Asset-Liability Management via Risk-Sensitive Control: Jump-Diffusion

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Asset and Liability Management

Asset and liability management (ALM) is critical for funded investors such as endowment funds and pension funds, but also for investors who have the ability to grow their asset base by borrowing: banks and hedge funds.

In this talk we solve an ALM problem using Risk-Sensitive Control methods. Our work is related to the articles on surplus management by Rudolf and Ziemba [13] and Benk [2].

In our approach, the investor's objective is to jointly select an optimal amount of leverage and an optimal asset allocation to maximise the expected utility of the equity or surplus of his/her portfolio. Also, we allow securities prices and the value of the liability to be influenced by a number of underlying factors.
Balance Sheet of a Funded Investor

Assets at time $t$:
- Valued at $V(t)$
- Constituted of a portfolio of securities.

Liabilities at time $t$:
- Valued at $L(t)$

Equity at time $t$:
- Values at $E(t)$

$L(t) = V(t) - E(t)$

Leverage ratio: $\rho(t) = \frac{V(t)}{E(t)}$
Risk Sensitive Control: A Definition

Risk-sensitive control is a generalization of classical stochastic control in which the degree of risk aversion or risk tolerance of the optimizing agent is explicitly parameterized in the objective criterion and influences directly the outcome of the optimization.

In risk-sensitive control, the decision maker’s objective is to select a control policy $h(t)$ to maximize the criterion

$$J(x, t, h; \theta) := -\frac{1}{\theta} \ln \mathbb{E} \left[ e^{-\theta F(t, x, h)} \right]$$

where $t$ is the time, $x$ is the state variable, $F$ is a given reward function, and the risk sensitivity $\theta \in (-1, 0) \cup (0, \infty)$ is an exogenous parameter representing the decision maker’s degree of risk aversion.
An Intuitive View of the Criterion

A Taylor expansion of the previous expression around $\theta = 0$ evidences the vital role played by the risk sensitivity parameter:

$$J(x, t, h; \theta) = \mathbb{E}[F(t, x, h)] - \frac{\theta}{2} \text{Var}[F(t, x, h)] + O(\theta^2)$$ (2)

- $\theta \to 0$, “risk-null”: corresponds to classical stochastic control;
- $\theta < 0$: “risk-seeking” case corresponding to a maximization of the expectation of a convex decreasing function of $F(t, x, h)$;
- $\theta > 0$: “risk-averse” case corresponding to a minimization of the expectation of a convex increasing function of $F(t, x, h)$. 
From Risk-Sensitive Control to Risk-Sensitive Asset Management

If we chose $F(t, x) = \ln \text{Wealth}_T$, the Taylor expression is tantamount to a dynamic version of Markowitz’ mean-variance analysis, but with a built-in correction for higher moments.

This leads us to the risk-sensitive asset management model:

$$J(x, t, h; \theta) := -\frac{1}{\theta} \ln \mathbb{E} \left[ e^{-\theta \ln \text{Wealth}_T} \right]$$

$$= -\frac{1}{\theta} \ln \mathbb{E} \left[ \text{Wealth}^{-\theta}_T \right]$$

$$= \mathbb{E} \left[ \ln \text{Wealth}_T \right] - \frac{\theta}{2} \text{Var} \left[ \ln \text{Wealth}_T \right] + O(\theta^2)$$
Emergence of a Risk-Sensitive Asset Management (RSAM) Theory

- Jacobson [9], Whittle [14], Bensoussan [3] led the theoretical development of risk sensitive control.
- Bielecki and Pliska [4]: first to apply continuous time risk-sensitive control as a practical tool to solve ‘real world’ portfolio selection problems.
- Major contribution by Kuroda and Nagai [10]: elegant solution method based on a change of measure argument.
The Risk-Sensitive Asset Management (RSAM) theory was developed based on diffusion models. In a jump-diffusion setting,

- Davis and Lleo [5] consider a finite time horizon problem with random jumps in the asset prices. They prove the existence of an optimal control and showed that the value function is a smooth (strong) solution of the Hamilton Jacobi Bellman Partial Differential Equation (HJB PDE).

- Davis and Lleo [6] consider a finite time horizon problem with random jumps in both asset prices and factors. Under standard control assumptions they prove the existence of an optimal control and showed that the value function is a smooth (strong) solution of the Hamilton Jacobi Bellman Partial Differential Equation (HJB PDE).

