

Chapter 2

Introduction of a Quantum of Time (“chronon”), and its Consequences for the Electron in Quantum and Classical Physics [☆]

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1. INTRODUCTION

In this paper we discuss the consequences of the introduction of a quantum of time τ_0 in the formalism of non-relativistic quantum mechanics, by referring ourselves, in particular, to the theory of the *chronon* as proposed by P. Caldirola. Such an interesting "finite difference" theory, forwards — at the classical level — a selfconsistent solution for the motion in an external electromagnetic field of a charged particle like an electron, when its charge cannot be regarded as negligible, overcoming all the known difficulties met by Abraham–Lorentz's and Dirac's approaches (and even allowing a clear answer to the question whether a free falling electron does or does not emit radiation), and — at the quantum level — yields a remarkable mass spectrum for leptons.

After having briefly reviewed Caldirola's approach, our first aim will be to work out, discuss, and compare to one another the *new* formulations of Quantum Mechanics (QM) resulting from it, in the Schrödinger, Heisenberg and density–operator (Liouville–von Neumann) pictures, respectively.

Moreover, for each picture, we show that three (*retarded*, *symmetric* and *advanced*) formulations are possible, which refer either to times t and $t-\tau_0$, or to times $t-\tau_0/2$ and $t+\tau_0/2$, or to times t and $t+\tau_0$, respectively. We shall see that, when the chronon tends to zero, the ordinary QM is obtained as the limiting case of the "symmetric" formulation only; while the "retarded" one does naturally appear to describe QM with friction, i.e., to describe *dissipative* quantum systems (like a particle moving in an absorbing medium). In this sense, *discretized* QM is much richer than the ordinary one.

We are also going to obtain the (retarded) finite–difference Schrödinger equation within the Feynman path integral approach, and study some of its relevant solutions. We then derive the time–evolution operators of this discrete theory, and use them to get the finite–difference Heisenberg equations.

When discussing the mutual compatibility of the various pictures listed above, we find that they *can* be written down in a form such that they result to be equivalent (as it happens in the “continuous” case of ordinary QM), even if our Heisenberg picture cannot be derived by “discretizing” directly the ordinary Heisenberg representation.

Afterwards, some typical applications and examples are studied, as the free particle (electron), the harmonic oscillator and the hydrogen atom; and various cases are pointed out, for which the predictions of discrete QM differ from those expected from “continuous” QM.

At last, the density matrix formalism is applied for a possible solution of the *measurement problem* in QM, with interesting results, as for instance a natural explication of “decoherence”, which reveal the power of discretized (in particular, *retarded*) QM.

The idea of a discrete temporal evolution is not a new one and, as with almost all physical ideas, has from time to time been recovered from oblivion.¹ For instance, in classical Greece this idea came to light as part of the atomistic thought. In the Middle Ages, belief in the discontinuous character of time was at the basis of the “theistic atomism” held by the Arabic thinkers of the Kalām (Jammer, 1954). In Europe, discussions about the discreteness of space and time can be found in the writings of Isidore of Sevilla, Nicolaus Boneti and Henry of Harclay, investigating the nature of continuum. In more recent times, the idea of the existence of a fundamental interval of time was rejected by Leibniz, because it was incompatible with his rationalistic philosophy. Within modern physics, however, Planck’s famous work on black-body radiation inspired a new view of the subject. In fact, the introduction of the quanta opened a wide range of new scientific possibilities regarding how the physical world can be conceived, including considerations, like those in this chapter, on the discretization of time within the framework of quantum mechanics.

In the early years of the twentieth century, Mach regarded the concept of *continuum* as a consequence of our physiological limitations: “... le temps et l’espace ne représentent, au point de vue physiologique, qu’un continue apparent, qu’ils se composent très vraisemblablement

¹ Historical aspects related to the introduction of a fundamental interval of time in physics can be found in Casagrande (1977).

d'elements discontinus, mais qu'on ne peut distinguer nettement les uns des autres" (Arzeliès, 1966, p. 387). Also Poincaré (1913) took into consideration the possible existence of what he called an "atom of time": the minimum amount of time that allows distinguishing between two states of a system. Finally, in the 1920s, J. J. Thomson (1925–26) suggested that the electric force acts in a discontinuous way, producing finite increments of momentum separated by finite intervals of time. Such a seminal work has since inspired a series of papers on the existence of a fundamental interval of time, the *chronon*, although the overall repercussion of that work was small at that time. A further seminal article was written by Ambarzumian and Ivanenko (1930), which assumed a discrete nature for space-time and also stimulated many subsequent papers.

It is important to stress that, in principle, time discretization can be introduced in two distinct (and completely different) ways:

1. By attributing to time a discrete structure, that is, by regarding time not as a continuum, but as a one-dimensional "lattice".
2. By considering time as a continuum, in which events can take place (discontinuously) only at discrete instants of time.

Almost all attempts to introduce a discretization of time followed the first approach, generally as part of a more extended procedure in which space-time as a whole is considered intrinsically discrete (a four-dimensional lattice). Recently, Lee (1983) introduced a time discretization on the basis of the *finite* number of experimental measurements performable in any finite interval of time.² For an early approach in this direction, see Tati (1964) and references therein, such as Yukawa (1966) and Darling (1950). Similarly, formalizations of an intrinsically discrete physics have also been proposed (McGoveran and Noyes, 1989).

The second approach was first adopted in the 1920s (e.g., by Levi, 1926, and by Pokrowski, 1928) after Thomson's work, and resulted in the first real example of a theory based on the existence of a fundamental interval of time: the one set forth by Caldirola (1953, 1956) in the 1950s.³ Namely, Caldirola formulated a theory for the *classical* electron, with the aim of providing a consistent (classical) theory for its motion in an electromagnetic field. In the late 1970s, Caldirola (1976a) extended its procedure to nonrelativistic QM.

It is known that the classical theory of the electron in an electromagnetic field (despite the efforts by Abraham, 1902; Lorentz, 1892,1904; Poincaré,

² See also Lee (1987), Friedberg and Lee (1983), and Bracci *et al.* (1983).

³ Further developments of this theory can be found in Caldirola (1979a) and references therein. See also Caldirola (1979c, 1979d; 1984b) and Caldirola and Recami (1978), as well as Petzold and Sorg (1977), Sorg (1976), and Mo and Papas (1971).

1906; and Dirac, 1938a,b; as well as Einstein, 1915; Frenkel, 1926, 1926–28; Lattes *et al.*, 1947; and Ashauer, 1949, among others) actually presents many serious problems *except when the field of the particle is neglected*.⁴ By replacing Dirac's differential equation with two finite-difference equations, Caldirola developed a theory in which the main difficulties of Dirac's theory were overcome. As seen later, in Caldirola's relativistically invariant formalism the chronon characterizes the changes experienced by the dynamical state of the electron when submitted to external forces. The electron is regarded as an (extended-like) object, which is pointlike only at discrete positions x_n (along its trajectory) such that the electron takes a quantum of proper time to travel from one position to the following one (or, rather, two chronons; see the following). It is tempting to examine extensively the generalization of such a theory to the quantum domain, and this will be performed herein. Let us recall that one of the most interesting aspects of the discretized Schrödinger equations is that the mass of the muon and of the tau lepton follows as corresponding to the two levels of the first (degenerate) excited state of the electron.

In conventional QM there is a perfect equivalence among its various pictures: the ones from Schrödinger, Heisenberg's, and the density matrices formalism. When discretizing the evolution equations of these different formalisms, we succeed in writing them in a form such that they are still equivalent. However, to be compatible with the Schrödinger representation, our Heisenberg equations *cannot*, in general, be obtained by a *direct* discretization of the continuous Heisenberg equation.

This work is organized as follows. In Section 2 we present a brief review of the main classical theories of the electron, including Caldirola's. In Section 3 we introduce the three discretized forms (retarded, advanced, and symmetrical) of the Schrödinger equation, analyze the main characteristics of such formulations, and derive the retarded one from Feynman's path integral approach. In Section 4, our discrete theory is applied to some simple quantum systems, such as the harmonic oscillator, the free particle, and the hydrogen atom. The possible experimental deviations from the predictions of ordinary QM are investigated. In Section 5, a new derivation of the discretized Liouville-von Neumann equation, starting from the coarse-grained hypothesis, is presented. Such a representation is then adopted to tackle the measurement problem in QM, with rather interesting results. Finally, a discussion on the possible interpretation of our discretized equations is found in Section 6.

⁴ It is interesting to note that all those problems have been—necessarily—tackled by Yaghjian (1992) in his book when he faced the question of the relativistic motion of a charged, macroscopic sphere in an external electromagnetic field (see also Yaghjian, 1989, p. 322).

2. THE INTRODUCTION OF THE CHRONON IN THE CLASSICAL THEORY OF THE ELECTRON

Almost a century after its discovery, the electron continues to be an object still awaiting a convincing description, both in classical and quantum electrodynamics.⁵ As Schrödinger put it, the electron is *still* a stranger in electrodynamics. Maxwell's electromagnetism is a field theoretical approach in which no reference is made to the existence of material corpuscles. Thus, one may say that one of the most controversial questions of twentieth-century physics, the wave-particle paradox, is not characteristic of QM only. In the electron classical theory, matching the description of the electromagnetic fields (obeying Maxwell equations) with the existence of charge carriers like the electron is still a challenging task.

The hypothesis that electric currents could be associated with charge carriers was already present in the early "particle electrodynamics" formulated in 1846 by Fechner and Weber (Rohrlich, 1965, p. 9). But this idea was not taken into consideration again until a few decades later, in 1881, by Helmholtz. Till that time, electrodynamics had developed on the hypothesis of an electromagnetic *continuum* and of an *ether*.⁶ In that same year, Thomson (1881) wrote his seminal paper in which the electron mass was regarded as purely electromagnetic in nature. Namely, the energy and momentum associated with the (electromagnetic) fields produced by an electron were held entirely responsible for the energy and momentum of the electron itself (Belloni, 1981).

Lorentz's electrodynamics, which described the particle-particle interaction via electromagnetic fields by the famous force law

$$\mathbf{f} = \rho \left(\mathbf{E} + \frac{1}{c} \mathbf{v} \wedge \mathbf{B} \right) \quad (1)$$

where ρ is the charge density of the particle on which the fields act, dates back to the beginning of the 1890 decade. The electron was finally discovered by Thomson in 1897, and in the following years various theories appeared. The famous (prerelativistic) theories by Abraham, Lorentz, and Poincaré regarded it as an extended-type object, endowed again with a purely electromagnetic mass. As is well known, in 1902 Abraham proposed the simple (and questionable) model of a rigid sphere, with a uniform electric charge density on its surface. Lorentz's (1904) was quite similar and tried to improve the situation with the mere introduction of the effects resulting from the Lorentz-Fitzgerald contraction.

