Generalized Lorentz Transformations in Four Dimensions and Superluminal Objects.

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Summary. — A new group $G$ of Lorentz transformations (LT) in four dimensions, generalized also for Superluminal frames, is introduced and particularly studied in its physical implications. With the help of a principle of duality — implied by $G$ — between subluminal and Superluminal frames, the meanings of «inertial frame», «equivalence», «principle of relativity», «covariance» may be correspondingly extended. A biunivocal correspondence exists between bradyonic and tachyonic velocities, which we find to be a particular conformal mapping (inversion). Since the group $G$ consists of generic rotations in space-time, it includes, e.g., also the total-inversion operation $(PT)$. Moreover (for a non «charge»-free universe), it is shown that our generalized special relativity requires covariance under $CPT$. A «tachyonization principle» is formulated, on the basis of which relativistic physical laws (of mechanics and electrodynamics, at least) can be easily extended to tachyons. Many simple applications are performed of the generalized LT’s (velocity composition law, comparison of the length and time units, Doppler effect, refraction index,...), either useful to clarify our problem or interesting in astrophysics.

1. — Introduction.

In a recent paper of ours \textsuperscript{(1)}, a generalization of the Lorentz transformations (LT) has been performed \textit{in four dimensions}, for reference frames $S$

travelling faster than light with respect to the usual class of inertial frames $s$ (*). The philosophical analysis developed in paper I showed that, if Superluminal observers $S$ are to be physical (in particular, if standard space-time measurements may actually be performed by $S$), then a «symmetry» or «duality» between frames $s$ and $S$ must hold, in the sense that particles behaving as tachyons ($s$) with respect to subluminal observers $s$ will behave as bradyons ($S$) (*) with respect to subluminal observers $s$ will behave as bradyons ($S$).

(*) After the completion of paper I, we became aware of the existence of papers (2-4), which approached our problem too. Criticizing ref. (2) is the content of ref. (3). Paper (4), by ANTIPPA, puts forward a theory yielding the same results as Parker’s (7), both works being confined to bidimensional space-time. The original features of ref. (4), i.e. introducing unidirectionality in time for bradyons and in space for tachyons, seems extraneous to our own philosophy.

(4) A. F. ANTIPPA: A one-dimensional causal theory of tachyons, preprint UQTR-TH-3, University of Quebec (Trois Rivieres, 1971). See previous footnote. In a preceding paper (4*), ANTIPPA and EVERETT went back to Feinberg’s philosophy (4*), taking advantage of the fact that Lorentz-covariance violation implicit in it does not come out in a bidimensional space-time.


with respect to Superluminal observers $S$, and viceversa. Actually, the words bradyons (B), tachyons (T), frames $s$ and frames $S$ have no absolute meaning, but only a relative one ($^1$) ("principle of duality").

Besides, in paper I it was shown that, if a frame $S$ is Superluminal as seen from a frame $s$, it will still be Superluminal when seen by any other $s$ subluminal frame; and so on. The velocity of light $c$ preserves of course its character of invariant quantity for both $s$ and $S$ frames ($^{18}$).

We shall call "inertial" all the (physical) frames $s$ and $S$, with relative velocities $u \geq c$.

But, according to paper I, we shall for the moment regard the reference frames moving with the invariant velocity $c$ as (the only ones) "unphysical", because of their singular characteristics ($^{18}$).

In paper I it has been argued that a "principle" of relativity does actually hold for our whole class $\{I\}$ of "inertial" frames, since the physical laws (when generalized also for tachyons) are to be covariant for the whole class $\{I\}$ ($^*$).

We shall come back to this point. It is worth-while to add that—due to the "principle of duality"—frames $S$ are assumed to have at their disposal exactly the same physical objects as frames $s$ have.

Owing to the last two facts, we can get the "equivalence of reference frames" extended to all our "inertial frames". In the following we shall show how the comparison may be performed between measurements taken in systems $s$ and $S$.

The main point of our generalization is the following: when passing from a subluminal frame $s$ to a Superluminal one $S$ (or vice versa, since those terms

$^*$ As we know, particles that look like B's (T's) to frames $s$ will look like T's (B's) to frames $S$, and vice versa. Therefore—and we shall come back to this point in the following—if both $s$ and $S$ observe the same object as required in relativity, bradyonic laws will transform into tachyonic laws under a Superluminal LT, and vice versa. On the contrary, each observer (either $s$ or $S$) will use the same law to describe different objects: namely, objects having velocity $u < c$ (or $u > c$) relative to him himself: i.e. to describe "bradyons" or "tachyons". In this second sense, the whole of relativistic physical laws—when written in a form holding for both B's and T's—has to be $G$-covariant. An example may be easily worked out, e.g., in relation to eq. (7) of the text. More in general, one may be convinced of the last assertion by remembering the possibility of writing laws in tensorial form.
have no absolute meaning), we have to admit that space (time)-like intervals with respect to $s$ must be time (space)-like with respect to $S$ \(^{(1)}\). Such an «inversion» is required in order to realize the above-mentioned symmetry between $s$ and $S$ frames \(^{(1,s)}\). Actually, that «inversion» operates the symmetry with respect to the light-cone (both in the configuration and tetraimpulse spaces).

Namely, in paper I it has been shown that LT's between two inertial frames must be such that \(^{(1,s)}\)

\[
e^{2}t'^{2} - x'^{2} = \pm (e^{2}t^{2} - x^{2})
\]

for $u \leq c$, respectively.