- Davis and Lleo [7] extend these results to a benchmarked asset management problem.
The Risk-Sensitive ALM Problem - General Model

Let \((\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})\) be the underlying probability space. Take a market with

1. A money market asset \(S_0\) with dynamics

\[
\frac{dS_0(t)}{S_0(t)} = a_0(t, X(t)) \, dt, \quad S_0(0) = s_0 \tag{3}
\]

2. \(m\) risky assets following jump-diffusion SDEs

\[
\frac{dS_i(t)}{S_i(t^-)} = \left[ a_i(t, X(t^-)) \right] \, dt + \sum_{k=1}^{N} \Sigma_{ik}(t, X(t)) \, dW_k(t) + \int_{Z} \gamma_i(t, z) \, \tilde{N}(dt, dz), \\
S_i(0) = s_i, \quad i = 1, \ldots, m \tag{4}
\]

3. A liability given by the same type of jump-diffusion process:

\[
\frac{dL(t)}{L(t)} = c(t, X(t^-)) \, dt + \varsigma'(t, X(t)) \, dW(t) + \int_{Z} \eta(t, z) \, \tilde{N}(dt, dz), \quad L(0) = l \tag{5}
\]

4. A \(n\)-dimensional vector of factors \(X(t)\) following

\[
dX(t) = b(t, X(t^-)) \, dt + \Lambda(t, X(t)) \, dW(t) + \int_{Z} \xi(t, X(t^-), z) \, \tilde{N}(dt, dz), \\
X(0) = x_0 \in \mathbb{R}^n. \tag{6}
\]
Note:

- $W(t)$ is a $\mathbb{R}^{m+n+1}$-valued $(\mathcal{F}_t)$-Brownian motion with components $W_k(t)$, $k = 1, \ldots, (m + n + 1)$.

- $\tilde{N}_p(dt, dz)$ is a Poisson random measure (see e.g. Ikeda and Watanabe [8]) defined as

$$
\tilde{N}_p(dt, dz) = \begin{cases} 
N_p(dt, dz) - \nu(dz)dt =: \tilde{N}_p(dt, dz) & \text{if } z \in \mathbb{Z}_0 \\
N_p(dt, dz) & \text{if } z \in \mathbb{Z} \setminus \mathbb{Z}_0
\end{cases}
$$
Two Implementations

We consider two different implementations of this general model:

1. **Affine dynamics**, with jumps in assets and liabilities only;

2. **Standard control** assumptions with jumps in assets, liabilities and factors.
Assumption 1: Affine dynamics with jumps in assets and liabilities

1. The money market asset $S_0$ has a dynamics

\[
\frac{dS_0(t)}{S_0(t)} = (a_0 + A'_0 X(t)) \, dt, \quad S_0(0) = s_0
\]  

(7)

2. $m$ risky assets following jump-diffusion SDEs

\[
\frac{dS_i(t)}{S_i(t^-)} = (a + AX(t))_i \, dt + \sum_{k=1}^{N} \Sigma_{ik} dW_k(t) + \int_{Z} \gamma_i(z) \bar{N}(dt, dz), \\
S_i(0) = s_i, \quad i = 1, \ldots, m
\]  

(8)

3. A liability given by the same type of jump-diffusion process:

\[
\frac{dL(t)}{L(t)} = (c + C'X(t)) \, dt + \varsigma' dW(t) + \int_{Z} \eta(z) \bar{N}(dt, dz), \quad L(0) = l
\]  

(9)

4. A $n$-dimensional vector of factors $X(t)$ following

\[
dX(t) = (b + BX(t)) \, dt + \Lambda dW(t), \\
X(0) = x_0 \in \mathbb{R}^n.
\]  

(10)
Assumption 2: Standard control assumptions

Under standard control assumptions, our model has the same form as the general model:

1. A money market asset $S_0$ with dynamics

   \[
   \frac{dS_0(t)}{S_0(t)} = a_0(t, X(t)) \, dt, \quad S_0(0) = s_0
   \]  

2. $m$ risky assets following jump-diffusion SDEs

   \[
   \frac{dS_i(t)}{S_i(t)} = \left[a_i(t, X(t^-))\right] \, dt + \sum_{k=1}^{N} \Sigma_{ik}(t, X(t))dW_k(t) + \int_{Z} \gamma_i(t, z) \tilde{N}(dt, dz),
   \]
   \[S_i(0) = s_i, \quad i = 1, \ldots, m\]  