⁵ Compare, for example, the works by Recami and Salesi (1994, 1996, 1997a, 1997b, 1998a, 1998b) and references therein. See also Pavsic *et al.* (1993, 1995) and Rodrigues, Vaz, and Recami (1993).

⁶ For a modern discussion of a similar topic, see Likharev and Claeson (1992).

2.1. The Abraham–Lorentz’s Theory of the Electron

A major difficulty in accurately describing the electron motion was the inclusion of the radiation reaction (i.e., of the effect produced on such a motion by the fields radiated by the particle itself). In the model proposed by Abraham–Lorentz the assumption of a purely electromagnetic structure for the electron implied that

$$\mathbf{F}_p + \mathbf{F}_{\text{ext}} = \mathbf{0} \quad (2)$$

where \mathbf{F}_p is the self-force due to the self-fields of the particle, and \mathbf{F}_{ext} is the external force. According to Lorentz’s law, the self-force was given by

$$\mathbf{F}_p = \int \rho \left(\mathbf{E}_p + \frac{1}{c} \mathbf{v} \wedge \mathbf{B}_p \right) d^3\mathbf{r}$$

where \mathbf{E}_p and \mathbf{B}_p are the fields produced by the charge density ρ itself, according to the Maxwell-Lorentz equations. For the radiation reaction force, Lorentz obtained the following expression:

$$\mathbf{F}_p = -\frac{4}{3c^2} W_{\text{el}} \mathbf{a} + \frac{2k e^2}{3c^3} \dot{\mathbf{a}} - \frac{2e^2}{3c^3} \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \frac{1}{c^n} \frac{d^n \mathbf{a}}{dt^n} O(R^{n-1}), \quad (3)$$

where $k \equiv (4\pi\epsilon_0)^{-1}$ (in the following, whenever convenient, we shall assume units such that numerically $k = 1$), and where

$$W_{\text{el}} \equiv \frac{1}{2} \iint \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r} d^3\mathbf{r}'$$

is the electrostatic self-energy of the considered charge distribution, and R is the radius of the electron. All terms in the sum are structure dependent. They depend on R and on the charge distribution. By identifying the electromagnetic mass of the particle with its electrostatic self-energy

$$m_{\text{el}} = \frac{W_{\text{el}}}{c^2}$$

it was possible to write Eq. (2) as

$$\frac{4}{3} m_{\text{el}} \dot{\mathbf{v}} - \Gamma = \mathbf{F}_{\text{ext}} \quad (4)$$

so that

$$\Gamma = \frac{2e^2}{3c^3} \dot{\mathbf{a}} (1 + O(R)) \quad (5)$$

which was the equation of motion in the Abraham–Lorentz model. Quantity Γ is the radiation reaction force, the reaction force acting on the

electron. One problem with Eq. (4) was constituted by the factor $\frac{4}{3}$. In fact, if the mass is supposed to be of electromagnetic origin only, then the total momentum of the electron would be given by

$$\mathbf{p} = \frac{4}{3} \frac{W_{\text{el}}}{c^2} \mathbf{v} \quad (6)$$

which is *not* invariant under Lorentz transformations. That model, therefore, was nonrelativistic. Finally, we can observe from Eq. (3) that the structure-dependent terms are functions of higher derivatives of the acceleration. Moreover, the resulting differential equation is of the *third order*, so that initial position and initial velocity are not enough to single out a solution. To suppress the structure terms, the electron should be reducible to a point, ($R \rightarrow 0$), but in this case the self-energy W_{el} and mass m_{el} would diverge!

After the emergence of the special theory of relativity, or rather, after the publication by Lorentz in 1904 of his famous transformations, some attempts were made to adapt the model to the new requirements.⁷ Abraham himself (1905) succeeded in deriving the following generalization of the radiation reaction term [Eq. (5)]:

$$\Gamma_{\mu} = \frac{2e^2}{3c} \left(\frac{d^2 u_{\mu}}{ds^2} + \frac{u_{\mu} u^{\nu}}{c^2} \frac{d^2 u_{\nu}}{ds^2} \right). \quad (7)$$

A solution for the problem of the electron momentum noncovariance was proposed by Poincaré in 1905 by the addition of cohesive forces of nonelectromagnetic character. This, however, made the nature of the electron no longer purely electromagnetic.

On the other hand, electrons could not be considered pointlike because of the obvious divergence of their energy when $R \rightarrow 0$; thus, a description of the electron motion could not dismiss the structure terms. Only Fermi (1922) succeeded in showing that the correct relation for the momentum of a purely electromagnetic electron could be obtained without Poincaré's cohesive forces.

2.2. Dirac's Theory of the Classical Electron

Notwithstanding its inconsistencies, the Abraham–Lorentz's theory was the most accepted theory of the electron until the publication of Dirac's theory in 1938. During the long period between these two theories, as well as afterward, various further attempts to solve the problem were set forth,

⁷ See, for example, von Laue (1909), Schott (1912), Page (1918, 1921), and Page and Adams (1940).

either by means of extended-type models (Mie, Page, Schott and so on⁸), or by trying again to treat the electron as a pointlike particle (Fokker, Wentzel and so on).⁹

Dirac's approach (1938a) is the best-known attempt to describe the classical electron. It bypassed the critical problem of the previous theories of Abraham and Lorentz by devising a solution for the pointlike electron that *avoided* divergences. By using the conservation laws of energy and momentum and Maxwell equations, Dirac calculated the flux of the energy-momentum four-vector through a tube of radius $\varepsilon \ll R$ (quantity R being the radius of the electron at rest) surrounding the world line of the particle, and obtained

$$m \frac{du_\mu}{ds} = F_\mu + \Gamma_\mu \quad (8)$$

where Γ_μ is the Abraham four-vector [Eq. (7)], that is, the reaction force acting on the electron itself, and F_μ is the four-vector that represents the external field acting on the particle:

$$F_\mu = \frac{e}{c} F_{\mu\nu} u^\nu. \quad (9)$$

According to such a model, the rest mass m_0 of the electron is the limiting, finite value obtained as the difference of two quantities tending to infinity when $R \rightarrow 0$

$$m_0 = \lim_{\varepsilon \rightarrow 0} \left(\frac{1}{2} \frac{e^2}{c^2 \varepsilon} - k(\varepsilon) \right),$$

the procedure followed by Dirac was an early example of elimination of divergences by means of a subtractive method.

At the nonrelativistic limit, Dirac's equation tends to the one previously obtained by Abraham-Lorentz:

$$m_0 \frac{d\mathbf{v}}{dt} - \frac{2e^2}{3c^3} \frac{d^2\mathbf{v}}{dt^2} = e \left(\mathbf{E} + \frac{1}{c} \mathbf{v} \wedge \mathbf{B} \right) \quad (10)$$

⁸ There were several attempts to develop an extended-type model for the electron. See, for example, Compton (1919) and references therein; also Mie (1912), Page (1918), Schott (1912), Frenkel (1926), Schrödinger (1930), Mathisson (1937), Hönl and Papapetrou (1939, 1940), Bhabha and Corben (1941), Weyssenhof and Raabe (1947), Pryce (1948), Huang (1952), Hönl (1952), Proca (1954), Bunge (1955), Gursev (1957), Corben (1961, 1968, 1977, 1984, 1993), Fleming (1965), Liebowitz (1969), Gallardo *et al.* (1967), Kálnay (1970, 1971), Kálnay and Torres (1971), Jehle (1971), Riewe (1971), Mo and Papas (1971), Bonnor (1974), Marx (1975), Perkins (1976), Cvijanovich and Vigier (1977), Gutkowski *et al.* (1977), Barut (1978a), Lock (1979), Hsu and Mac (1979), Coleman (1960), McGregor (1992) and Rodrigues *et al.* (1993).

⁹ A historical overview of these different theories of electron can be found in Rohrlich (1965) and references therein and also Rohrlich (1960).

except that in the Abraham–Lorentz’s approach m_0 diverged. Equation (10) shows that the reaction force equals $\frac{2}{3} \frac{e^2}{c^3} \frac{d^2 \mathbf{v}}{dt^2}$.

Dirac’s dynamical equation [Eq. (8)] was later reobtained from different, improved models.¹⁰ Wheeler and Feynman (1945), for example, rederived Eq. (8) by basing electromagnetism on an action principle applied to particles only via their own absorber hypothesis. However, Eq. (8) also presents many problems, related to the many infinite nonphysical solutions that it possesses. Actually, as previously mentioned, it is a third-order differential equation, requiring three initial conditions for singling out one of its solutions. In the description of a *free* electron, for example, it even yields “self-accelerating” solutions (runaway solutions), for which velocity and acceleration increase spontaneously and indefinitely (see Eliezer, 1943; Zin, 1949; and Rohrlich, 1960, 1965). Selection rules have been established to distinguish between physical and nonphysical solutions (for example, Schenberg, 1945 and Bhabha, 1946). Moreover, for an electron submitted to an electromagnetic pulse, further nonphysical solutions appear, related this time to *pre-accelerations* (Ashauer, 1949). If the electron comes from infinity with a uniform velocity v_0 and at a certain instant of time t_0 is submitted to an electromagnetic pulse, then it starts accelerating *before* t_0 . Drawbacks such as these motivated further attempts to determine a coherent model for the classical electron.

2.3. Caldirola’s Theory for the Classical Electron

Among the various attempts to formulate a more satisfactory theory, we want to focus attention on the one proposed by Caldirola. Like Dirac’s, Caldirola’s theory is also Lorentz invariant. Continuity, in fact, is not an assumption required by Lorentz invariance (Snyder, 1947). The theory postulates the existence of a universal interval τ_0 of proper time, even if time flows continuously as in the ordinary theory. When an external force acts on the electron, however, the reaction of the particle to the applied force is not continuous: The value of the electron velocity u_μ should jump from $u_\mu(\tau - \tau_0)$ to $u_\mu(\tau)$ only at certain positions s_n along its world line; these discrete positions are such that the electron takes a time τ_0 to travel from one position s_{n-1} to the next s_n .

In this theory¹¹ the electron, in principle, is still considered pointlike but the Dirac relativistic equations for the classical radiating electron are replaced: (1) by a corresponding *finite-difference* (retarded) equation in the velocity $u^\mu(\tau)$

¹⁰ See Schenberg (1945), Havas (1948), and Loinger (1955).