The linear transformations connecting inertial frames and satisfying eq. (1) are—roughly speaking—the usual proper, orthochronous (homogeneous) Lorentz transformations $A_{<}$ for $u < c$ \(^{(*)}\), and Lorentz transformations $iA_{<} \equiv \equiv iA_{>}$ for $u > c$ \(^{(*)}\). \(^{(1)}\)

For instance, in paper I we have shown that—in the simple case of collinear motion along the $x$-axis—condition (1) is satisfied \(^{(**)}\) by

\[
\begin{align*}
x' &= \pm \frac{x - ut}{\sqrt{1 - \beta^{2}}}, & t' &= \pm \frac{t - ux/c^{2}}{\sqrt{1 - \beta^{2}}}, \\
y' &= \pm y \sqrt{\frac{1 - \beta^{2}}{1 - \beta^{2}}}, & z' &= \pm z \sqrt{\frac{1 - \beta^{2}}{1 - \beta^{2}}}.
\end{align*}
\]

(1 bis)

Let us rewrite relation (1) for $\beta^{2} > 1$ (i.e. for transition from a $s$ to a $S$) as

\[
e^{2}t'^{2} + (ix')^{2} + (iy')^{2} + (iz')^{2} = (ict)^{2} + x^{2} + y^{2} + z^{2},
\]

and explicitly notice the following. Since we considered for simplicity the case of collinear motion along the $x$-axis, our eqs. (1 bis) must be—as they are—such that

\[
(1')
\]

\[
e^{2}t'^{2} + (ix')^{2} = (ict)^{2} + x^{2}
\]

and that

\[
(1^{'})
\]

\[
(iy')^{2} = y^{2}; \quad (iz')^{2} = z^{2}.
\]

\(^{(1)}\) And the ones times the operator $-1 = PT$. See Sect. 2.
\(^{(**)\)} See, R. MIGNANI, E. RECAMI and U. LOMBARDO: Lett. Nuovo Cimento, 4, 624 (1972). We shall see in the following (see Fig. 4 and 2) that in our eqs. (1 bis) the sign minus has to be taken only for «transfinite» transformations, bypassing the point $P_{\infty}$ according to the observer.
Of course, both \( s \) and \( S \) will observe only real quantities! The imaginary units merely record the relative sign of the various couples of the corresponding-component squares.

One of the aims of the present work is extensively illustrating the new generalized LT group \( G \), since it was scarcely mentioned in paper I. Another aim is clarifying some points of paper I.

In our terminology \( G \)-covariant means covariant under the whole group \( G \).

By the way, let us remember that only measurement results are supposed to be expressed by real quantities, but the generalized Lorentz transformations (GLT) may well be represented by matrices built up with imaginary quantities too (\(^1\)).

2. – The group \( G \) of the generalized Lorentz transformations in four dimensions.

In order to fix our ideas, let us first consider a universe free of \( \ast \) charges \( (*) \) and let us represent the \( A \)'s by \( 4 \times 4 \) matrices. If we put

\[
A_< \equiv A(|\beta| < 1), \quad A_> \equiv A(|\beta| > 1),
\]

where \( A(|\beta| > 1) \) are formally identical with usual proper, orthochronous LT's but correspond to values \(|\beta| > 1\), it is possible to see that

\[
A_<^{-1}(\beta) = A_<(-\beta), \quad [iA_>(\beta)]^{-1} = -iA_>(-\beta) = -iA_>^{-1}(\beta).
\]

Thence

\[
[iA_>(\beta)][-iA_<^{-1}(\beta)] = 1,
\]

but

\[
[iA_>(\beta)][iA_>^{-1}(\beta)] = -1,
\]

and the generalized group (\(^1\)) \( G \) of paper I has in particular to contain also the total-inversion operator

\[
PT = -1.
\]

Precisely, by considering successive applications of GLT's of type \( A_< \) and \( iA_> \), it is easy to realize that the group \( G \) of the GLT's consists of four subsets:

\[
G \equiv (L_1) \cup (L_2) \cup (L_3) \cup (L_4), \tag{2}
\]

\( (*) \) In this work, the word \( \ast \) charge \( \ast \) is used in its widest sense.
where

\[ \mathcal{L}_1 \equiv \{ A_\beta \}, \quad \mathcal{L}_2 \equiv \{ -A_\beta \}, \]
\[ \mathcal{L}_3 \equiv \{ iA_\beta \}, \quad \mathcal{L}_4 \equiv \{ -iA_\beta \}. \]

In fact, if \( A_\beta \) is a generic LT for \( |\beta| < 1 \), i.e. a subluminal LT, \( (LT) \), then the generic GLT for \( |\beta| > 1 \), i.e. the generic Superluminal LT, \( (STL) \), will be \( iA_\beta \). Moreover, if \( L \in G \), then also \( -L \equiv (PT)L \in G \). In the following, we shall indicate simply by \( L \) the generic element (GLT) of \( G \).

By the way

\[ \det L = +1, \quad \forall L \in G. \]

It is easy to recognize that a correspondence exists between subluminal LT's from a frame \( s_0 \) (*) to a frame \( s \) moving with velocity \( u \) (\( 0 < u < c \)) and SLT's connecting \( s_0 \) to a frame \( S \) travelling in the same direction with velocity \( U = c^2/u \) (\( U > c \)), in the sense that

\[ (3) \quad iA_\beta(U) = K \cdot A_\beta(c^2/U), \]

\( K \) being a matrix independent of the velocity magnitudes \( u, U \). For simplicity we consider eq. (3) only for collinear motion.