3. A liability given by the same type of jump-diffusion process:

   \[
   \frac{dL(t)}{L(t)} = c(t, X(t^-))dt + \varsigma'(t, X(t))dW(t) + \int_{Z} \eta(t, z) \tilde{N}(dt, dz), \quad L(0) = l
   \]  

4. A $n$-dimensional vector of factors $X(t)$ following

   \[
   dX(t) = b(t, X(t^-)) \, dt + \Lambda(t, X(t))dW(t) + \int_{Z} \xi(t, X(t^-), z) \tilde{N}(dt, dz),
   \]
   \[X(0) = x_0 \in \mathbb{R}^n.\]
The functions $a_0, a, b, c, \Sigma = [\sigma_{ij}], \varsigma, \Lambda$ are Lipschitz continuous, bounded with bounded derivatives in terms of the variables $t$ and $x$.

- Ellipticity condition:

$$\Sigma \Sigma' > 0 \quad (15)$$

- The jump intensities $\xi(z)$ and $\gamma(z)$ satisfies appropriate well-posedness conditions.

- Independence of systematic (factor-driven) and idiosyncratic (asset-driven and liability-driven) jump: $\forall (t, x, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{Z}$,

$$\gamma(t, z)\xi'(t, x, z) = \eta(t, z)\xi'(t, x, z) = 0.$$
... plus an extra condition:

**Assumption**

The vector valued function $\gamma(t, z)$ satisfy:

$$\int_{\mathbb{Z}} |\xi(t, x, z)| \nu(dz) < \infty, \quad \forall (t, x) \in [0, T] \times \mathbb{R}^n$$

(16)

Note that the minimal condition on $\xi$ under which the factor equation (14) is well posed is

$$\int_{\mathbb{Z}_0} |\xi(t, x, z)|^2 \nu(dz) < \infty,$$

However, for this paper it is essential to impose the stronger condition (16) in order to connect the viscosity solution of HJB partial integro-differential equation (PIDE) with the viscosity solution of a related parabolic PDE.
Next Steps: Find the Dynamics of The Assets and Equity

Assets at time $t$:
- Valued at $V(t)$
- Constituted of a portfolio of $m$ securities.

Liabilities at time $t$:
- Valued at $L(t)$

Equity at time $t$:
- Values at $E(t)$

E(t) = V(t) - L(t)

Leverage ratio: $\rho(t) = V(t)/E(t)$
Wealth Dynamics

The wealth, $V(t)$ of the investor in response to an investment strategy $h(t) \in \mathcal{H}$, follows the dynamics

$$
\frac{dV(t)}{V(t^-)} = (a_0(t, X(t))) \, dt + h'(t) \hat{a}(t, X(t)) \, dt + h'(t) \Sigma(t, X(t))dW_t
$$

$$
+ \int_Z h'(t) \gamma(t, z) \tilde{N}_{p}(dt, dz)
$$

(17)

with initial endowment $V(0) = v$, where $\hat{a} := a - a_0 \mathbf{1}$ and $\mathbf{1} \in \mathbb{R}^m$ denotes the $m$-element unit column vector.
Equity and Leverage

The time $t$ equity or surplus, $E(t)$, is the wealth belonging directly to the investor. It is defined as the difference between the value of the assets and of the liabilities, i.e.

$$E(t) = V(t) - L(t), \quad E(0) := e_0 = v - l > 0$$

WLOG, we assume that $e_0 = 1$.

The dynamics of the equity is given in differential form by

$$dE(t) = dV(t) - dL(t)$$
The time $t$ leverage ratio, $\rho(t)$, is defined as the ratio of asset value to equity value:

$$\rho(t) = \frac{V(t)}{E(t)}$$

Thus,

$$V(t) = \rho(t)E(t) \quad L(t) = (\rho(t) - 1)E(t)$$

In our case, leverage is a control variable: the investor’s objective is to choose both an optimal level of leverage and an optimal investment strategy.
The dynamics of the equity in response to an ALM policy \((h(t), \rho(t))\) can be expressed as

\[
dE(t) = V(t^-) \left[ (a_0(t, X(t))) \, dt + h'(t) \hat{a}(t, X(t)) \, dt + h'(t) \Sigma(t, X(t)) \, dW_t \\
+ \int_Z h'(t) \gamma(t, z) \bar{N}(dt, dz) \right] \\
- L(t^-) \left[ c(t, X(t^-))dt + \varsigma'(t, X(t))dW(t) + \int_Z \eta(t, z) \bar{N}(dt, dz) \right]
\]
Rewriting in terms of equity and leverage only, we get