¹¹ Caldirola presented his theory of electron in a series of papers in the 1950s, such as his 1953 and 1956 works. Further developments of his theory can be found in Caldirola (1979a) and references therein. See also Caldirola (1979c; 1979d; 1984b) and Caldirola and Recami (1978).

$$\frac{m_0}{\tau_0} \left\{ u_\mu(\tau) - u_\mu(\tau - \tau_0) + \frac{u_\mu(\tau)u_\nu(\tau)}{c^2} [u_\nu(\tau) - u_\nu(\tau - \tau_0)] \right\} = \frac{e}{c} F_{\mu\nu}(\tau)u_\nu(\tau), \quad (11)$$

which reduces to the Dirac equation [Eq. (8)] when $\tau_0 \rightarrow 0$, but cannot be derived from it (in the sense that it cannot be obtained by a simple discretization of the time derivatives appearing in Dirac's original equation); and: (2) by a second equation, this time connecting the "discrete positions" $x^\mu(\tau)$ along the world line of the particle; in fact, the dynamical law in Eq. (11) is by itself unable to specify univocally the variables $u_\mu(\tau)$ and $x_\mu(\tau)$, which describe the motion of the particle. Caldirola named it the *transmission law*:

$$x_\mu(n\tau_0) - x_\mu[(n-1)\tau_0] = \frac{\tau_0}{2} \{ u_\mu(n\tau_0) - u_\mu[(n-1)\tau_0] \}, \quad (12)$$

which is valid inside each discrete interval τ_0 , and describes the internal or microscopic motion of the electron.

In these equations, $u^\mu(\tau)$ is the ordinary four-vector velocity satisfying the condition

$$u_\mu(\tau)u^\mu(\tau) = -c^2 \quad \text{for } \tau = n\tau_0$$

where $n = 0, 1, 2, \dots$ and $\mu, \nu = 0, 1, 2, 3$; $F^{\mu\nu}$ is the external (retarded) electromagnetic field tensor, and the quantity

$$\frac{\tau_0}{2} \equiv \theta_0 = \frac{2}{3} \frac{ke^2}{m_0c^3} \simeq 6.266 \times 10^{-24} \text{s} \quad (13)$$

is defined as the chronon associated with the electron (as justified below). The chronon $\theta_0 = \tau_0/2$ depends on the particle (internal) properties, namely, on its charge e and rest mass m_0 .

As a result, the electron happens to appear eventually as an extended-like particle¹², with an internal structure, rather than as a pointlike object (as initially assumed). For instance, one may imagine that the particle does not react instantaneously to the action of an external force because of its finite extension (the numerical value of the chronon is of the same order as the time spent by light to travel along an electron classical diameter). As noted, Eq. (11) describes the motion of an object that happens to be pointlike only at discrete positions s_n along its trajectory,

¹² See, for example, Salesi and Recami (1995, 1996, 1997a,b, 1998). See also the part on the field theory of leptons in Recami and Salesi (1995) and on the field theory of the extended-like electron in Salesi and Recami (1994, 1996).

Caldirola, 1956, 1979a, even if both position and velocity are still continuous and well-behaved functions of the parameter τ , since they are differentiable functions of τ .

It is essential to notice that a discrete character is assigned to the electron merely by the introduction of the fundamental quantum of time, with no need of a “model” for the electron. As is well known, many difficulties are encountered with both the strictly pointlike models and the extended-type particle models (spheres, tops, gyroscopes, and so on). In Barut’s words (1991), “If a spinning particle is not quite a point particle, nor a solid three dimensional top, what can it be?” We deem the answer lies in a third type of model, the “extended-like” one, as the present theory; or as the (related) theoretical approach in which the center of the pointlike charge is spatially distinct from the particle center of mass (see Salesi and Recami, 1994, and ensuing papers on this topic, like Recami and Salesi, 1997a,b, 1998a, and Salesi and Recami, 1997b). In any case, it is not necessary to recall that the worst troubles in quantum field theory (e.g., in quantum electrodynamics), like the presence of divergencies, are due to the pointlike character still attributed to (spinning) particles, since the problem of a suitable model for elementary particles was transported, without a suitable solution, from classical to quantum physics. In our view that particular problem may still be the most important in modern particle physics.

Equations (11) and (12) provide a full description of the motion of the electron. Notice that the global “macroscopic” motion can be the same for different solutions of the transmission law. The behavior of the electron under the action of external electromagnetic fields is completely described by its macroscopic motion.

As in Dirac’s case, the equations are invariant under Lorentz transformations. However, as we shall see, they are *free* of pre-accelerations, self-accelerating solutions, and the problems with the hyperbolic motion that had raised great debates in the first half of the twentieth century.

In the *nonrelativistic limit* the previous (retarded) equations reduces to the form

$$\frac{m_0}{\tau_0} [\mathbf{v}(t) - \mathbf{v}(t - \tau_0)] = e \left[\mathbf{E}(t) + \frac{1}{c} \mathbf{v}(t) \wedge \mathbf{B}(t) \right], \quad (14)$$

$$\mathbf{r}(t) - \mathbf{r}(t - \tau_0) = \frac{\tau_0}{2} [\mathbf{v}(t) - \mathbf{v}(t - \tau_0)], \quad (15)$$

which *can* be obtained, this time, from Eq. (10) by directly replacing the time derivatives by the corresponding finite-difference expressions. The macroscopic Eq. (14) had already been obtained by other authors for the dynamics of extended-type electrons¹³.

¹³ Compare, for example, Schott (1912), Page (1918), Page and Adams (1940), Bohm and Weinstein (1948), and Eliezer (1950).

The important point is that Eqs. (11) and (12), or (14) and (15), allow difficulties met with the Dirac classical Eq. (8) to be overcome. In fact, the electron *macroscopic* motion is completely determined once velocity and initial position are given. Solutions of the relativistic Eqs. (11) and (12) for the radiating electron—or of the corresponding non-relativistic Eqs. (14) and (15)—were obtained for several problems. The resulting motions never presented unphysical behavior, so the following questions can be regarded as solved [Caldirola, 1956, 1979a](#):

- Exact relativistic solutions:
 - Free electron motion
 - Electron under the action of an electromagnetic pulse ([Cirelli, 1955](#))
 - Hyperbolic motion ([Lanz, 1962](#))
- Non-relativistic approximate solutions:
 - Electron under the action of time-dependent forces
 - Electron in a constant, uniform magnetic field ([Prosperetti, 1980](#))
 - Electron moving along a straight line under the action of an elastic restoring force ([Caldirola et al., 1978](#))

Before we proceed, it is interesting to briefly analyze the electron radiation properties as deduced from the finite-difference relativistic Eqs. (11) and (12) to show the advantages of the present formalism with respect to the Abraham–Lorentz–Dirac one. Such equations can be written ([Lanz, 1962; Caldirola, 1979a](#)) as

$$\frac{\Delta Q_\mu(\tau)}{\tau_0} + R_\mu(\tau) + S_\mu(\tau) = \frac{e}{c} F_{\mu\nu}(\tau) u^\nu(\tau), \quad (16)$$

where

$$\Delta Q_\mu \equiv m_0 [u_\mu(\tau) - u_\mu(\tau - \tau_0)] \quad (17)$$

$$R_\mu(\tau) \equiv -\frac{m_0}{2\tau_0} \left\{ \frac{u_\mu(\tau) u_\nu(\tau)}{c^2} [u^\nu(\tau + \tau_0) + u^\nu(\tau - \tau_0) - 2u^\nu(\tau)] \right\} \quad (18)$$

$$S_\mu(\tau) = -\frac{m_0}{2\tau_0} \left\{ \frac{u_\mu(\tau) u_\nu(\tau)}{c^2} [u^\nu(\tau + \tau_0) - u^\nu(\tau - \tau_0)] \right\}. \quad (19)$$

In Eq. (16), the first term $\Delta Q_0/\tau_0$ represents the variation per unit of proper time (in the interval $\tau - \tau_0$ to τ) of the particle energy-momentum vector. The second one, $R_\mu(\tau)$, is a dissipative term because it contains only even derivatives of the velocity as can be proved by expanding $u^\nu(\tau + \tau_0)$ and $u^\nu(\tau - \tau_0)$ in terms of τ_0 ; furthermore, it is never negative [Caldirola, 1979a; Lanz, 1962](#) and can therefore represent the energy-momentum *radiated* by the electron in the unit of proper time. The third term, $S_\mu(\tau)$, is conservative and represents the rate of change in proper time of the electron *reaction* energy-momentum.

The time component ($\mu = 0$) of Eq. (16) is written as

$$\frac{T(\tau) - T(\tau - \tau_0)}{\tau_0} + R_0(\tau) + S_0(\tau) = P^{\text{ext}}(\tau), \quad (20)$$

where quantity $T(\tau)$ is the kinetic energy

$$T(\tau) = m_0 c^2 \left(\frac{1}{\sqrt{1 - \beta^2}} - 1 \right) \quad (21)$$

so that in Eq. (20) the first term replaces the proper-time derivative of the kinetic energy, the second one is the energy radiated by the electron in the unit of proper time, $S_0(\tau)$ is the variation rate in proper time of the electron reaction energy (radiative correction), and $P^{\text{ext}}(\tau)$ is the work done by the external forces in the unit of proper time.

We are now ready to show that Eq. (20) yields a clear explanation for the origin of the so-called acceleration energy (Schott energy), appearing in the energy-conservation relation for the Dirac equation. In fact, expanding in power series with respect to τ_0 the left-hand sides of Eqs.(16–19) for $\mu = 0$, and keeping only the first-order terms, yields

$$\frac{T(\tau) - T(\tau - \tau_0)}{\tau_0} \simeq \frac{dT}{d\tau} - \frac{2e^2}{3c^2} \frac{da_0}{d\tau} \quad (22)$$

$$R_0(\tau) \simeq \frac{1}{\sqrt{1 - \beta^2}} \frac{2e^2}{3c^3} a_\mu a^\mu \quad (23)$$

$$S_0(\tau) \simeq 0 \quad (24)$$

where a^μ is the four-acceleration

$$a^\mu \equiv \frac{du^\mu}{d\tau} = \gamma \frac{du^\mu}{dt}$$

quantity γ being the Lorentz factor. Therefore, Eq. (20) to the first order in τ_0 becomes

$$\frac{dT}{d\tau} - \frac{2e^2}{3c^2} \frac{da_0}{d\tau} + \frac{2e^2}{3c^3} \frac{a_\mu a^\mu}{\sqrt{1 - \beta^2}} \simeq P^{\text{ext}}(\tau), \quad (25)$$

or, passing from the proper time τ to the observer's time t :

$$\frac{dT}{dt} - \frac{2e^2}{3c^2} \frac{da_0}{dt} + \frac{2e^2}{3c} a_\mu a^\mu \simeq P^{\text{ext}}(\tau) \frac{d\tau}{dt}. \quad (26)$$

The last relation is identical with the energy-conservation law found by [Fulton and Rohrlich \(1960\)](#) for the Dirac equation. In [Eq. \(26\)](#) the derivative of $(2e^2/3c^2)a_0$ appears, which is simply the acceleration energy. Our approach clearly shows that it arises only by expanding in a power series of τ_0 the kinetic energy increment suffered by the electron during the fundamental proper-time interval τ_0 , while such a Schott energy (as well as higher-order energy terms) does not need show up explicitly when adopting the full formalism of finite-difference equations. We return to this important point in [subsection 2.4](#).