The matrix \( K \) represents a \( \text{transcendental SLT} \). In fact, for \( U \to \pm \infty \), eq. (3) becomes

\[ (3 \text{ bis}) \quad L_{\pm\infty} = iA_\beta(\pm\infty) = K. \]

This accords with the observation that, if a tachyon moves with velocity \( v = v_z \) relative to us, it will appear with \textit{divergent} velocity to the observer \( s'' \) having (collinear) velocity \( u = u_z = c^2/v \) relative to us (see Fig. 3). For instance, in the simple case of collinear motion, \( u = u_z, \ U = U_z \),

\[ L_{\infty} \equiv K = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}. \]

(*) For simplicity, in the following we shall consider \textit{ourselves} as the observer \( s_0 \).
At last, in the collinear case, the matrix $K$ operates the exchanges

$$
x(u) \rightarrow -t(c^2/u),
\quad t(u) \rightarrow -x(c^2/u)
$$

and $y(u), z(u) \rightarrow iy(c^2/u), iz(c^2/u)$, that accords with eqs. (1')-(1'').

In the collinear case, for infinite speed, as well as for zero speed, the notion of direction becomes meaningless (*) (†).

It is already realizable that the tools of pseudo-Euclidean geometry are not the best ones for dealing with our problems; we shall therefore borrow a bit from projective geometry.

From what precedes, it is apparent that our « duality principle » is characterized by the fact that—with reference to a frame $s_0$—there is a biunivocal correspondence between observers with velocity $u$ and those with velocity $U = c^2/u$. Precisely, the « symmetry » (see Fig. 1) between subluminal and Superluminal frames is a particular conformal mapping (« inversion »):

$$u \mapsto c^2/u.$$  

![Fig. 1. - Representation of the conformal mapping (inversion) $u \leftrightarrow c^2/u$ in the simplified case of collinear velocities $u \equiv u_0 \geq 0$ relative to a frame $s_0$. Since $|u| \leq c$ and we have to deal also with the transcendental frame, we project from a pole $P$ the axis $u$ onto the circle having $u = \pm c$ as diametral points. The chords $AB$ (where $A, B$ refer respectively to a subluminal velocity $u$ and a Superluminal velocity $U = c^2/u$, i.e. to corresponding velocitics in the conformal mapping) are normal to the axis $u$. In such a mapping, the velocities $u = U = c$ are the united ones (as required), and velocities zero, infinite correspond to each other:

$$(4 \text{ bis}) \quad u = c \leftrightarrow U = c, \quad u = 0 \leftrightarrow U = \infty.$$  

(*) We might also define the LT to a « luminal frame ». But, as already said, we cannot consider the luminal frames $s_\infty = S$, as physical, even if mathematical use of « infinite-momentum frames » spreads out. In any case, from such frames the space-time should appear as a bidimensional space free of photons (projection of the instantaneous 3-dimensional space onto a plane normal to the ray direction).
The latter tells us the known fact that the divergent velocity plays for tachyons the same role as played by the null velocity for bradyons. The mapping (4) was realized also by previous authors (1-4,7,9) in different ways.

In order to illustrate the physical meaning of the four subsets in eq. (2), let us for simplicity confine ourselves to the case of frames with collinear velocities. We can represent such velocities \( u = \pm c \) (relative to a frame \( s_0 \)) along an axis; see Fig. 2. But, since \( |u| \geq c \) and we have to meet also the divergent velocity, it is convenient to project the axis \( u \) from the pole \( P = P_\infty \) onto the circle having \( u = \pm c \) as diametral points (see Fig. 1, where chords \( AB \parallel u \)).

The characteristic feature of Fig. 2 is that our circle \( \gamma \) must be considered as a «double circle», with a «branching point» at \( P(u = \infty) \) (such a point reflects the existence of the double sign in eqs. (1 bis).) In fact, along \( \gamma \) we pass with continuity from frames \( s^k \) (e.g. with a right-handed spatial frame) to totally inverted frames \( s^k \equiv PTs^k \) (with a left-handed spatial frame and a reversed time axis). This could have been expected, since the total inversion \( PT \) is nothing but a particular «rotation» in (four-dimensional) space-time, and such a rotation may be achieved when we do not restrict our attention any more to subluminal relative velocities (*).

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Let us be clearer. If $A_\lt$ is the LT from $s_0$ to frame $s_1 = 1$ moving with velocity $u$, then the (conformally correspondent) transformation $iA_\lt$ will connect $s_0$ to the frame $S_2 = 2$ moving with velocity $U = c^2/u$ with respect to $s_0$ and with infinite velocity (*) with respect to $s_1$. Now, we may go back from $S_2$ to the same $s_0$ by applying the inverse transformation $(A_\lt)^{-1} = -iA_\lt^{-1}$, so that $(iA_\lt) \cdot (-iA_\lt^{-1}) = 1$. But we might also decide to come back from $S_2$ to $s_0$ by bypassing the transcendental frame (relative to $s_0$), *i.e.* by applying the transformation $+ iA_\lt^{-1}$; however, in this case we would go back to the frame $s_0$ with all its axes reversed, since $(iA_\lt) \cdot (iA_\lt^{-1}) = -1 \cdot PT$. More generally, if $L(s_0 \rightarrow f)$ is the GLT connecting frames $s_0$ and $f$, we have