\[
\frac{dE(t)}{E(t^-)} = c(t, X(t))dt \\
+ \rho(t) \left[ h'(t)\hat{a}(t, X(t)) - \hat{c}(t, X(t)) \right] dt \\
+ \left( \varsigma'(t, X(t)) + \rho(t)[h'(t)\Sigma(t, X(t)) - \varsigma'(t, X(t)) \right] dW(t), \\
+ \int_{Z} \left\{ \eta(t, z) + \rho(t) \left[ h'(t)\gamma(t, z) - \eta(t, z) \right] \right\} \tilde{N}(dt, dz) \\
= \alpha(t, X(t), h(t), \rho(t))dt + \beta(t, X(t), h(t), \rho(t))dW(t), \\
+ \int_{Z} \zeta((t, z, h(t), \rho(t)))\tilde{N}(dt, dz)
\] (18)

where

\[
\alpha(t, x, h, \rho) := c(t, x) + \rho \left[ h'\hat{a}(t, x) - \hat{c}(t, x) \right] \\
\beta(t, x, h, \rho) := \varsigma'(t, x) + \rho(h'\Sigma(t, x) - \varsigma'(t, x)) \\
\zeta(t, z, h, \rho) := \eta(t, z) + \rho(t) \left[ h'(t)\gamma(t, z) - \eta(t, z) \right]
\]

and \( \hat{c} := c - a_0 \).
Investment and Leverage Constraints

We also consider \( r \in \mathbb{N} \) fixed investment constraints expressed in the form

\[
\Upsilon' h(t) \leq \upsilon
\]  

(19)

where \( \Upsilon \in \mathbb{R}^m \times \mathbb{R}^r \) is a matrix and \( \upsilon \in \mathbb{R}^r \) is a column vector.

Assumption

The system

\[
\Upsilon' y \leq \upsilon
\]

for the variable \( y \in \mathbb{R}^m \) admits at least two solutions.

We also introduce the following constraints on leverage:

\[
\mathcal{K} := \{ \rho \in \mathbb{R} : -\infty < \rho_- \leq \rho(t) \leq \rho^+ < \infty \}
\]

(20)

where \( \rho_- , \rho^+ \) are two real constants.

These assumptions guarantee that there will be at least one ALM policy satisfying the constraints.
Problem Formulation

The investor’s objective is to maximise the risk-sensitive criterion $J(h, \rho)$

$$J(h, \rho; \theta) := -\frac{1}{\theta} \ln \mathbb{E} \left[ e^{-\theta \ln E_T} \right] = -\frac{1}{\theta} \ln \mathbb{E} \left[ \ln E_T^{-\theta} \right]$$

(21)

where $\ln E_T(h)$ can be interpreted as the log return on equity.
From (17) and the general Itô formula we find that the term $e^{-\theta \ln E(T)}$ can be expressed as

$$e^{-\theta \ln E(T)} = \exp \left\{ \theta \int_0^T g(t, X_t, h(t)) dt \right\} \chi^h(T) \quad (22)$$

where

$$g(t, x, h) = \frac{1}{2} (\theta + 1) \beta \beta'(t, x, h, \rho) - \alpha(t, x, h, \rho)$$

$$+ \int_Z \left\{ \frac{1}{\theta} \left[ (\zeta(t, z, h, \rho))^{-\theta} - 1 \right] + \zeta(t, z, h, \rho) 1_{Z_0}(z) \right\} \nu(dz)$$

$$= \frac{1}{2} (\theta + 1) \left[ \zeta'(t, x) + \rho(h'\Sigma(t, x) - \zeta'(t, x)) \right] \left[ \zeta'(t, x) + \rho(h'\Sigma(t, x) - \zeta'(t, x)) \right]'$$

$$- c(t, x) - \rho \left[ h'\hat{a}(t, x) - \hat{c}(t, x) \right]$$

$$+ \int_Z \left\{ \frac{1}{\theta} \left[ (\eta(t, z) + \rho(t) [h'(t)\gamma(t, z) - \eta(t, z))]^{-\theta} - 1 \right]$$

$$+ (\eta(t, z) + \rho(t) [h'(t)\gamma(t, z) - \eta(t, z)]) 1_{Z_0}(z) \right\} \nu(dz) \quad (23)$$
and the Doléans exponential $\chi^h_t$ is given by