Let us finally observe ([Caldirola, 1979a](#), and references therein) that, when setting

$$\frac{m_0}{ec\tau_0} [u_\mu(\tau)u_\nu(\tau - \tau_0) - u_\mu(\tau - \tau_0)u_\nu(\tau)] \equiv F_{\mu\nu}^{\text{self}}, \quad (27)$$

the relativistic equation of motion [[Eq. \(11\)](#)] becomes

$$\frac{e}{c} \left(F_{\mu\nu}^{\text{self}} + F_{\mu\nu}^{\text{ext}} \right) u^\nu = 0, \quad (28)$$

confirming that $F_{\mu\nu}^{\text{self}}$ represents the (retarded) self-field associated with the moving electron.

2.4. The Three Alternative Formulations of Caldirola's Theory

Two more (alternative) formulations are possible with Caldirola's equations, based on different discretization procedures. In fact, [Eqs. \(11\)](#) and [\(12\)](#) describe an intrinsically radiating particle. And, by expanding [Eq. \(11\)](#) in terms of τ_0 , a radiation reaction term appears. Caldirola called those equations the *retarded* form of the electron equations of motion.

By rewriting the finite-difference equations, on the contrary, in the form

$$\frac{m_0}{\tau_0} \left\{ u_\mu(\tau + \tau_0) - u_\mu(\tau) + \frac{u_\mu(\tau)u_\nu(\tau)}{c^2} [u_\nu(\tau + \tau_0) - u_\nu(\tau)] \right\} = \frac{e}{c} F_{\mu\nu}(\tau)u_\nu(\tau), \quad (29)$$

$$x_\mu[(n+1)\tau_0] - x_\mu(n\tau_0) = \tau_0 u_\mu(n\tau_0), \quad (30)$$

one gets the *advanced* formulation of the electron theory, since the motion—according to [eqs. \(29\)](#) and [\(30\)](#)—is now determined by advanced actions. In contrast with the retarded formulation, the advanced one describes an electron that absorbs energy from the external world.

Finally, by adding the retarded and advanced actions, Caldirola derived the *symmetric* formulation of the electron theory:

$$\frac{m_0}{2\tau_0} \left\{ u_\mu(\tau + \tau_0) - u_\mu(\tau - \tau_0) + \frac{u_\mu(\tau)u_\nu(\tau)}{c^2} [u_\nu(\tau + \tau_0) - u_\nu(\tau - \tau_0)] \right\} \\ = \frac{e}{c} F_{\mu\nu}(\tau)u_\nu(\tau), \quad (31)$$

$$x_\mu[(n + 1)\tau_0] - x_\mu((n - 1)\tau_0) = 2\tau_0 u_\mu(n\tau_0), \quad (32)$$

which does not include any radiation reaction terms and describes a nonradiating electron.

Before closing this brief introduction to Caldirola's theory, it is worthwhile to present two more relevant results derived from it. The second one is described in the next subsection. If we consider a free particle and look for the "internal solutions" of the Eq. (15), we then get—for a periodical solution of the type

$$\dot{x} = -\beta_0 c \sin\left(\frac{2\pi\tau}{\tau_0}\right) \\ \dot{y} = -\beta_0 c \cos\left(\frac{2\pi\tau}{\tau_0}\right) \\ \dot{z} = 0$$

which describes a uniform circular motion, and by imposing the kinetic energy of the internal rotational motion to equal the intrinsic energy m_0c^2 of the particle—that the amplitude of the oscillations is given by $\beta_0^2 = \frac{3}{4}$. Thus, the magnetic moment corresponding to this motion is exactly the *anomalous magnetic moment* of the electron, obtained here in a purely classical context (Caldirola, 1954):

$$\mu_a = \frac{1}{4\pi} \frac{e^3}{m_0c^2}.$$

This shows that the anomalous magnetic moment is an intrinsically classical, and not quantum, result; and the absence of \hbar in the last expression is a confirmation of this fact.

2.5. Hyperbolic Motions

In a review paper on the theories of electron including radiation-reaction effects, Erber (1961) criticized Caldirola's theory for its results in the case of hyperbolic motion.

Let us recall that the opinion of Pauli and von Laue (among others) was that a charge performing uniformly accelerated motions—for example, an electron in free fall—could not emit radiation (Fulton and Rohrlich, 1960). That opinion was strengthened by the invariance of Maxwell equations under the group of conformal transformations (Cunningham, 1909; Bateman, 1910; Hill, 1945), which in particular includes transformations from rest to uniformly accelerated motions. However, since the first decades of the twentieth century, this had been—however—an open question, as the works by Born and Schott had on the contrary suggested a radiation emission in such a case (Fulton and Rohrlich, 1960). In 1960, Fulton and Rohrlich, using Dirac's equation for the classical electron, demonstrated that the electron actually emits radiation when performing a hyperbolic motion (see also Leiter, 1970).

A solution of this paradox is possible within Caldirola's theory, and it was derived by Lanz (1962). By analyzing the energy-conservation law for an electron submitted to an external force and following a procedure similar to that of Fulton and Rohrlich (1960), Lanz obtained Eq. (20). By expanding it in terms of τ and keeping only the first-order terms, he arrived at Eq. (25), identical to the one obtained by Fulton and Rohrlich, in which (we repeat) the Schott energy appears. A term that Fulton and Rohrlich (having obtained it from Dirac's expression for the radiation reaction) interpreted as a part of the internal energy of the charged particle.

For the particular case of hyperbolic motion, it is

$$a_\mu a^\mu = \frac{da_0}{d\tau}$$

so that there is *no* radiation reaction [compare with Eq. (25) or (26)]. However, neither the acceleration energy, nor the energy radiated by the charge per unit of proper time, $\frac{2}{3}e^2 a_\mu a^\mu$, is zero.

The difference is that in the discrete case this acceleration energy does not exist as such. It comes from the discretized expression for the charged particle kinetic energy variation. As seen in Eq. (22), the Schott term appears when the variation of the kinetic energy during the fundamental interval of proper time is expanded in powers of τ_0 :

$$\frac{T(\tau) - T(\tau - \tau_0)}{\tau_0} \simeq \frac{d}{d\tau} T - \frac{2e^2}{3c^2} \frac{d}{d\tau} a_0.$$

This is an interesting result, since it was not easy to understand the physical meaning of the Schott acceleration energy. With the introduction of the fundamental interval of time, as we know, the changes in the kinetic energy are no longer continuous, and the Schott term merely expresses, to first order, the variation of the kinetic energy when passing from one discrete instant of time to the subsequent one.

In Eqs. (22) and (25), the derivative $dT/d\tau$ is a point function, forwarding the kinetic energy slope *at* the instant τ . And the dissipative term $\frac{2}{3}e^2a_\mu a^\mu$ is simply a relativistic generalization of the Larmor radiation law: if there is acceleration, then there is also radiation emission.

For the hyperbolic motion, however, the energy dissipated (because of the acceleration) has only the same magnitude as the energy gain due to the kinetic energy increase. We are not forced to resort to pre-accelerations to justify the origin of such energies (Plass, 1960, 1961). Thus, the present theory provides a clear picture of the physical processes involved in the uniformly accelerated motion of a charged particle.

3. THE HYPOTHESIS OF THE CHRONON IN QUANTUM MECHANICS

Let us now address the main topic of this chapter: the chronon in quantum mechanics. The speculations about the discreteness of time (on the basis of possible physical evidences) in QM go back to the first decades of the twentieth century, and various theories have proposed developing QM on a space-time lattice.¹⁴ This is not the case with the hypothesis of the chronon, where we do not actually have a discretization of the time coordinate. In the 1920s, for example, Pokrowski (1928) suggested the introduction of a fundamental interval of time, starting from an analysis of the shortest wavelengths detected (at that time) in cosmic radiation. More recently, for instance, Ehrlich (1976) proposed a quantization of the elementary particle lifetimes, suggesting the value 4.4×10^{-24} s for the *quantum* of time.¹⁵ However, a time discretization is suggested by the very foundations of QM. There are physical limits that prevent the distinction of arbitrarily close successive states in the time evolution of a quantum system. Basically, such limitations result from the Heisenberg relations such that, if a discretization is introduced in the description of a quantum system, it cannot possess a *universal* value, since those limitations depend on the characteristics of the particular system under consideration. In other words, the value of the fundamental interval of time must change *a priori* from system to system. All these points make the extension of Caldirola's procedure to QM justifiable.

In the 1970s, Caldirola (1976a,b, 1977a,b,c, 1978a) extended the introduction of the chronon to QM, following the same guidelines that had led him to his theory of the electron. So, time is still a continuous variable, but the evolution of the system along its world line is discontinuous. As for

¹⁴ See, for example, Cole (1970) and Welch (1976); also compare with Jackson (1977), Meessen (1970), Vasholz (1975) and Kitazoe *et al.* (1978).

¹⁵ See also Golberger and Watson (1962), Froissart *et al.* (1963), DerSarkissian and Nelson (1969), Cheon (1979), and Ford (1968).

the electron theory in the nonrelativistic limit, one must substitute the corresponding finite-difference expression for the time derivatives; for example

$$\frac{df(t)}{dt} \rightarrow \frac{f(t) - f(t - \Delta t)}{\Delta t} \quad (33)$$

where proper time is now replaced by the local time t . Such a procedure was then applied to obtain the finite-difference form of the Schrödinger equation. As for the electron case, there are three different ways to perform the discretization, and three “Schrödinger equations” can be obtained (Caldirola and Montaldi, 1979):

$$i \frac{\hbar}{\tau} [\Psi(\mathbf{x}, t) - \Psi(\mathbf{x}, t - \tau)] = \hat{H}\Psi(\mathbf{x}, t), \quad (34)$$

$$i \frac{\hbar}{2\tau} [\Psi(\mathbf{x}, t + \tau) - \Psi(\mathbf{x}, t - \tau)] = \hat{H}\Psi(\mathbf{x}, t), \quad (35)$$

$$i \frac{\hbar}{\tau} [\Psi(\mathbf{x}, t + \tau) - \Psi(\mathbf{x}, t)] = \hat{H}\Psi(\mathbf{x}, t), \quad (36)$$

which are, respectively, the *retarded*, *symmetric*, and *advanced* Schrödinger equations, all of them transforming into the (same) continuous equation when the fundamental interval of time (which can now be called just τ) goes to zero. It can be immediately observed that the symmetric equation is of the second order, while the other two are first-order equations. As in the continuous case, for a finite-difference equation of order n a single and complete solution requires n initial conditions to be specified.