$$L(s_0 \rightarrow 1) = A_\lt(u), \quad L(s_0 \rightarrow 2) = iA_\lt(c^2/u),$$

$$L(s_0 \rightarrow 3) = -A_\lt(-c^2/u) = -iA_\lt^{-1}(c^2/u),$$

$$L(s_0 \rightarrow 4) = -A_\lt(-u) = -A_\lt^{-1}(u),$$

$$L(s_0 \rightarrow 5) = -A_\lt(u), \quad L(s_0 \rightarrow 6) = -iA_\lt(c^2/u),$$

$$L(s_0 \rightarrow 7) = iA_\lt^{-1}(c^2/u), \quad L(s_0 \rightarrow 8) = A_\lt^2(u),$$

$$L(s_0 \rightarrow 9 = 1) = A_\lt(u).$$

The relative velocity between $s_1 = 1$ and $S_2 = 2$, or between $s_8 = 8$ and $S_7 = 7$, is infinite; whilst the relative velocity between $S_3 = 3$ and $S_2 = 2$, or between $s_8 = 8$ and $s_1 = 1$ is

$$\frac{2u}{1 + (u/c)^2} = \frac{2U}{1 + (U/c)^2}.$$  

Every time we cross the $u$-axis, we have the exchange *spacelike* $\leftrightarrow$ *timelike*, or formally a multiplication by $\pm i$ (in the sense of Fig. 2). In other words, at $u = +c$ we get transition from $\{ \pm A_\lt \}$ to $\{ \pm iA_\lt \}$, and at $u = -c$ transition from $\{ \pm iA_\lt \}$ to $\{ \pm A_\lt \}$. Besides, when we cross the pole $P_\infty$, we get the $PT$-inversion, or formally a multiplication by $-1$.

Before going on, we want to re-emphasize that the principle of relativity—in its wider sense—asks that physical laws (at least in their form valid for both B’s and T’s) be covariant under all the elements of $G$. Therefore, it may seem that covariance also under the $PT$-symmetry is required.

Let us inspect the problem more closely. Let us now consider a *non-charge-free* universe. For going (from $s_0$, in the counter-clockwise sense: see Fig. 2) to a *PT-ed* frame, we must bypass the *branching point* $P_\infty$, *i.e.* considering GLT’s, eqs. (1 *bis*), with *negative* sign. Therefore, if we consider

(*) The corresponding frames $s, S$ such that $\vec{SS} \perp \vec{u}$, have infinite relative velocity. The SLT from $s$ to $S$ will be in this case $L = L_\infty$ (see Fig. 2).

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a particle, also its transformed energy will be negative. This further fact has
to be taken into account: together with $PT$, it allows the transition (*) from
that particle to its antiparticle (*). In other words, extended relativity tells
us that elementary-particle laws are expected not to change only under the
combined operation: i) parity, ii) time reversal, iii) particle-antiparticle
exchange. We shall come back to this question later on.

3. - Extended velocity composition law.

At this point we want to spend some more words about the generalization
of the velocity composition law for speeds exceeding the speed of light. Following
paper I, with reference to two collinear systems moving with relative velocity
$u = u_z$, $|u| > c$, we have

$$v' = c^2 \frac{u_z - u}{c^2 - u u_z}, \quad v_{1} = c^2 \frac{v_z + \sqrt{1 - \beta^2}}{c^2 - u u_z}, \quad \left(\beta^2 = \frac{u^2}{c^2} \geq 1\right).$$

Let us confine ourselves only to $v = v_z$; eqs. (5) hold for $u \geq c$ (see Fig. 3).
It is already well known (**) that a tachyon $T$ may appear as a « particle » (e.g.
in the initial state of a certain reaction) to an observer $s$ and as an « antipar-
ticle » in the opposite state (i.e. outgoing from that interaction region) to
other observers $s'$. In fact, the first of eq. (5) yields, for $|u| < c$ and $v_z > c$,

$$v' \rightarrow \begin{cases} \infty, & \text{for } u = c^2/v_z \\ \text{sign } (v_z) = \text{sign } (v_z), & \text{for } c < u < c^2/v_z \\ \text{sign } (v_z) = \text{sign } (v_z), & \text{for } c > u > c^2/v_z \end{cases}$$

The same would of course happen for $v_z'$ and $v_z'$.

Now we want to underline another peculiarity of the first of eq. (5). Let us
consider our own rest frame $s_0$ and two collinear Superluminal frames $S, S'$
having (positive) velocities $U = U_z > c$ and $V = V_z > c$, respectively. In the
case that $S'$ is faster than $S$ (with respect to us), the frame $S$ will not

(*) If we introduce in relativity the notion of « particle », then relativity itself leads
to the concept of « antiparticle »: in such a sense, the concept of antiparticle is a purely
relativistic one. Remember the «switching principle»—which assumes that « physical
signals are transported only by positive-energy objects»—and related problems: see, e.g.,
30, 718 (1962), and ref. (10).

and references therein. See also, e.g., E. C. G. Sudarshan: Ark. Phys., 39 (40), 585
(1969); in Symposia on Theoretical Physics and Mathematics, Vol. 10 (New York, 1970),
p. 129.