\begin{align*}
\chi^h(t) &:= \exp \left\{ -\theta \int_0^t \beta(s, X(s), h(s), \rho(s)) dW_s \\
&\quad - \frac{1}{2} \theta^2 \int_0^t \beta(s, X(s), h(s), \rho(s)) \beta(s, X(s), h(s), \rho(s))' ds \\
&\quad + \int_0^t \int_{\mathbb{R}^+} \ln (1 - G(s, z, h(s), \rho(s))) \tilde{N}(ds, dz) \\
&\quad + \int_0^t \int_{\mathbb{R}^+} \{ \ln (1 - G(s, z, h(s), \rho(s))) + G(s, z, h(s), \rho(s)) \} \nu(dz) ds \right\},
\end{align*}

(24)

with

\begin{align*}
G(t, z, h, \rho) &= 1 - (\zeta(t, z, h, \rho))^{-\theta} \\
&= 1 - (\eta(t, z) + \rho(t) [h'(t)\gamma(t, z) - \eta(t, z)])^{-\theta}
\end{align*}

(25)
Change of Measure

For $h \in A$, $\rho \in R$ and $\theta > 0$ let $P^h$ be the measure on $(\Omega, \mathcal{F}_T)$ defined via the Radon-Nikodým derivative

$$\frac{dP^h}{dP} = \chi^h(T),$$

(26)

and let $E^h$ denote the corresponding expectation. Then

$$J(h, \rho) = -\frac{1}{\theta} \ln E^h \left[ \exp \left( \theta \int_0^T g(t, X_t, h(t)) \, dt \right) \right].$$

(27)

Moreover, under $P^h$,

$$W_t^h = W_t + \theta \int_0^t \beta(s, X(s), h(s), \rho(s)) \, ds$$

$$= W_t + \theta \int_0^t \varsigma'(s, X(s)) + \rho(s)(h'(s)\Sigma(s, X(s)) - \varsigma'(s, X(s))) \, ds$$

is a standard Brownian motion and...
... the $\mathbb{P}^h$-compensated Poisson random measure is given by

$$
\int_0^t \int_{\mathbb{Z}_0} \tilde{N}^h(ds, dz)
= \int_0^t \int_{\mathbb{Z}_0} N(ds, dz) - \int_0^t \int_{\mathbb{Z}_0} \{1 - G(s, z, h(s))\} \nu(dz)ds
= \int_0^t \int_{\mathbb{Z}_0} N(ds, dz) - \int_0^t \int_{\mathbb{Z}_0} \left\{ (\zeta(s, z, h(s), \rho(s)))^{-\theta} \right\} \nu(dz)ds
= \int_0^t \int_{\mathbb{Z}_0} N(ds, dz) - \int_0^t \int_{\mathbb{Z}_0} \left\{ (\eta(s, z) + \rho(s) [h'(s)\gamma(s, z) - \eta(s, z)])^{-\theta} \right\} \nu(dz)ds
$$
As a result, under \( \mathbb{P}^h \) the factor process \( X(s), 0 \leq s \leq t \) satisfies the SDE:

\[
dX(s) = f(s, X(s), h(s))ds + \Lambda(s, X(s))dW^\theta_s + \int_{\mathbb{R}} \xi(s, X(s^), z) \tilde{N}^h(ds, dz), \quad X(0) = x_0
\]

where

\[
f(t, x, h) \\
\quad := b(t, x) - \theta \Lambda \beta(t, X(t), h, \rho) + \int_{\mathbb{R}} \xi(t, x, z) \left[ (\zeta(t, z, h, \rho))^{-\theta} \right] \nu(dz)
\]

\[
\quad = b(t, x) - \theta \Lambda \left[ \zeta'(t, x) + \rho(h' \Sigma(t, x) - \zeta'(t, x)) \right] \\
\quad \quad + \int_{\mathbb{R}} \xi(t, x, z) \left[ (\eta(t, z) + \rho(t) \left[ h'(t) \gamma(t, z) - \eta(t, z) \right])^{-\theta} - 1_{z_0}(z) \right] \nu(dz)
\]

and \( b \) is the \( \mathbb{P} \)-measure drift of the factor process.
Following the change of measure we introduce two auxiliary criterion functions under $\mathbb{P}_h^\theta$:

- the risk-sensitive control problem:

$$I(v, x; h; t, T; \theta) = -\frac{1}{\theta} \ln \mathbb{E}_{t,x}^{h,\theta} \left[ \exp \left\{ \theta \int_t^T g(X_s, h(s); \theta) ds \right\} \right]$$  (30)

where $\mathbb{E}_{t,x} [\cdot]$ denotes the expectation taken with respect to the measure $\mathbb{P}_h^\theta$ and with initial conditions $(t, x)$.