The equations are different, and the solutions they provide are also fundamentally different. There are two basic procedures to study the properties of such equations. For some special cases, they can be solved by one of the various existing methods for solving finite-difference equations or by means of an attempt solution, an *ansatz*. The other method is to find a new Hamiltonian \tilde{H} such that the new continuous Schrödinger equation,

$$i\hbar \frac{\partial \Psi(\mathbf{x}, t)}{\partial t} = \tilde{H}\Psi(\mathbf{x}, t), \quad (37)$$

reproduces, at the points $t = n\tau$, the same results obtained from the discretized equations. As shown by Casagrande and Montaldi (1977; 1978), it is always possible to find a continuous generating function that makes it possible to obtain a differential equation equivalent to the original finite-difference one, such that at every point of interest their solutions are identical. This procedure is useful because it is generally difficult to work with the finite-difference equations on a qualitative basis. Except for some special cases, they can be solved only numerically. This equivalent

Hamiltonian \tilde{H} is, however, non-hermitean and is frequently very difficult to obtain. When the Hamiltonian is time-independent, the equivalent Hamiltonian is quite easy to calculate. For the symmetric equation, for example, it is given by

$$\tilde{H} = \frac{\hbar}{\tau} \sin^{-1} \left(\frac{\tau}{\hbar} \hat{H} \right) \quad (38)$$

As expected, $\tilde{H} \rightarrow \hat{H}$ when $\tau \rightarrow 0$. One can use the symmetric equation to describe the nonradiating electron (bound electron) since for Hamiltonians explicitly independent of time its solutions are always of oscillating character:

$$\Psi(\mathbf{x}, t) = \exp \left(-i \frac{t}{\tau} \sin^{-1} \left(\frac{\tau}{t} \hat{H} \right) \right) f(\mathbf{x}).$$

In the classical theory of electrons, the symmetric equation also represents a nonradiating motion. It provides only an approximate description of the motion without considering the effects due to the self-fields of the electron. However, in the quantum theory it plays a fundamental role. In the discrete formalism, it is the only way to describe a bound nonradiating particle.

The solutions of the advanced and retarded equations show completely different behavior. For a Hamiltonian explicitly independent of time the solutions have a general form given by

$$\Psi(\mathbf{x}, t) = \left[1 + i \frac{\tau}{\hbar} \hat{H} \right]^{-t/\tau} f(\mathbf{x}),$$

and, expanding $f(\mathbf{x})$ in terms of the eigenfunctions of \hat{H} ,

$$\begin{aligned} \hat{H}u_n(\mathbf{x}) &= W_n u_n(\mathbf{x}) \\ f(\mathbf{x}) &= \sum_n c_n u_n(\mathbf{x}) \end{aligned}$$

with

$$\sum_n |c_n|^2 = 1,$$

it can be obtained that

$$\Psi(\mathbf{x}, t) = \sum_n c_n \left[1 + i \frac{\tau}{\hbar} W_n \right]^{-t/\tau} u_n(\mathbf{x}).$$

In particular, the norm of this solution is given by

$$|\Psi(\mathbf{x}, t)|^2 = \sum_n |c_n|^2 \exp(-\gamma_n t)$$

with

$$\gamma_n = \frac{1}{\tau} \ln \left(1 + \frac{\tau^2}{\hbar^2} W_n^2 \right) = \frac{W_n^2}{\hbar^2} \tau + O(\tau^3).$$

The presence of a damping factor, depending critically on the value τ of the chronon, must be noted.

This *dissipative* behavior originates from the retarded character of the equation. The analogy with the electron theory also holds, and the retarded equation possesses intrinsically dissipative solutions representing a radiating system. The Hamiltonian has the same status as in the continuous case. It is an observable since it is a Hermitean operator and its eigenvectors form a basis of the state space. However, due to the damping term, the norm of the state vector is no longer constant. An opposite behavior is observed for the solutions of the advanced equation in the sense that they increase exponentially.

Before proceeding, let us mention that the discretized QM (as well as Caldirola and coworkers' approach to "QM with friction" as, for example, in Caldirola and Montaldi (1979)) can find room within the theories based on the so-called Lie-admissible algebras (Santilli, 1979a,b, 1981a,b,c, 1983).¹⁶ For a different approach to decaying states see Agodi *et al.* (1973) and Recami and Farias (2009).

3.1. The Mass of the Muon

The most impressive achievement related to the introduction of the chronon hypothesis in the realm of QM comes from the description of a bound electron using the new formalism. Bound states are described by the symmetric Schrödinger equation and a Hamiltonian that does not depend explicitly on time. A general solution can be obtained by using a convenient *ansatz*:

$$\Psi(\mathbf{x}, t) = \sum_n u_n(\mathbf{x}) \exp(-i\alpha_n t),$$

where $\hat{H} u_n(\mathbf{x}) = E_n u_n(\mathbf{x})$ gives the spectrum of eigenvalues of the Hamiltonian. If the fundamental interval of time τ corresponds to the chronon θ_0 associated with the classical electron, it can be straightforwardly obtained that

$$\alpha_n = \frac{1}{\theta_0} \sin^{-1} \left(\frac{E_n \theta_0}{\hbar} \right).$$

¹⁶ Extensive related work (not covered in the present paper) can also be found in Jannussis (1985a,b, 1990, 1984a), Jannussis *et al.* (1990; 1983a; 1983b) and Mignani (1983); see also Jannussis *et al.* (1982a; 1982b; 1981a; 1981b; 1980a; 1980b), Jannussis (1984b,c), and Montaldi and Zanon (1980).

This solution gives rise to an upper limit for the eigenvalues of the Hamiltonian due to the condition

$$\left| \frac{E_n \theta_0}{\hbar} \right| \leq 1.$$

Since θ_0 is finite, there is a maximum value for the energy of the electron given by

$$E_{\max} = \frac{\hbar}{\theta_0} = \frac{2 \hbar m_0 c^3}{3 e^2} \approx 105.04 \text{ MeV}.$$

Now, including the rest energy of the electron, we finally get

$$E = E_{\max} + E_0^{\text{electron}} \approx 105.55 \text{ MeV},$$

which is *very* close (an error of 0.1%) to the measured value of the rest mass of the *muon*. The equivalent Hamiltonian method allows extending the basis of eigenstates beyond the critical limit. However, for the eigenvalues above the critical limit, the corresponding eigenstates are unstable and decay in time:

$$\Psi(\mathbf{x}, t) = \sum_n c_n u_n(\mathbf{x}) \exp(-i\gamma_n t) \exp(-k_n t),$$

As for the retarded equation, the norm of the state vector is not constant and decays exponentially with time for those eigenstates outside the stability range. [Caldirola \(1976a,b, 1977c\)](#) interpreted this norm as indicating the probability of the existence of the particle in its original Hilbert space, and associated a mean lifetime with these states.

The considerations regarding the muon as an excited state of the electron can be traced back to the days of its discovery. Particularly, it has already been observed that the ratio between the masses of the two particles is almost exactly $3/(2\alpha)$, where α is the fine structure constant ([Nambu, 1952](#)). It has already been noted also that $\frac{2}{3}\alpha$ is the coefficient of the radiative reaction term in Dirac's equation for the classical electron ([Rosen, 1964, 1978](#)). [Bohm and Weinstein \(1948\)](#) put forward the hypothesis that various kinds of "mesons" could be excited states of the electron. [Dirac \(1962\)](#) even proposed a specific model for an extended electron to interpret the muon as an excited state of the electron.¹⁷

[Caldirola \(1978a; 1977a; 1977b; see also Fryberger, 1981\)](#) observed that by means of the Heisenberg uncertainty relations it is possible to associate

¹⁷ On this point, also compare the following references: [Barut \(1978a,b\)](#), [Motz \(1970\)](#), [Ouchi and Ohmae \(1977\)](#), [Nishijima and Sato \(1978\)](#), [Sachs \(1972a,b\)](#), [Pavsic \(1976\)](#), [Matore \(1981\)](#), [Sudarshan \(1961\)](#) and [Kitazoe \(1972\)](#).

the existence of the muon as an excited state of the electron with the introduction of the chronon in the theory of electron. The relation

$$\Delta\tau \Delta E \geq \hbar/2$$

imposes limitations in the determination, at a certain instant, of the energy E associated with the internal motion of the electron. If excited states of the particle corresponding to larger values of mass exist, then it is possible to speak of an "electron with rest mass m_0 " only when $\Delta E \leq (\mu_0 - m_0)c^2$, where μ_0 is the rest mass of the internal excited state. Such internal states could be excited in the presence of sufficiently strong interactions. From the uncertainty relation, we have that

$$\Delta\tau \geq \frac{\hbar}{2(\mu_0 - m_0)c^2},$$

and, supposing the muon as an excited state, we get

$$(\mu_0 - m_0)c^2 \cong \frac{3\hbar c}{2e^2} m_0 c^2.$$

Thus, it can be finally obtained that

$$\Delta\tau \geq \frac{1}{3} \frac{e^2}{m_0 c^2} = \frac{\tau_0}{2}.$$

That is, the value of the rest mass of an interacting electron can be taken only inside an interval of the proper time larger than half a chronon. So, when we take into account two successive states, each one endowed with the same uncertainty $\Delta\tau$, they must then be separated by a time interval of at least $2\Delta\tau$, which corresponds exactly to the chronon τ_0 .

3.2. The Mass Spectrum of Leptons

To obtain the mass of the next particle, a possibility to be considered is to take the symmetric equation as describing the muon. According to this naïve argumentation, the equation also foresees a maximum limit for the energy of the eigenstates of the muon. By assuming the equation as successively describing the particles corresponding to these maxima, an expression can be set up for the various limit values, given by

$$E_0^{(n)} = m_0 c^2 \left[\frac{3\hbar c}{2e^2} + 1 \right]^n = m_0 c^2 \left[\frac{3}{2\alpha} + 1 \right]^n, \quad (39)$$

such that, for

$$\begin{aligned} n = 0 &\rightarrow E^{(0)} = 0.511 \text{ MeV} && \text{(electron)} \\ n = 1 &\rightarrow E^{(1)} = 105.55 \text{ MeV} && \text{(muon)} \\ n = 2 &\rightarrow E^{(2)} = 21801.54 \text{ MeV} && \text{(heavy lepton?)}, \end{aligned}$$

the masses for the first excited states can be obtained, including a possible heavy lepton which, according to the experimental results until now, does not seem to exist.

Following a suggestion by [Barut \(1979\)](#); see also [Tennakone and Pakvasa, 1971, 1972](#)), according to which it should be possible to obtain the excited states of the electron from the coupling of its intrinsic magnetic moment with its self-field, [Caldirola \(1978b, 1979b, 1980, 1984a\)](#) and [Benza and Caldirola \(1981\)](#), considering a model of the extended electron as a micro-universe ([Recami, 2002](#)), also succeeded in evaluating the mass of the lepton τ .