(**) See, e.g., ref. (10).
observe the frame $S'$ moving with positive velocity, as suggested by usual intuition! On the contrary, in fact, if we call $v' = v'_z$ the $S'$ velocity as seen by $S$, we have

$$v' < 0 \quad \text{when } V > U,$$

$$v' > 0 \quad \text{when } V < U.$$

This fact has to be borne in mind in order to understand Fig. 2 better. Let us analyse its simplified version, Fig. 4, in which only «right-handed» frames

![Diagram](image-url)
appear. Let our rest frame be $s_0$, and our transcendent frame $S_\infty$. Then, if $A_\prec$ is the LT connecting $s_0$ and the frame $s_1 \equiv 1$, travelling with velocity $u > 0$ (relative to $s_0$), the same LT $= A_\prec$ connects $S_\infty$ and the frame $S_2 \equiv 2$, travelling with velocity $U = c^2/u$ with respect to $s_0$. This is required by the group properties of $G$, as we have exploited them (*). But this means that $S_\infty$ must judge $S_2$ to move with the positive velocity $u$, even if according to us frame $S_2$ is slower than $S_\infty$ (remember that $s_0, s_1, S_2, S_\infty$ are all collinear frames!). By the way, $L(8 \rightarrow 1) = A_\prec^2$.

Fig. 5. – Where it is seen that, when $S'$ travels faster than $S$ relative to $s_0$, i.e. to us (all the frames are collinear), then $S'$ appears to $S$ moving with negative velocity (even if it moves with positive velocity with respect to us). See the vector corresponding to $S'$. Analogously, frames $s', s'', S''$ will be seen by $S$ as moving forwards (even if they would be expected to be seen travelling backwards by $S$, according to the usual intuition).

The above-mentioned fact may be seen also on the basis of Fig. 5, where $S'$ is a (Superluminal) frame travelling faster than $S$ relative to $s_0$, i.e. to us (all the frames are still collinear). By inspection of the components of a generic space-time displacement vector associated with the motion of $S'$ (relative to $s_0$), one realizes that $S'$ appears to $S$ as moving with negative velocity (even if it moves with positive velocity with respect to us). Analogously, frames $s', s'', S''$ will be seen by $S$ as moving forwards (even if they would be expected to be seen by $S$ travelling backwards, according to the usual intuition).

4. – «Tachyonization» principle.

Let us assume that we know, besides the class $\mathcal{A}$ of usual physical laws (e.g. of mechanics and electrodynamics) for bradyons, also the class $\mathcal{B}$ of the

(*) However, things may be probably done in a more smooth, continuous fashion by using a bit more the elementary concepts of projective geometry: namely, by suitably inverting the signs in eqs. (1 bis) at $u = \pm c$, besides at $u = \infty$. This will be discussed on another occasion.
physical laws for tachyons. When we pass from a subluminal frame $s$ to a superluminal one $S$, class $\mathcal{A}$ will have of course \(^{(1,2)}\) to transform into class $\mathcal{B}$ and vice versa. *In this sense* the totality of physical laws ($\mathcal{A} \cup \mathcal{B}$) will be covariant under the whole group $G$, or simply "$G$-covariant".

Moreover, a preliminary inspection suggests that physical laws (of special relativity) may be written in a form valid for both bradyons and tachyons: a form obviously coinciding with the usual one in the bradyonic case. As an example let us remember \(^{(1)}\) the law

$$m = \frac{m_0}{\sqrt{1 - \beta^2}} \quad (\beta^2 \leq 1),$$

which reads

$$m = \frac{m_0}{\sqrt{1 - \beta^2}} \quad \text{for } \beta^2 < 1, \quad \text{and} \quad m = \frac{\text{i}m_0}{\sqrt{1 - \beta^2}} \quad \text{for } \beta^2 > 1.$$

Let us explicitly re-emphasize that the $\text{i}$ appearing for $\beta^2 > 1$ comes from the SLT's, and does not mean at all that tachyons have an imaginary proper mass (as we know, a tachyon behaves as a bradyon with reference to its rest frame, and therefore all its proper quantities are real). The same happens for proper time and proper length; for example, since $d\tau = d\tau' \cdot \sqrt{|1 - \beta^2|}$, when passing from the tachyon rest system to our frame, one has

$$d\tau' = \frac{d\tau_0}{\sqrt{|1 - \beta^2|}} = \frac{d\tau_0}{\sqrt{\beta^2 - 1}} = \frac{d\tau_0}{-\text{i} \sqrt{1 - \beta^2}} = \frac{\text{i} d\tau_0}{\sqrt{1 - \beta^2}} \quad (\beta^2 > 1),$$

where of course $d\tau$, $d\tau_0$ are both real.

It is worth-while to notice that, when generalizing \(^{(1)}\) physical laws for tachyons ($\beta^2 > 1$), one should pay attention that *a priori* $\sqrt{\beta^2 - 1} = \pm \text{i} \sqrt{1 - \beta^2}$, since $(\pm \text{i})^2 = -1$. Always we consistently choose \(^{*}\) the sign *minus*, in order, *e.g.*, to get positive values of $m$ in eq. (7). It is understood that $+ \sqrt{1 - \beta^2}$ represents the upper half-plane solution for $\beta^2 > 1$. See, *e.g.*, our eq. (5).