- the exponentially transformed criterion

$$\tilde{I}(v, x, h; t, T; \theta) := \mathbb{E}_{t,x}^{h,\theta} \left[ \exp \left\{ \theta \int_t^T g(s, X_s, h(s); \theta) ds \right\} \right]$$  (31)
How to Solve a Stochastic Control Problem

Our objective is to solve the control problem in a classical sense.

The process involves

1. deriving the HJB P(I)DE;
2. identifying a (unique) candidate optimal control;
3. proving existence of a $C^{1,2}$ (classical) solution to the HJB P(I)DE.
4. proving a verification theorem;
The HJB P(I)DEs

The HJB PIDE associated with the risk-sensitive control criterion (30) is

$$\frac{\partial \Phi}{\partial t} + \sup_{h \in \mathcal{J}, \rho \in \mathcal{K}} L^h \left( t, x, \Phi, D\Phi, D^2\Phi \right) = 0 \quad (32)$$

where

$$L^h(t, x, u, p, M) = f(t, x, h)'p + \frac{1}{2} \text{tr} \left( \Lambda \Lambda'(t, x)M \right) - \frac{\theta}{2} p'\Lambda\Lambda'(t, x)p$$

$$- g(t, x, h) + \mathcal{I}_{NL} [t, x, u, p] \quad (33)$$

with

$$\mathcal{I}_{NL} [t, x, u, p] = \int_Z \left\{ -\frac{1}{\theta} \left( e^{-\theta[u(t,x+\xi(t,x,z))-u(t,x)]]} - 1 \right) - \xi(t, x, z)'p \right\} \nu(d\xi)$$

and subject to the terminal condition (recall our normalization $e_0 = 1$)

$$\Phi(T, x) = 0, \quad x \in \mathbb{R}^n. \quad (35)$$

Condition (16) ensures that $\mathcal{I}_{NL}$ is well defined, at least for bounded $u$. 
To remove the quadratic growth term, we consider the PIDE associated with the exponentially-transformed problem (31):

\[
\frac{\partial \tilde{\Phi}}{\partial t}(t, x) + \frac{1}{2} \text{tr} \left( \Lambda \Lambda'(t, x) D^2 \tilde{\Phi}(t, x) \right) + H(t, x, \tilde{\Phi}, D\tilde{\Phi}) \\
+ \int_Z \left\{ \tilde{\Phi}(t, x + \xi(t, x, z)) - \tilde{\Phi}(t, x) - \xi(t, x, z)' D\tilde{\Phi}(t, x) \right\} \nu(dz) = 0 \tag{36}
\]

subject to terminal condition

\[
\tilde{\Phi}(T, x) = 1 \tag{37}
\]

where for \( r \in \mathbb{R}, \ p \in \mathbb{R}^n \)

\[
H(s, x, r, p) = \inf_{h \in \mathcal{H}} \left\{ f(s, x, h)' p + \theta g(s, x, h)r \right\} \tag{38}
\]

In particular \( \tilde{\Phi}(t, x) = \exp \{-\theta \Phi(t, x)\} \).
Some Insights...

The presence (or absence) of jumps plays a crucial role in our problem. If we think about our general model as a meta model for diffusion and jump-diffusion problems we observe that:

1. **Pure diffusion**: we have a pure diffusion problem and the hope of finding an analytical solution for the optimal policy pair \((h^*, \rho^*)\). The HJB equation is a parabolic PDE;

2. **Jumps in asset and liabilities only**: because of the jumps we will not generally be able to analytical solution for the optimal policy pair \((h^*, \rho^*)\). However, the HJB equation remains a parabolic PDE;

3. **Jumps in factors only**: we can potentially find an analytical solution for the optimal policy pair \((h^*, \rho^*)\), but now the jumps in the factor level have transformed the HJB equation is a parabolic PIDE;

4. **Full jump diffusion**: because of the jumps in assets and liabilities we will not generally be able to analytical solution for the optimal policy pair \((h^*, \rho^*)\). Moreover, the jumps in the factor level have transformed the HJB equation is a parabolic PIDE;
Identifying a (Unique) Candidate Optimal Control