Caldirola took into account, for the electron, a model of a point-object moving around in a four-dimensional *de Sitter* micro-universe characterized by

$$c^2t^2 - x^2 - y^2 - z^2 = c^2\tau_0^2,$$

where τ_0 is the chronon associated with the electron and the radius of the micro-universe is given by $a = c\tau_0$. Considering the spectrum of excited states obtained from the naïve argumentation above, we find that each excited state determines a characteristic radius for the micro-universe. Thus, for each particle, the trajectory of the point-object is confined to a spherical shell defined by its characteristic radius and by the characteristic radius of its excited state. For the electron, for example, the point-object moves around, inside the spherical shell defined by its corresponding radius and by the one associated with its excited state: the muon. Such radii are given by

$$a^{(n)} = \tau_0 c \left[\frac{3}{2} \frac{1}{\alpha} + 1 \right]^{-n}. \tag{40}$$

According to the model—supposing that the intrinsic energy of the lepton $e^{(n)}$ is given by $m^{(n)}c^2$ —the lepton moves in its associated micro-universe along a circular trajectory with a velocity $\beta = \frac{\sqrt{3}}{2}$, to which corresponds an intrinsic magnetic moment

$$\mu_a^{(n)} = \frac{1}{4\pi} \frac{e^2}{m^{(n)}c^2}. \tag{41}$$

Starting from [Barut's suggestion \(1979\)](#), Caldirola obtained for the lepton $e^{(n)}$ an extra self-energy given by

$$E^{(n,p)} = (2p)^4 m^{(n)}c^2.$$

The condition set down on the trajectory of the point-object, so that it remains confined to its corresponding spherical shell, is given by

$$E^{(n,p)} \leq \left[\frac{31}{2\alpha} + 1 \right] m_0 c^2,$$

and the values attainable by p are $p = 0$ for $n = 0$, and $p = 0, 1$ for $n \neq 0$. The spectrum of mass is then finally given by

$$m^{(n,p)} = \left[1 + (2p)^4 \right] m^{(n)} = m_0 \left[1 + (2p)^4 \right] \left[\frac{31}{2\alpha} + 1 \right]^n. \quad (42)$$

Thus, for different values of n and p we have the following:

n	p	$m^{(n)}$	
0	0	0.511 MeV	electron
1	0	105.55 MeV	muon
	1	1794.33 MeV	tau

It must be noted that the tau appears as an internal excited state of the muon and its mass is in fair agreement with the experimental values (Hikasa *et al.*, 1992): $m_\tau \approx 1784$ MeV. The difference between these values is less than 1%. Which is remarkable given the simplicity of the model. The model foresees the existence of other excited states that do not seem to exist. This is to some extent justifiable once the muon is obtained as an excited electron and the description of the electron does not imply the existence of any other state. To obtain the lepton tau it was necessary to introduce into the formalism the coupling of the intrinsic magnetic moment with the self-field of the electron.

3.3. Feynman Path Integrals

The discretized Schrödinger equations can easily be obtained using Feynman's path integral approach. This is particularly interesting since it gives a clearer idea of the meaning of these equations. According to the hypothesis of the chronon, time is still a continuous variable and the introduction of the fundamental interval of time is connected only with the reaction of the system to the action of a force. It is convenient to restrict the derivation to the one-dimensional (1D) case, considering a particle under the action of a potential $V(x, t)$. Although the time coordinate is continuous, we assume a discretization of the system (particle) position corresponding to instants separated by time intervals τ (Figure 1).

The transition amplitude for a particle going from an initial point (x_1, t_1) of the space-time to a final point (x_n, t_n) is given by the propagator

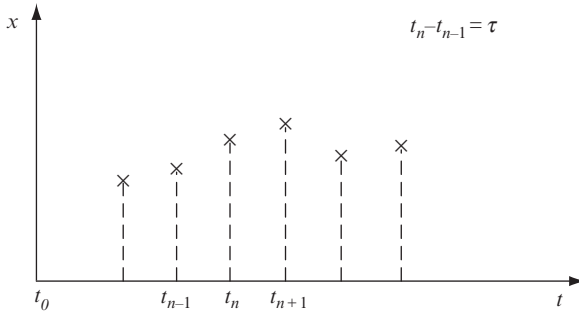


FIGURE 1 Discrete steps in the time evolution of the considered system (particle).

$$K(x_n, t_n; x_1, t_1) = \langle x_n, t_n | x_1, t_1 \rangle. \tag{43}$$

In Feynman's approach this transition amplitude is associated with a path integral, where the classical action plays a fundamental role. It is convenient to introduce the notation

$$S(n, n - 1) \equiv \int_{t_{n-1}}^{t_n} dt \mathcal{L}(x, \dot{x}) \tag{44}$$

such that $\mathcal{L}(x, \dot{x})$ is the classical Lagrangian and $S(n, n_1)$ is the classical action. Thus, for two consecutive instants of time, the propagator is given by

$$K(x_n, t_n; x_{n-1}, t_{n-1}) = \frac{1}{A} \exp\left(\frac{i}{\hbar} S(x_n, t_n; x_{n-1}, t_{n-1})\right). \tag{45}$$

The path integral is defined as a sum over all the paths that can be possibly traversed by the particle and can be written as

$$\langle x_n, t_n | x_1, t_1 \rangle = \lim_{N \rightarrow \infty} A^{-N} \int dx_{N-1} \int dx_{N-2} \dots \int dx_2 \prod_{n=2}^N \exp\left(\frac{i}{\hbar} S(n, n - 1)\right), \tag{46}$$

where A is a normalization factor. To obtain the discretized Schrödinger equations we must consider the evolution of a quantum state between two consecutive configurations (x_{n-1}, t_{n-1}) and (x_n, t_n) . The state of the system at t_n is denoted as

$$\Psi(x_n, t_n) = \int_{-\infty}^{+\infty} K(x_n, t_n; x_{n-1}, t_{n-1}) \Psi(x_{n-1}, t_{n-1}) dx_{n-1}. \tag{47}$$

On the other hand, it follows from the definition of the classical action (Eq. 44) that

$$S(x_n, t_n; x_{n-1}, t_{n-1}) = \frac{m}{2\tau}(x_n - x_{n-1})^2 - \tau V\left(\frac{x_n + x_{n-1}}{2}, t_{n-1}\right). \quad (48)$$

Thus, the state at t_n is given by

$$\Psi(x_n, t_n) = \int_{-\infty}^{+\infty} \exp\left\{\frac{im}{2\hbar\tau}(x_n - x_{n-1})^2 - i\frac{\tau}{\hbar}V\left(\frac{x_n + x_{n-1}}{2}, t_{n-1}\right)\right\} \times \Psi(x_{n-1}, t_{n-1}) dx_{n-1}. \quad (49)$$

When $\tau \approx 0$, for x_n slightly different from x_{n-1} , the integral due to the quadratic term is rather small. The contributions are considerable only for $x_n \approx x_{n-1}$. Thus, we can make the following approximation:

$$x_{n-1} = x_n + \eta \rightarrow dx_{n-1} \equiv d\eta,$$

such that

$$\Psi(x_{n-1}, t_{n-1}) \cong \Psi(x_n, t_{n-1}) + \left(\frac{\partial\Psi(x_n, t_{n-1})}{\partial x}\right)\eta + \left(\frac{\partial^2\Psi}{\partial x^2}\right)\eta^2.$$

By inserting this expression into Eq. (49), supposing that¹⁸

$$V\left(x + \frac{\eta}{2}\right) \approx V(x),$$

and taking into account only the terms to the first order in τ , we obtain

$$\Psi(x_n, t_n) = \frac{1}{A} \exp\left(-\frac{i}{\hbar}\tau V(x_n, t_{n-1})\right) \left(\frac{2i\hbar\pi\tau}{m}\right)^{1/2} \left(\Psi(x_n, t_{n-1}) + \frac{i\hbar\tau}{2m} \frac{\partial^2\Psi}{\partial x^2}\right).$$

Notwithstanding the fact that $\exp(-i\tau V(x_n, t_n)/\hbar)$ is a function defined only for certain well-determined values, it can be expanded in powers of τ , around an arbitrary position (x_n, t_n) . Choosing $A = (2i\hbar\pi\tau/m)^{-1/2}$, such that $\tau \rightarrow 0$ in the continuous limit, we derive

$$\begin{aligned} \Psi(x_n, t_{n-1} + \tau) - \Psi(x_n, t_{n-1}) &= -\frac{i}{\hbar}\tau V(x_n, t_{n-1})\Psi(x_n, t_{n-1}) \\ &+ \frac{i\hbar\tau}{2m} \frac{\partial^2\Psi}{\partial x^2} + \mathcal{O}(\tau^2). \end{aligned} \quad (50)$$

¹⁸ The potential is supposed to vary slowly with x .

By a simple reordering of terms, we finally obtain

$$i \frac{\Psi(x_n, t_{n-1} + \tau) - \Psi(x_n, t_{n-1})}{\tau} = \left\{ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x_n, t_{n-1}) \right\} \Psi(x_n, t_{n-1})$$

Following this procedure we obtain the advanced finite-difference Schrödinger equation, which describes a particle performing a 1D motion under the effect of potential $V(x, t)$.

The solutions of the advanced equation show an amplification factor that may suggest that the particle absorbs energy from the field described by the Hamiltonian in order to evolve in time. In the continuous classical domain the advanced equation can be simply interpreted as describing a *positron*. However, in the realm of the (discrete) nonrelativistic QM, it is more naturally interpreted as representing a system that absorbs energy from the environment.

To obtain the discrete Schrödinger equation only the terms to the first order in τ have been taken into account. Since the limit $\tau \rightarrow 0$ has not been accomplished, the equation thus obtained is only an approximation. This fact may be related to another one faced later in this chapter, when considering the measurement problem in QM.

It is interesting to emphasize that in order to obtain the retarded equation one may formally regard the propagator as acting backward in time. The conventional procedure in the continuous case always provides the advanced equation: therefore, the potential describes a mechanism for transferring energy from a field to the system. The retarded equation can be formally obtained by assuming an inversion of the time order, considering the expression

$$\Psi(x_{n-1}, t_{n-1}) = \int_{-\infty}^{+\infty} \frac{1}{A} \exp \left\{ \frac{i}{\hbar} \int_{t_n}^{t_{n-1}} \mathcal{L} dt \right\} \Psi(x_n, t_n) dx_n, \quad (51)$$

which can be rigorously obtained by merely using the closure relation for the eigenstates of the position operator and then redefining the propagator in the inverse time order. With this expression, it is possible to obtain the retarded Schrödinger equation. The symmetric equation can easily be obtained by a similar procedure.