In the first paragraph of this Section we expressed in which sense all the "inertial frames" (with relative velocities $|u| \geq c$) are equivalent. From such considerations, it is immediate to get the "principle of tachyonization", which extends Parker's principle \(^{(1)}\) to the four-dimensional space-time: *The re-

\(^{(1)}\) See, *e.g.*, eq. (4') in paper I. But a misprint entered eqs. (7) of paper I, regarding the SLT for velocity; these equations should read

$$v_{y,z}' = \frac{iv_{y,z} \sqrt{1 - \beta^2}}{1 - uw_{y,z}/c^2} = \frac{iv_{y,z} \sqrt{\beta^2 - 1}}{1 - uw_{y,z}/c^2} = \frac{iv_{y,z} (-\text{i}) \sqrt{1 - \beta^2}}{1 - uw_{y,z}/c^2} = \frac{v_{y,z} \sqrt{1 - \beta^2}}{1 - uw_{y,z}/c^2}. $$
lativistic laws (of mechanics and electrodynamics) for tachyons follow by applying a SLT (e.g. the transcendent one) to the corresponding laws for bradyons ».

In Sect. 3 we have seen that, since $PT$ is a chronotopical rotation, relativistic physical laws are expected to be covariant under the $CPT$-symmetry. Since the pure $PT$ operation onto space-time brings from a subluminal frame to another subluminal frame, the previous statement refers to usual relativistic laws (even if not written in the $G$-covariant form). Due to its importance, let us derive it in a second way (for a non « charge »-free universe).

As pointed out in ref. (10), let us consider a tachyon and a succession of subluminal frames (for simplicity, all moving collinearly with the tachyon). Let us call $s_\infty$ the frame in which the tachyon appears with divergent velocity. If a frame moving slower than $s_\infty$ sees the tachyon travelling in a certain direction, then a frame moving faster than $s_\infty$ will actually see the « tachyon » as an antitachyon travelling in the opposite direction: this is precisely what is already expressed by our relations (6). What we want to underline here is that, when the tachyon appears with reversed velocity, it will also show the opposite charge (10). Therefore, bypassing the « infinite-velocity frame » $s_\infty$ (in the above sense) implies the operation of charge conjugation ($C$). With reference to Fig. 4, when we consider our frame $s_0$ as seen by a succession of Superluminal frames, and we « cross » the transcendental frame $S_\infty$, we have that all our charges appear reversed. In other words, for « transfinite transformations » we get a $C$-symmetry. Therefore, when we operate a rotation (in the four-dimensional space-time) with the aim to reach the totally inverted frame $PT s_0$ (see Fig. 2), really we reach the frame $CPT s_0$.

Briefly, the right way for doing $PT$ is doing $CPT$. The $CPT$-covariance is required by our mere « extended relativity principle » (when we do not confine ourselves to subluminal relative velocities).

Analogous considerations are to be kept in mind when investigating the physical law transformations for transition between two generalized inertial frames.

5. - Comparison of time and space intervals.

With the standard procedure, from the GLT’s for the four-vector components one gets the GLT’s for the measure units of space (standard-rod length) and time (standard-clock time interval).

If $\Delta x_0$ and $\Delta t_0$ are the proper intervals, the magnitudes of the observed ones are

$$\Delta x = |\Delta x| = \Delta x_0 \cdot \sqrt{1 - \beta^2}, \quad \Delta t = |\Delta t| = \Delta t_0 / \sqrt{1 - \beta^2} \quad (|\beta| > 1).$$
For simplicity, we considered two collinear frames. Of course, eqs. (9) do not depend on the sign of $\beta$. Figure 6, which depicts eqs. (9), shows that for $\beta^2 > 1$ we can have both Lorentz contraction and dilatation of space or time intervals. The fact that $\Delta x = \Delta x_0$, $\Delta t = \Delta t_0$ not only for $\beta = 0$ but also for the value $\beta = \sqrt{2}$ will be geometrically interpreted in Fig. 8.

Let us now consider the SLT's for space or time intervals by extending the usual geometrical interpretation of the LT's (see Fig. 7). From an algebraic point of view, following ref. (7,9), we may observe that for $\beta^2 \approx 1$

$$x = \frac{x' + ct'}{\sqrt{1 - 1/\beta^2}}, \quad t = \frac{t' + x'/c}{\sqrt{1 - 1/\beta^2}}.$$

Therefore, if we put $\tan \psi = 1/\beta$ and $K = \sqrt{(\beta^2 + 1)/(\beta^2 - 1)}$, we get

$$x = K(x' \sin \psi + ct' \cos \psi), \quad t = K\left(\frac{x'}{c} \cos \psi + t' \sin \psi\right).$$

Equations (10), by the standard reasoning (11), allow us to interpret these SLT's as in Fig. 8.

