About a Dirac-Like Equation for the Photon according to Ettore Majorana (*).

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New interest is increasing in the possible interpretations of quantum mechanics, and particularly of the quantum wave function \( \psi \) (1). In this context it may be of help developing a hint originally due to MAJORANA (2,3) (Fig. 1), i.e. writing a Dirac-like equation for the photon.

Moreover, in doing so we shall be able to stress once again (4) the importance in electrodynamics of the complex quantity \( F = F_{\text{c}} - iH \); in fact, it results to be an essential tool in the present attempt (as MAJORANA (2) realized almost fifty years ago). Subsequently, its importance has been recognized as well by WEINBERG (5) and many others (6-7).

Macroscopically, and at a statistical level, the electric and magnetic fields \( E, H \) are known to be connected (though the quantity \( E^2 + H^2 \)) to the local mean number of

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(1) See, e.g., proceedings of 1974 Frascati Meeting on Foundations of Quantum Mechanics, to be published.
(2) E. MAJORANA: Scientific manuscripts, unpublished, deposited at the «Domus Galilaeana», Pisa, quaderno 2, p. 101/1; 3, p. 11, 160; 15, p. 16; 17, p. 83, 159. They were written in a year between 1928-32. See also E. AMALDI: In La vita e l'opera di Ettore Majorana (Roma, 1966).
(3) See also E. MAJORANA: Nuovo Cimento, 9, 335 (1932).
(equal) photons. Majorana's idea seems to have been that of analogously expressing the probability quantum wave $\psi$ of a photon in terms of $E, H$, thus giving it a more direct meaning than usually (when introducing the electromagnetic four-potential). What is

$$\psi = E + iH$$

Let us introduce the quantities

$$\psi_i = E_i - iH_i \quad (i = 1, 2, 3).$$

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**Fig. 1.** The most important (and most clear) page, handwritten by Majorana in a year between 1928-'32, that—among other ones—suggested us the present work.

More, Majorana wrote for the photon an equation similar to the Dirac one, thus possibly suggesting an analogous interpretation even for the electron wave function.

Let us introduce the quantities

$$\psi_i = E_i - iH_i \quad (i = 1, 2, 3).$$
Then the standard Maxwell equations may be written (*)

\[
\text{div } \mathbf{\Phi} = \mathbf{\epsilon}, \quad i \text{ rot } \mathbf{\Phi} = \mathbf{j} + \frac{\partial \mathbf{\Phi}}{\partial t},
\]

where \( \mathbf{\Phi} = (\psi_1, \psi_2, \psi_3) \). In absence of electric charges and currents, we can pass to quantize eqs. (2) by using the correspondence principle \((- i\partial/\partial x_1 \rightarrow p_1; + i\partial/\partial t \rightarrow W)\), and we get

\[
\begin{align*}
W\psi_1 + ip_2 \psi_3 - ip_3 \psi_2 &= 0, \\
W\psi_2 + ip_3 \psi_1 - ip_1 \psi_3 &= 0, \\
W\psi_3 + ip_1 \psi_2 - ip_2 \psi_1 &= 0,
\end{align*}
\]

with the (transversality) condition

\[
(p \cdot \mathbf{\Phi}) = 0.
\]

Equations (3) may immediately assume the Dirac-type form (2)

\[
(W + \mathbf{\alpha} \cdot \mathbf{p}) \mathbf{\Phi} = 0
\]

with the above condition, provided that the three matrices (2)

\[
\alpha_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

are introduced satisfying the angular-momentum commutation rules

\[
[\alpha_i, \alpha_k] = -i\epsilon_{ikl} \alpha_l (i, k, l = 1, 2, 3),
\]

the quantity \( \epsilon_{ikl} \) being the Ricci tensor normalized so that \( \epsilon_{123} = 1 \).

Thus, now \( \mathbf{\Phi} \) may be considered as a (quantum) wave function for the photon, satisfying a massless Dirac-like equation. By the way, if we consider only \( \mathbf{\Phi} \) eigenfunctions (**) of \( W \) and \( \mathbf{p} \), then the resolubility condition of the system of equations (4),

\[
\begin{vmatrix}
W & -ip_2 & ip_2 \\
ip_2 & W & -ip_1 \\
-ip_2 & ip_1 & W
\end{vmatrix} = 0,
\]

\( (*) \) We are using rationalized Gaussian units. We shall use also natural units \((c = 1, \ h = 1)\) when convenient.