An interesting characteristic related to these apparently opposed equations is the impossibility of obtaining one from the other by a simple time inversion. The time order in the propagators must be related to the inclusion, in these propagators, of something like the advanced and retarded potentials. Thus, to obtain the retarded equation we can formally consider effects that act backward in time. Considerations such as these, that led to the derivation of the three discretized equations, can supply useful guidelines for comprehension of their meaning.

3.4. The Schrödinger and Heisenberg Pictures

In discrete QM, as well as in the “continuous” one, the use of discretized Heisenberg equations is expected to be preferable for certain types of problems. As for the continuous case, the discretized versions of the Schrödinger and Heisenberg pictures are also equivalent. However, we show below that the Heisenberg equations *cannot*, in general, be obtained by a direct discretization of the continuous equations.

First, it is convenient to introduce the discrete time evolution operator for the symmetric

$$\hat{U}(t, t_0) = \exp \left[-\frac{i(t-t_0)}{\tau} \sin^{-1} \left(\frac{\tau \hat{H}}{\hbar} \right) \right] \quad (52)$$

and for the retarded equation,

$$\hat{U}(t, t_0) = \left[1 + \frac{i}{\hbar} \tau \hat{H} \right]^{-(t-t_0)/\tau}. \quad (53)$$

To simplify the equations, the following notation is used throughout this section:

$$\Delta f(t) \leftrightarrow \frac{f(t+\tau) - f(t-\tau)}{2\tau} \quad (54)$$

$$\Delta_R f(t) \leftrightarrow \frac{f(t) - f(t-\tau)}{\tau}. \quad (55)$$

For both operators above it can easily be demonstrated that, if the Hamiltonian \hat{H} is a Hermitean operator, the following equations are valid:

$$\Delta \hat{U}(t, t_0) = \frac{1}{i\hbar} \hat{U}(t, t_0) \hat{H}, \quad (56)$$

$$\Delta \hat{U}^\dagger(t, t_0) = \frac{1}{i\hbar} \hat{U}^\dagger(t, t_0) \hat{H}. \quad (57)$$

In the Heisenberg picture the time evolution is transferred from the state vector to the operator representing the observable according to the definition

$$\hat{A}^H \equiv \hat{U}^\dagger(t, t_0 = 0) \hat{A}^S \hat{U}(t, t_0 = 0). \quad (58)$$

In the symmetric case, for a given operator \hat{A}^S , the time evolution of the operator $\hat{A}^H(t)$ is given by

$$\begin{aligned} \Delta \hat{A}^H(t) &= \Delta \left[\hat{U}^\dagger(t, t_0 = 0) \hat{A}^S \hat{U}(t, t_0 = 0) \right] \\ \Delta \hat{A}^H(t) &= \frac{1}{i\hbar} \left[\hat{A}^H, \hat{H} \right] \end{aligned} \quad (59)$$

which has exactly the same form as the equivalent equation for the continuous case. The important feature of the time evolution operator that is used to derive the expression above is that it is a *unitary* operator. This is true for the symmetric case. For the retarded case, however, this property is no longer satisfied. Another difference from the symmetric and continuous cases is that the state of the system is also time-dependent in the retarded Heisenberg picture:

$$|\Psi^H(t)\rangle = \left[1 + \frac{\tau^2 \hat{H}^2}{\hbar^2} \right]^{-(t-t_0)/\tau} |\Psi^S(t_0)\rangle. \quad (60)$$

By using the property $[\hat{A}, f(\hat{A})] = 0$, it is possible to show that the evolution law for the operators in the retarded case is given by

$$\Delta \hat{A}^H(t) = \left\{ \frac{1}{i\hbar} [\hat{A}^S(t), \hat{H}^S(t)] + \Delta \hat{A}^S(t) \right\}^H. \quad (61)$$

In short, we can conclude that the discrete symmetric and the continuous cases are formally quite similar and the Heisenberg equation can be obtained by a direct discretization of the continuous equation. For the retarded and advanced cases, however, this does not hold. The compatibility between the Heisenberg and Schrödinger pictures is analyzed in the appendices.

Here we mention that much parallel work has been done by Jannussis *et al.* For example, they have studied the retarded, dissipative case in the Heisenberg representation, then studying in that formalism the (normal or damped) harmonic oscillator. On this subject, see [Jannussis *et al.* \(1982a,b, 1981a,b, 1980a,b\)](#) and [Jannussis \(1984b,c\)](#).

3.5. Time-Dependent Hamiltonians

We restricted the analysis of the discretized equations to the time-independent Hamiltonians for simplicity. When the Hamiltonian is explicitly time-dependent, the situation is similar to the continuous case. It is always difficult to work with such Hamiltonians but, as in the continuous case, the theory of small perturbations can also be applied. For the symmetric equation, when the Hamiltonian is of the form

$$\hat{H} = \hat{H}_0 + \hat{V}(t), \quad (62)$$

such that \hat{V} is a small perturbation related to \hat{H}_0 , the resolution method is similar to the usual one. The solutions are equivalent to the continuous solutions followed by an exponentially varying term. It is always possible to solve this type of problem using an appropriate ansatz.

However, another factor must be considered and is related to the existence of a limit beyond which \hat{H} does not have stable eigenstates. For the symmetric equation, the equivalent Hamiltonian is given by

$$\tilde{H} = \frac{\hbar}{\tau} \sin^{-1} \left(\frac{\tau}{\hbar} \hat{H} \right). \quad (63)$$

Thus, as previously stressed, beyond the critical value the eigenvalues are not real and the operator \tilde{H} is no longer Hermitean. Below that limit, \tilde{H} is a densely defined and self-adjoint operator in the $\mathcal{L} \subset L^2$ subspace defined by the eigenfunctions of \tilde{H} . When the limit value is exceeded, the system changes to an excited state and the previous state loses physical meaning. In this way, it is convenient to restrict the observables to self-adjoint operators that keep invariant the subspace \mathcal{L} . The perturbation \hat{V} is assumed to satisfy this requirement.

In usual QM it is convenient to work with the interaction representation (Dirac's picture) in order to deal with time-dependent perturbations. In this representation, the evolution of the state is determined by the time-dependent potential $\hat{V}(t)$, while the evolution of the observable is determined by the stationary part of the Hamiltonian \hat{H}_0 . In the discrete formalism, the time evolution operator defined for \hat{H}_0 , in the symmetric case, is given by

$$\hat{U}_0(t, t_0) = \exp \left[-\frac{i(t-t_0)}{\tau} \sin^{-1} \left(\frac{\tau \hat{H}_0}{\hbar} \right) \right]. \quad (64)$$

In the interaction picture the vector state is defined, from the state in the Schrödinger picture, as

$$|\Psi^I(t)\rangle = \hat{U}_0^\dagger(t) |\Psi^S(t_0)\rangle, \quad (65)$$

where $\hat{U}_0^\dagger(t) \equiv \hat{U}_0^\dagger(t, t_0 = 0)$. On the other hand, the operators are defined as

$$\hat{A}^I = \hat{U}_0^\dagger(t) \hat{A}^S \hat{U}_0(t). \quad (66)$$

Therefore, it is possible to show that, in the interaction picture, the evolution of the vector state is determined by the equation

$$i\hbar \Delta \Psi^I(\mathbf{x}, t) = \frac{i\hbar}{2\tau} [\Psi^I(\mathbf{x}, t + \tau) - \Psi^I(\mathbf{x}, t - \tau)] = \hat{V}^I \Psi^I(\mathbf{x}, t), \quad (67)$$

which is equivalent to a direct discretization of the continuous equation. For the operators, we determine that

$$\Delta \hat{A}^I(t) = \frac{\hat{A}^I(t + \tau) - \hat{A}^I(t - \tau)}{2\tau} = \frac{1}{i\hbar} [\hat{A}^I, \hat{H}_0], \quad (68)$$

which is also equivalent to the continuous equation.

Thus, for the symmetric case, the discrete interaction picture retains the same characteristics of the continuous case for the evolution of the operators and state vectors, once, obviously, the eigenstates of \hat{H} remain below the stability limit. We can adopt, for the discrete case, a procedure similar to that one commonly used in QM to deal with small time-dependent perturbations.

We consider, in the interaction picture, the same basis of eigenstates associated with the stationary Hamiltonian \hat{H}_0 , given by $|n\rangle$. Then,

$$|\Psi(t)\rangle^I = \sum_n \Psi(t) \langle n | \Psi(t) \rangle^I |n\rangle = \sum_n c_n(t) |n\rangle$$

is the expansion, over this basis, of the state of the system at a certain instant t . It must be noted that the evolution of the state of the system is determined once the coefficients $c_n(t)$ are known. Using the evolution equation [Eq. (67)], it can be obtained that

$$i\hbar\Delta \langle n | \Psi(t) \rangle^I = \sum_m \langle n | \hat{V}^I | m \rangle \langle m | \Psi(t) \rangle^I.$$

Using the evolution operator to rewrite the perturbation \hat{V} in the Schrödinger picture, we obtain

$$i\hbar\Delta c_n(t) = \sum_m c_m(t) V_{nm}(t) \exp(i\omega_{nm}t), \quad (69)$$

such that

$$\omega_{nm} = \frac{1}{\tau} \left[\sin^{-1} \left(\frac{\tau E_n}{\hbar} \right) \sin^{-1} \left(\frac{\tau E_m}{\hbar} \right) \right],$$

and we obtain the evolution equation for the coefficients $c_n(t)$, the solution of which gives the time evolution of the system.

As in usual QM, it is also possible to work with the interaction picture evolution operator, $\hat{U}^I(t, t_0)$, which is defined as

$$|\Psi(t)\rangle^I = \hat{U}^I(t, t_0) |\Psi(t_0)\rangle^I,$$

such that Eq. (67) can be written as

$$i\hbar\Delta \hat{U}^I(t, t_0) = \hat{V}^I(t) \hat{U}^I(t, t_0). \quad (70)$$

The operator $\hat{U}^I(t, t_0)$ must satisfy the initial condition $\hat{U}^I(t, t_0) = 0$. Given this condition, for the finite-difference equation above we have the solution

$$\hat{U}^I(t, t_0) = \exp \left[\frac{-i(t-t_0)}{\tau} \sin^{-1} \left(\frac{\tau \hat{V}^I(t)}{\hbar} \right) \right].$$

A difference from the continuous case, where the approximate evolution operator is an infinite Dyson series, is that this approach provides a well determined expression. The solution to the problem is obtained by correlating the elements of the matrix associated with such operator to the evolution coefficients $c_n(t)$.

In general, the finite-difference equations are more difficult to analytically solve than the equivalent differential equations. In particular, this difficulty is much more stressed for the system of equations obtained from the formalism above.