For the extension of the geometrical interpretation, it is enough to remember (besides the standard definitions) that: i) the Lorentz-transformed space unit $\Delta x$ may be also derived from the time $a$ (relative to us) taken by the moving standard rod to pass «before our eyes»: $\Delta x = | + a\beta c|$; ii) the Lo-

---

Fig. 7. – The geometrical interpretation of the usual LT’s ($\beta < 1$), for space or time intervals, in the case of collinear frames and $\beta > 0$. One has length contraction and time dilatation. Besides, making recourse to the standard definitions, one must notice that: i) $\Delta x = -a\beta c = |a\beta c|$, where $|a| = -a$ is the time (relative to us) taken by the moving standard rod to pass «before our eyes»; ii) $\Delta t = b/\beta c = |b/\beta c|$, where $|b| = b$ is the space travelled (in our frame) by a «lamp» which is switched on for a unitary time (this time being measured in the co-moving frame). Notice that $OA = \Delta x = -a\beta c = |a\beta c| < \Delta x_0$, $OB \equiv \Delta t = b/\beta c = |b/\beta c| > \Delta t_0$, $OU_t' = OU_t$ ($\Delta t, \Delta x > 0$); $0 < \beta < 1$.

recentz-transformed time unit $\Delta t$ may be also derived from the space $b$ travelled (in our frame) by a «lamp» which is switched on for a unitary time (this time being measured in the co-moving frame): $\Delta t = |b/\beta c|$. Moreover, the ex-

$$\begin{align*}
\infty &> \beta > 1 \\
\Delta t & = |b/\beta c| > \Delta t_0 \quad \text{for } 1 < \beta < \sqrt{2} \\
&< \Delta t_0 \quad \text{for } \beta > \sqrt{2} \\
\Delta x & = |a/\beta c| < \Delta x_0 \quad \text{for } 1 < \beta < \sqrt{2} \\
&> \Delta x_0 \quad \text{for } \beta > \sqrt{2}
\end{align*}$$

Fig. 8. – The geometrical interpretation of our SLT’s (for $\beta > 1$). The fact that one has a change in the (spacelike or timelike) nature of intervals when passing from frames $s$ to frames $S$ reflects in the exchanged use of hyperbolas, when considering space and time intervals respectively. Notice that one has still $\Delta t = |b/\beta c|$, and $\Delta x = |a\beta c|$, but now—according to Fig. 6—we get both Lorentz contractions and dilatations. In particular (for $\beta > 1$), we have $\Delta t = \Delta t_0$ and $\Delta x = \Delta x_0$ when $\beta = \sqrt{2}$. Moreover, for $1 < \beta < \sqrt{2}$, we have $\Delta t > \Delta t_0$ and $\Delta x < \Delta x_0$, whilst, for $\beta > \sqrt{2}$, we have $\Delta t < \Delta t_0$ and $\Delta x > \Delta x_0$. 
changed use (for $\beta^2 > 1$) of the hyperbola, when considering space and time intervals respectively, reflects the change in the (spacelike or timelike) nature of intervals that we have when passing from frames $s$ to frames $S$ (see Fig. 8). Consistently with Fig. 6, in Fig. 8 we have both Lorentz contractions and dilatations as $\beta$ varies. In particular, from the equations of our hyperbolas it is immediate to see (for $\beta > 1$) that $\Delta t = \Delta t_0$, $\Delta x = \Delta x_0$ when, and only when, $\beta = \sqrt{2}$.

An analogous procedure can be used for interpreting the other cases. For instance, the case $-\infty < \beta < -1$ results to be symmetric to the case in Fig. 8.

Moreover, if we proceed beyond the case $+ 1 < \beta < \infty$, we get again the whole previous succession of cases, but—according to Fig. 2—it will now regard totally inverted frames. See Fig. 9.

6. The generalized Doppler effect and cosmology.

Let us spend some words on the generalization of the Doppler-effect formula for Superluminal sources, because of its possible astrophysical interest.

From the SLT's for a time interval, one gets that

$$v = \frac{v_0 \cdot \sqrt{1 - \beta^2}}{1 + \beta \cos \alpha},$$

(11)
where \( u = \beta c \) is the relative velocity and \( \alpha = \hat{u} \cdot \hat{l} \), the vector \( l \) being directed from the observer to the source. In the particular case of relative motion strictly along the \( x \)-axis, since

\[
\text{sign}(u) \cdot \text{sign}(\cos \alpha) = \begin{cases} 
- & \Rightarrow \text{approach}, \\
+ & \Rightarrow \text{recession}, 
\end{cases}
\]

we obtain the behaviour represented in Fig. 10, where the dashed lines refer to approach and the solid ones to recession. The two solid curves (recession)

\[ \nu \]

Fig. 10. – Doppler-effect extension: observed frequency vs. relative velocity for motion along the \( x \)-axis. The sign minus refers to approach (dashed line) and plus to recession (solid line). The interpretation of the negative values appearing for the Superluminal approach is given in Fig. 11.

are one the conformal correspondent of the other, as expected, in the sense that the same frequency will be observed both for \( u = v \) and for \( u = c^2/v \). The same is true for the two dashed curves (approach), except for the sign. The interpretation of the negative sign appearing for Superluminal approach is easy in the spirit of what precedes (1); see Fig. 11.

In fact, a (subluminal) observer will receive the radio-emission of a Superluminal source in the reversed chronological order. It is immediate to think the following. If a macroscopic phenomenon produces a known radio-emission obeying a certain chronological law, and one happens to detect a reversed radio-emission, we could believe a superluminal source has been observed.

Extrapolating our «duality principle» even for gravitation, we would have a way to deduce also the behaviour of tachyons in a gravitational field.
As regards the Čerenkov effect of tachyons in homogeneous matter (of course, we shall not have such an effect in vacuum (12)), they will radiate Čerenkov light every time they travel faster than light in that medium. This is always verified for tachyons in a "bradyonic medium". On the contrary, tachyon's behaviour in a "tachyonic medium" will be symmetric to the behaviour of a bradyon in usual matter (remember the duality principle). Such considerations may be substantiated by bearing in mind how the medium

![Diagram](image)

Fig. 11. - The radio emission of a Superluminal source, approaching the observer along the x-axis, will be received in reversed chronological order. This is the meaning of the negative frequencies entering Fig. 10. The line S is the Superluminal world line.

