} + (1 - \alpha) \| (X - x)^- \|_{\Phi_2},$$

while Jaworski (2006) considered instead loss functions of the type

$$\pi_\alpha (X, x) := E \left[ \Phi_1 (X - x)^+ \right] + \Phi_2(x),$$

that depend only on the right tail of the distribution.
Some comparisons between the different formulations:

- Orlicz quantiles are always positively homogeneous, while generalized quantiles have this property only in the case $\Phi_1(x) = \Phi_2(x) = X^p$.

- Orlicz quantiles lack the property of monotonicity with respect to the $\leq_{st}$ order, as it was shown by means of a counterexample in Bellini and Rosazza Gianin (2012).

- Jaworski quantiles are monotonic with respect to the $\leq_{st}$ order and (under additional assumptions) convex, but they do not satisfy translation invariance or positive homogeneity.
Expectiles

Thus the expectiles, introduced by Newey and Powell (1987) as

\[ e_\alpha(X) = \arg \min_{x \in \mathbb{R}} E[\alpha(X - x)^2 + (1 - \alpha)(X - x)^{-2}] \]

are the only example of a generalized quantile that is also a coherent risk measure (for \( \alpha \geq \frac{1}{2} \)). When \( \alpha = \frac{1}{2} \) the expectile coincides with the mean, while when \( \alpha \leq \frac{1}{2} \) expectiles are superadditive, in the sense that

\[ e_\alpha(X + Y) \geq e_\alpha(X) + e_\alpha(Y). \]

Moreover, we have the following elementary properties:

\[ \begin{align*}
& \mathbb{P}(X \leq Y) > 0 \Rightarrow e_\alpha(X) < e_\alpha(Y) \\
& e_\alpha(X) = -e_{1-\alpha}(-X).
\end{align*} \]
First order condition

The first order condition can be written as

\[(1 - \alpha) \int_{-\infty}^{e_\alpha} F(t) dt = \alpha \int_{e_\alpha}^{+\infty} \bar{F}(t) dt\]

or equivalently

\[e_\alpha(X) - E[X] = \frac{2\alpha - 1}{1 - \alpha} E[(X - e_\alpha(X))^+]\]

or also

\[\alpha = \frac{E[(X - e_\alpha)^-]}{E[|X - e_\alpha|]},\]

that shows that expectiles can be seen as the usual quantiles of a transformed distribution (Jones, 1994).
Dual representation

Theorem

Let $X \in L^1$, and $\alpha \in (0, 1)$. Then

$$e_\alpha(X) = \begin{cases} 
\max_{\varphi \in M_\alpha} E[\varphi X] & \text{if } \alpha \geq \frac{1}{2} \\
\min_{\varphi \in M_\alpha} E[\varphi X] & \text{if } \alpha \leq \frac{1}{2}
\end{cases},$$

where

$$M_\alpha = \left\{ \varphi \in L^\infty, \varphi \geq 0 \text{ a.s., } E_P[\varphi] = 1, \frac{\text{ess sup } \varphi}{\text{ess inf } \varphi} \leq \beta \right\},$$

with $\beta = \max \left\{ \frac{\alpha}{1-\alpha}, \frac{1-\alpha}{\alpha} \right\}$. When $\alpha \geq \frac{1}{2}$, the optimal scenario is

$$\overline{\varphi} := \frac{\alpha 1\{X>e_\alpha\} + (1-\alpha) 1\{X\leq e_\alpha\}}{E[\alpha 1\{X>e_\alpha\} + (1-\alpha) 1\{X\leq e_\alpha\}]}.$$
Kusuoka representation

The Kusuoka representation of a coherent risk measure is its representation as a supremum of additively comonotone risk measures, that are convex combinations or integrals of CVaR with different $\alpha$ (see Kusuoka, 2001, Pflug, 2007, Shapiro, 2013).

**Theorem**

Let $X \in L^1$, $\frac{1}{2} \leq \alpha < 1$ and $\beta = \frac{\alpha}{1-\alpha}$. Then

$$e_\alpha(X) = \max_{\gamma \in \left[\frac{1}{\beta} , 1\right]} \{(1-\gamma)CVaR_{\beta-\frac{1}{\gamma}} + \gamma E[X]\}.$$ 

In particular,

$$e_\alpha(X) \geq \frac{E[X]}{2\alpha} + \left(1 - \frac{1}{2\alpha}\right) CVaR_\alpha(X).$$

Expectiles are an example of coherent risk measures that are not additively comonotone.
Comparison between quantiles and expectiles

In the statistical literature it is usually argued that typically expectiles are closer to the centre of the distribution than the corresponding quantiles. This is indeed the case for common distributions as the uniform or the normal, as the following figure shows:
We are not aware of general comparisons results. Koenker (1993) provided an example of a distribution with infinite variance for which expectiles $e_{\alpha}$ and quantiles $q_{\alpha}$ coincide for every $\alpha$. We can prove the following:

**Theorem**

Let $X$ be a Pareto-like distribution with tail index $\beta > 1$. Then

$$\frac{1}{\beta - 1} F(e_{\alpha}(X)) \sim 1 - \alpha \sim F(q_{\alpha}(X))$$

as $\alpha \to 1$.

If $\beta < 2$, then there exists some $\alpha_0 < 1$ such that for all $\alpha > \alpha_0$ it holds $e_{\alpha}(X) > q_{\alpha}(X)$; if $\beta > 2$, the reverse ordering applies.