\( (**) \) Essentially, plane waves in the energy-momentum space.
yields the correct energy-momentum relation for the photon \(^{(2)}\)

\[ W = \pm |p| . \]

At this point, let us observe that the three eqs. \((4)\) may be rewritten as a Schrödinger equation with the Hamiltonian

\[ \mathcal{H} = -\alpha \cdot p . \]

Then, by following the standard procedure, \textit{i.e.} by evaluating the commutator of \(\mathcal{H}\) with the orbital angular momentum \(L\)

\[ [\mathcal{H}, L]_- = -\alpha \wedge p , \]

the total angular momentum \(J\) follows to be

\[ J = L + \Sigma , \]

the quantity \(\Sigma\) being the intrinsic (spin) angular-momentum for the photon

\[ \Sigma = -i\alpha \wedge \alpha . \]

Equation \((9)\) is quite analogous to the expression \(\Sigma_n = -(i/2)(\alpha_n \wedge \alpha_n)/2\) of the spin operator for the Dirac electron, except for the factor \(\frac{1}{2}\) transformed into the factor 1 as expected. As expected, the operator \((9)\) has eigenvalues 0, \(\pm 1\). Moreover, with definition \((9)\), our momentum eigenfunctions \(\psi\) happen to be eigenfunctions also of the photon helicity operator \(s:\)

\[ s = \Sigma \cdot \hat{p} , \]

where, as usual, \(\hat{p} = p/|p|\).

Afterwards, it is worthwhile mentioning that, by introducing the quantities complex conjugate of eqs. \((1)\)

\[ \psi_i^* = E_i + i\Pi_i \quad \quad [i = 1, 2, 3] , \]

we can write the equations complex conjugate of eqs. \((4)\)

\[ (W + \alpha \cdot p)\psi^* = 0 . \]

Since eqs. \((4')\) appear as identical to eqs. \((4)\), we immediately conclude that photons coincide with antiphotons \((^7)\). Let us observe that eqs. \((4)\) may be also written

\[ \overline{\psi}(W + \alpha \cdot p) = 0 , \]

which are analogous in form to the corresponding equations for the Dirac-adjoint wave function (apart, of course, for a sign!). In eqs. \((4'')\) we understand that \(p = -i\partial/\partial x\),
and that $\overline{\Psi}$ is the Hermitian conjugate of $\Psi$

$$(1') \quad \overline{\Psi} = (\Psi^*)^T.$$ 

Let us explicitly mention that, by means of the formalism (1) and (2), one can reproduce the Maxwellian (classical) electromagnetism, even when charges are present (2) (e.g., one can deal with the problem of wave radiation (2)).

Here we want only to go back to the quantum treatment, and to show, e.g., how quantities as the energy-momentum tensor read in the new formalism (2), i.e. by using eqs. (1), (1'). Namely, by defining (2)

$$2x_{00} = 1, \quad 2x_{01} = x_1, \quad 2x_{02} = x_2, \quad 2x_{03} = x_3,$$

$$(11) \quad \begin{cases} 2x_{11} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & 2x_{12} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & 2x_{13} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \\ 2x_{22} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & 2x_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, & 2x_{33} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} \end{cases}$$

with, moreover (1),

$$x_{ik} = x_{ki} \quad (i, k = 0, 1, 2, 3),$$

we can write the (symmetric) energy-momentum tensor $T_{ik} = T_{ki}$ as follows (2):

$$T_{ik} = \overline{\Psi} x_{ik} \Psi. \quad (12)$$

Further notes on analogous subjects will possibly appear in the future.

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The idea for this work came to us from the manuscripts left by E. MAJORANA (who somewhat mysteriously disappeared in 1938, at thirtytwo), within a project for extracting from his manuscripts some hints interesting not only for history but, before all hopefully, for present-time physics. The authors are grateful to Profs. A. AGODI, E. AMALDI, C. CASTAGNOLI, E. FERRARI, I. F. QUERCIA, A. ZICHICHI for their interest, and to Profs. T. DERENZINI and C. MACCAGNI, Dr. M. TRICARICO and Mr A. GUERRI for the kind hospitality received at the "Domus Galilaeana", Pisa.

(*) Notice, by the way, that the nine matrices $x_{ik}$ (besides the identity $2x_{00}$) are nothing but a basis for the fundamental representation of $SU_3$. In fact, our $\Psi$ is a vector of a complex three-dimensional space.