![Diagram](image)

Fig. 12. - Generalization of the formula for the refraction index $N$ of a medium vs. the relative velocity $u$. The medium is supposed to move collinearly with respect to the observer. $N_0$ is the proper refraction index.

(12) Showing that a tachyon in vacuum does not emit Čerenkov radiation is a trivial application of our tachyonization principle. See also C. C. CHANG: preprint CPT-117 (Austin, Tex., 1971).
«refraction index» $N$ varies when the velocity $u$ (relative to the observer) varies, in special relativity. In the simple case of collinear motion, if $N_0$ is the proper refraction index, one has, for both $|u| \geq c$,

$$N(u) = \frac{N_0 c + u}{c + N_0 u},$$

and $v' = c/N$ will be the velocity of light in the medium (with respect to the observer). See Fig. 12.

Lastly, we want to mention that «complex transformations» (13) have been shown to provide a natural connection between electric and magnetic charge (13), as well as between bradyonic and tachyonic sources. Analogies between magnetic monopoles and tachyons had already been noticed in ref. (7).

* * *

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● RIASSUNTO

Un nuovo gruppo $G$ di trasformazioni di Lorentz (LT) in quattro dimensioni, generalizzato anche per sistemi di riferimento Superluminali, è introdotto e studiato particolarmente nelle sue implicazioni fisiche. Con l’aiuto di un «principio di dualità» — implicato da $G$ — tra sistemi subluminali e Superluminali, è possibile estendere il significato di «riferimento inerziale», «equivalenza», «principio di relatività», «covarianza». Tra velocità bradioniche e tachioniche esiste una corrispondenza biunivoca, che risulta essere una particolare corrispondenza conforme (inversione). Poiché il gruppo $G$ consiste di rotazioni generiche nello spazio-tempo, esso include per esempio anche l’operazione di inversione totale ($PT$). Inoltre (per un universo con «cariche»), si mostra che la nostra relatività ristretta generalizzata richiede la covarianza per $CPT$. Si formula un «principio di tachionizzazione», in base al quale le leggi fisiche relativistiche (quelle almeno della meccanica e dell’elettrodinamica) possono essere facilmente estese al caso dei tachioni. Si applicano le LT generalizzate ad alcuni semplici casi (legge di composizione delle velocità, confronto di unità di tempo e di lunghezza, effetto Doppler, indice di rifrazione, ...) utili per chiarire il nostro problema o di interesse in astrofisica.
Обобщенные преобразования Лоренцва четырех измерениях и сверхсветящихся объекты.

Резюме (*). — Вводится новая группа $G$ преобразований Лоренца в четырех измерениях, обобщенная также для сверхсветящихся систем отсчета. Исследуются физические применения новой группы $G$. Используя "принцип дуальности" между субсветящимися и сверхсветящимися системами отсчета, может быть расширен физический смысл понятий "инерциальной системы отсчета", "эквивалентности", "принципа относительности" и "ковariantности". Существует соответствие между скоростями брадионов и тахионов, которое получается как результат конкретного конформного отображения (инверсии). Так как группа $G$ содержит вращения в пространстве и времени, то она включает, например, также операцию полной инверсии (PT). В случае зарядовой инверсии наша обобщенная специальная теория относительности требует ковариантности относительно $CPT$. Формулируется "принцип тахионизации", на основе которого релятивистские физические законы (по крайней мере, механики и электродинамики) могут быть легко обобщены для тахионов. Рассмотрено много простых применений обобщенных преобразований Лоренца (закон сложения скоростей, сравнение единиц длины и времени, эффект Допплера, коэффициент преломления и т.д.), полезных либо для прояснения нашей проблемы, либо интересных в астрофизике.

(*) Переведено редакцией.
Generalized Lorentz Transformations in Four Dimensions and Superluminal Objects.

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(Nuovo Cimento, 14 A, 169 (1973))

Errata

<table>
<thead>
<tr>
<th>Page</th>
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<tr>
<td>170</td>
<td>10</td>
<td>paper</td>
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<td>174</td>
<td></td>
<td>eq. (3 bis)</td>
<td>$L_\infty = iA_\infty (+ \infty) = K$</td>
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<td>175</td>
<td></td>
<td>lines 5 and 6</td>
<td>eliminate</td>
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<tr>
<td>175</td>
<td>line 2 from bottom</td>
<td>free of photons</td>
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<td>176</td>
<td>line 2 of Fig. 2</td>
<td>groups $G$</td>
<td>group $G$</td>
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<td>186</td>
<td>line 7 from bottom</td>
<td>of a Superluminal</td>
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<td>187</td>
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Moreover, at p. 187, line 6, add the following sentence: «In fact, from the duality principle and Fig. 12, one may deduce that a) bradyons will emit Čerenkov radiation only in bradyonic media (when their speed is larger than the light speed in those bradyonic media), b) tachyons will emit Čerenkov radiation only in tachyonic media, when their speed is slower than the light speed in those tachyonic media (cf. Fig. 12)». 