The supremum in (32) can be expressed as

$$\sup_{h \in \mathcal{H}, \rho \in \mathcal{K}} L^h (t, x, u, p, M)$$

$$= b'(t, x)p + \frac{1}{2} \text{tr} (\Lambda\Lambda'(t, x)M) - \frac{\theta}{2} p'\Lambda\Lambda'(t, x)'p + c(t, x) + I_{NL} [t, x, u, p]$$

$$\sup_{h \in \mathcal{H}, \rho \in \mathcal{K}} \left\{ -\frac{1}{2} (\theta + 1) [\varsigma'(t, x) + \rho(h'\Sigma(t, x) - \varsigma'(t, x))] [\varsigma'(t, x) + \rho(h'\Sigma(t, x) - \varsigma'(t, x))] - \theta \Lambda [\varsigma'(t, x) + \rho(h'\Sigma(t, x) - \varsigma'(t, x))] p + \rho [h' \hat{a}(t, x) - \hat{c}(t, x)]$$

$$- \frac{1}{\theta} \int_{\mathcal{Z}} \left\{ (1 - \theta \xi(t, x, z)'p) \left[ (\eta(t, z) + \rho(t) [h'(t)\gamma(t, z) - \eta(t, z)])^{-\theta} - 1 \right]$$

$$+ (\rho(t) [h'(t)\gamma(t, z) - \eta(t, z)]) 1_{\mathcal{Z}_0}(z) \right\} \nu(dz) \right\}$$
This equation looks messy, but it is actually well structured:

- Because $\Sigma \Sigma' > 0$ and because systematic jumps are independent from idiosyncratic jumps, this problem corresponds to the maximization of a concave function on a convex set of constraints.

- By the Lagrange Duality (see for example Theorem 1 in Section 8.6 in [12]), we conclude that the supremum in (22) admits a unique maximizing pair $(\hat{h}, \hat{\rho})(t; x; p)$ for $(t; x; p) \in [0; T] \times R^n \times R^n$.

- By measurable selection, $(\hat{h}, \hat{\rho})$ can be taken as a Borel measurable function on $[0; T] \times R^n \times R^n$. 
Existence of a $C^{1,2}$ Solution to the HJB PDE

Once we have cleared the hurdle of having two controls, we are back to PDE territory:

- The number and properties of the controls do not matter (much);
- We can rely on our earlier results (see [5, 6])

Proving the existence of a strong, $C^{1,2}$, solution is the most difficult and intricate step in the process.

However, this is a necessary step if we want to use the Verification Theorem to conclusively solve our optimal investment problem.
Broadly speaking, the verification theorem states that if we have

- a $C^{1,2}([0, T] \times \mathbb{R}^n)$ bounded function $\phi$ which satisfies the HJB PDE (32) and its terminal condition;
- the stochastic differential equation

$$dX(s) = f(s, X(s), h(s), \rho(s); \theta)ds + \Lambda(s, X(s))dW^\theta_s + \int_Z \xi(s, X(s^-), z) \tilde{N}^\theta_p(ds, dz)$$

defines a unique solution $X(s)$ for each given initial data $X(t) = x$; and,
- there exists a Borel-measurable maximizing pair $(h^*(t, X_t), \rho^*(t, X_t))$ of $h \mapsto L^h \phi$ defined in (33);

then $\Phi$ is the value function and $(h^*(t, X_t), \rho^*(t, X_t))$ is the optimal pair of Markov control processes.

... and similarly for $\tilde{\Phi}$ and the exponentially-transformed problem.
Concluding Remarks

- Factors give us a lot of flexibility in the way we setup and parametrize the problem. We could even consider behavioural factors in our analysis (see [1])!

- Even with liabilities, leverage and jumps, we still manage to get a smooth value function and a convex optimisation problem for the controls. This is promising from a numerical perspective.

- The final major question we face to implement the model is how to populate the set of parameters sensibly.

- **Coming soon**: “Black-Litterman” for ALM (diffusion and affine jump-diffusion).
Thank you!

Any question?

3 Adapted from W. Krawcewicz, University of Alberta
Thank you!

Any question?

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4 Adapted from W. Krawcewicz, University of Alberta
Taming animal spirits: risk management with behavioural factors.

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Intertemporal surplus management with jump risk.

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