So for high $\alpha$ expectiles are more conservative than the quantiles for distributions with very heavy tails (infinite variance).
Robustness

In robust statistics, the notion of qualitative robustness of a statistical functional corresponds essentially to the continuity with respect to weak convergence. It is generally argued that this notion is not appropriate for financial risk measures; the mean itself it’s not robust. Stahl et al. (2012) suggest that a better notion of robustness might be continuity with respect to the Wasserstein distance, defined as

\[ d_W(F, G) := \inf\{ E[|X - Y|] : X \sim F, \ Y \sim G \}. \]

or equivalently

\[ d_W(F, G) = \int_{-\infty}^{\infty} |F(t) - G(t)|dt = \int_{0}^{1} |F^{-1}(\alpha) - G^{-1}(\alpha)|d\alpha. \]
As it is well known, convergence in the Wasserstein distance is stronger than weak convergence, indeed

\[ d_W(X_n, X) \to 0 \iff X_n \to X \text{ in distribution and } E[X_n] \to E[X]. \]

Thus convergence in the Wasserstein metric is a particular type of the so called \( \psi \)-weak convergence with \( \psi(x) = |x| \). We can prove that expectiles are Lipschitz with respect to the Wasserstein metric:

**Theorem**

For all \( X, Y \in L^1 \) and all \( \alpha \in (0, 1) \) it holds that

\[ |e_\alpha(X) - e_\alpha(Y)| \leq \beta d_W(X, Y), \text{ where } \beta = \max\{\frac{\alpha}{1-\alpha}, \frac{1-\alpha}{\alpha}\}. \]
Isotonicity results for generalized quantiles

A natural question is that there exist other generalized quantiles that are subadditive, or at least convex. Such a statistical functional would be an example of a convex law invariant risk measure, so from the results of Bauerle and Muller (2006), under weak technical additional conditions, it should be isotonic with respect to the convex order. In Bellini (2012) the general problem of isotonicity of generalized quantiles with respect to several stochastic orderings have been studied, by using a purely order-theoretic comparative static approach, in the spirit of Topkis (1978) and Milgrom and Shannon (1994).
The basic idea

In the defining minimization problem

$$x^*_\alpha(X) = \arg\min_{x \in \mathbb{R}} \pi_\alpha(X, x)$$

we consider $X$ as a parameter, belonging to the partially ordered set $(L, \preceq)$, where the role of the partial order $\preceq$ will be played in turn by $\leq_{st}$, $\leq_{cx}$ and $\leq_{icx}$.

The theory of monotone comparative statics provides necessary and sufficient conditions on the function $\pi_\alpha(X, x)$ for increasing optimal solutions, that is in order to have that $X \preceq Y \Rightarrow x^*_\alpha(X) \leq x^*_\alpha(Y)$. 
Submodularity and single crossing condition

Topkis (1978) showed that a sufficient condition for increasing optimal solutions in our situation is the submodularity of the function $\pi_\alpha(X, x)$, that is

$$\pi_\alpha(X, y) - \pi_\alpha(X, x) \geq \pi_\alpha(Y, y) - \pi_\alpha(Y, x),$$

for $y \geq x$ and $Y \succeq X$. Later, Milgrom and Shannon (1984) proved that a necessary condition for increasing optimal solutions is the so-called single crossing condition, defined as follows:

i) $\pi_\alpha(Y, y) - \pi_\alpha(Y, x) > 0 \Rightarrow \pi_\alpha(X, y) - \pi_\alpha(X, x) > 0$

ii) $\pi_\alpha(Y, y) - \pi_\alpha(Y, x) \geq 0 \Rightarrow \pi_\alpha(X, y) - \pi_\alpha(X, x) \geq 0$,

for $y \geq x$ and $Y \succeq X$. 
Isotonicity results

By applying these techniques, we can prove the following (Bellini, 2012):

Theorem

Let $\pi_\alpha(X, x) := \alpha E[\phi(X - x)^+] + (1 - \alpha) E[\phi(X - x)^-]$ with $X \in M^\Phi$ and $\alpha \in (0, 1)$.

a) If $\Phi(x) = x^2$ and $\alpha \geq \frac{1}{2}$, then $X \leq_{cx} Y \Rightarrow x^*_\alpha(X) \leq x^*_\alpha(Y)$; if instead $\alpha \leq \frac{1}{2}$ then $X \leq_{cx} Y \Rightarrow x^*_\alpha(X) \geq x^*_\alpha(Y)$

b) If $\Phi(x) = x^s$ with $s$ odd, then $X \leq_{s-cx} Y \Rightarrow x^*_\alpha(X)^* \leq x^*_\alpha(Y)^*$

b') If $\Phi(x) = x^s$ with $s$ even and $\alpha \geq \frac{1}{2}$, then $X \leq_{s-cx} Y \Rightarrow x^*_\alpha(X) \leq x^*_\alpha(Y)$; if instead $\alpha \leq \frac{1}{2}$ then $X \leq_{s-cx} Y \Rightarrow x^*_\alpha(X) \geq x^*_\alpha(Y)$

c) Let $\Phi \in C^2[0, +\infty)$. If $x^*_\alpha(X)$ is isotonic with respect to the $\leq_{cx}$ order, then $\Phi''(x) = k$. 
Some references

- Bellini, F., Klar, B., Müller, A., Rosazza Gianin, E. Generalized quantiles as risk measures, downloadable from SSRN