An alternative approach is to use the equivalent Hamiltonians (Caldirola, 1977a,b, 1978a; Fryberger, 1981). Once the equivalent Hamiltonian is found, the procedure is the same as for the continuous theory. If the perturbation term \hat{V} is small, the equivalent Hamiltonian can be written as

$$\tilde{H} = \frac{\hbar}{\tau} \sin^{-1}\left(\frac{\tau}{\hbar} \hat{H}_0\right) + \hat{V}(t) = \hat{H}_0 + \hat{V}(t).$$

In the interaction picture, the state of the system is now defined as

$$|\Psi^I(t)\rangle = \exp i \frac{\tilde{H}_0 t}{\hbar} |\Psi^S(t)\rangle, \quad (71)$$

and the operators are given by

$$\hat{A}^I = \exp\left(i \frac{\tilde{H}_0 t}{\hbar}\right) \hat{A}^S \exp\left(-i \frac{\tilde{H}_0 t}{\hbar}\right). \quad (72)$$

The state in Eq. (71) evolves according to the equation

$$i\hbar \frac{\partial}{\partial t} |\Psi^I(t)\rangle = \hat{V}^I |\Psi^I(t)\rangle, \quad (73)$$

where \hat{V}^I is obtained according to definition (72).

Now, small time-dependent perturbations can be handled by taking into account the time evolution operator defined by

$$|\Psi^I(t)\rangle = \hat{U}^I(t, t_0) |\Psi^I(t_0)\rangle. \quad (74)$$

According to the evolution law [Eq. (73)], we have

$$i\hbar \frac{d}{dt} \hat{U}^I(t, t_0) = \hat{V}^I(t) \hat{U}^I(t, t_0). \quad (75)$$

Thus, once given that $\hat{U}^I(t_0, t_0) = 1$, the time evolution operator is given by either

$$\hat{U}^I(t, t_0) = 1 - \frac{i}{\hbar} \int_{t_0}^t \hat{V}^I(t') \hat{U}^I(t', t_0) dt'$$

or

$$\hat{U}^I(t, t_0) = 1 + \sum_{n=1} \left(-\frac{i}{\hbar}\right)^n \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \dots \int_{t_0}^{t_{n-1}} dt_n \hat{V}^I(t_1) \hat{V}^I(t_2) \dots \hat{V}^I(t_n),$$

where the evolution operator is obtained in terms of a Dyson series.

Drawing a parallel, between the elements of the matrix of the evolution operator and the evolution coefficients $c_n(t)$ obtained from the continuous equation equivalent to Eq. (69), requires the use of the basis of eigenstates of the stationary Hamiltonian \hat{H}_0 . If the initial state of the system is an eigenstate $|m\rangle$ of that operator, then, at a subsequent time, we have

$$c_n(t) = \langle n | \hat{U}^I(t, t_0) | m \rangle.$$

The method of the equivalent Hamiltonian is simpler because it takes full advantage of the continuous formalism.

4. SOME APPLICATIONS OF THE DISCRETIZED QUANTUM EQUATIONS

Returning to more general questions, it is interesting to analyze the physical consequences resulting from the introduction of the fundamental interval of time in QM. In this section we apply the discretized equations to some typical problems.

4.1. The Simple Harmonic Oscillator

The Hamiltonian that describes a simple harmonic oscillator does not depend explicitly on time. The introduction of the discretization in the time coordinate does not affect the outputs obtained from the continuous equation for the spatial branch of the solution. This is always the case when the potential does not have an explicit time dependence.

For potentials like this, the solutions of the discrete equations are always formally identical, with changes in the numerical values that depend on the eigenvalues of the Hamiltonian considered and on the value of the chronon associated with the system described. We have the same spectrum of eigenvalues and the same basis of eigenstates but with the time evolution given by a different expression.

For the simple harmonic oscillator, the Hamiltonian is given by

$$\hat{H} = \frac{1}{2m} \hat{\mathbf{P}}^2 + \frac{m\omega^2}{2} \hat{\mathbf{X}}^2, \tag{76}$$

to which the eigenvalue equation corresponds:

$$\hat{H}|u_n\rangle = E_n|u_n\rangle, \quad (77)$$

so that E_n gives the energy eigenvalue spectrum of the oscillator.

As mentioned previously, since this Hamiltonian does not depend explicitly on time, there is always an upper limit for the possible values of its energy eigenvalues. In the basis of eigenfunctions of \hat{H} , a general state of the oscillator can be written as

$$|\Psi(t)\rangle = \sum_n c_n(0)|u_n\rangle \exp\left[-i\frac{t}{\tau} \sin^{-1}\left(\frac{E_n\tau}{\hbar}\right)\right],$$

with $c_n(0) = \langle u_n|\Psi(t=0)\rangle$. Naturally, when $\tau \rightarrow 0$, the solution above recovers the continuous expression with its time dependency given by $\exp\left(\frac{-iE_n t}{\hbar}\right)$. Therefore, there is only a small phase difference between the two expressions. For the mean value of an arbitrary observable,

$$\begin{aligned} \langle \Psi(t)|\hat{A}|\Psi(t)\rangle &= \sum_{m=0} \sum_{n=0} c_m^*(0)c_n(0)A_{mn} \exp\left[\frac{i}{\hbar}(E_m - E_n)t\right] \\ &\times \exp\left[i(E_m^3 - E_n^3)\frac{t\tau^2}{3!\hbar^3}\right] + O(\tau^4), \end{aligned}$$

with $A_{mn} = \langle u_m|\hat{A}|u_n\rangle$, we obtain an additional phase term that implies a small deviation of the resulting frequencies compared with the Bohr frequencies of the harmonic oscillator. To first approximation, this deviation is given by the term depending on τ^2 in the expression above.

Of note, the restrictions imposed on the spectrum of eigenvalues of \hat{H} mutilate the basis of eigenvectors: the number of eigenvectors becomes *finite* and does not constitute a complete set anymore. Therefore, it no longer forms a basis. For eigenstates beyond the upper limit the states are unstable and decay exponentially with time.

For a time-independent Hamiltonian, the retarded equation always furnishes damped solutions characteristic of radiating systems. In this case, there is neither stationary solutions nor an upper limit for the energy eigenvalues. The larger the eigenvalue, the larger the damping factor and more quickly its contribution to the state of the system tends to zero. If we write the state of the oscillator as

$$|\Psi(t)\rangle = \sum_n c_n(0)|u_n\rangle \left[1 + \frac{i}{\hbar}\tau E_n\right]^{\frac{t}{\tau}},$$

which has a norm decaying according to

$$\langle \Psi(t) | \Psi(t) \rangle = \sum_n c_n(0) |u_n\rangle \left[1 + \frac{\tau^2 E_n^2}{\hbar^2} \right]^{\frac{t}{\tau}}, \quad (78)$$

we have for an arbitrary observable that [with $\langle A(t) \rangle \equiv \langle A \rangle (t)$]:

$$\langle A(t) \rangle = \sum_m \sum_n c_m^*(0) c_n(0) A_{mn} \exp \left[-\frac{t}{\tau} \ln \left[1 + \frac{\tau^2}{\hbar^2} E_n E_m - \frac{i\tau}{\hbar} (E_m - E_n) \right] \right]$$

or, to the first order in τ ,

$$\langle A(t) \rangle = \sum_m \sum_n c_m^*(0) c_n(0) A_{mn} \exp \left[\frac{i}{\hbar} (E_m - E_n) t \right] \exp \left[-t (E_m^2 - E_n^2) \frac{\tau}{2\hbar^2} \right],$$

so that, in addition to the Bohr frequencies defining the emission and absorption frequencies of the oscillator, we obtain a damping term that causes the average value of the observable—which is explicitly independent of time—to tend to zero with time. A cursory analysis shows that even for very small eigenvalues, smaller than 1.0 eV, the damping factor is large, so the decay of the average values is very fast. The damping factor of the norm in Eq. (78) can be evaluated, and its behavior can be seen in Figure 2.

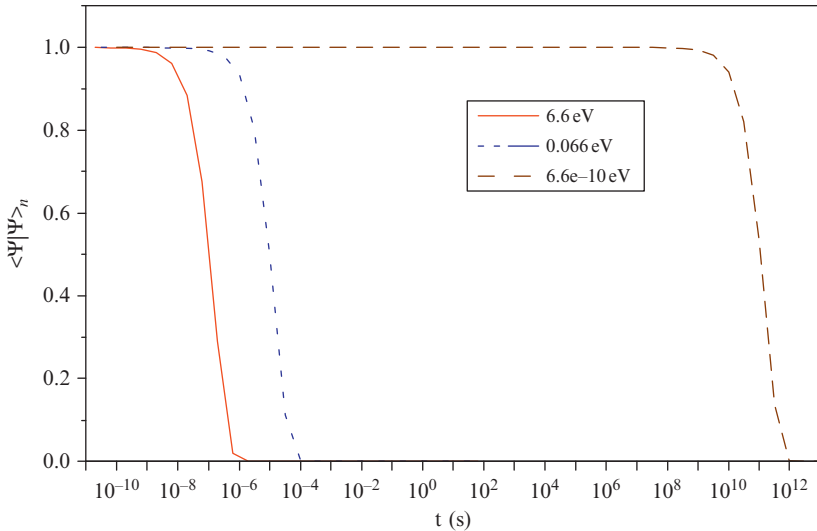


FIGURE 2 Typical behavior of the damping factor associated with different energy eigenvalues (retarded case).

4.2. Free Particle

For a free particle (an electron for example), the general solution of the symmetric Eq. (35) can be obtained, in the coordinate representation, using as an ansatz the solution for the continuous case. Thus, a spectrum of eigenfunctions (plane waves) is obtained given by

$$\Psi_p(\mathbf{x}, t) = (2\pi\hbar)^{-3/2} \exp\left(-i\alpha(|\mathbf{p}|)t + i\frac{(\mathbf{p}\cdot\mathbf{x})}{\hbar}\right).$$

Inserting this expression into the symmetric equation, we obtain for the frequency $\alpha(|\mathbf{p}|)$ that

$$\alpha(|\mathbf{p}|) = \frac{1}{c} \sin^{-1}\left(\frac{\tau}{\hbar} \frac{\mathbf{p}^2}{2m_0}\right). \quad (79)$$

When $\tau \rightarrow 0$, $\alpha(|\mathbf{p}|)\hbar$ coincides with the energy of the particle. As observed for the bound particle, here we also have an upper limit for the spectrum of eigenvalues. Thus, the upper limit for the possible values of momentum is given by

$$p \leq p_{\text{Max}} \equiv \sqrt{\frac{2m_0\hbar}{\tau}} = 10 \text{ MeV}/c \quad (80)$$

for the electron. In other words, there is a limit beyond which the frequencies cease to be real.

As in the continuous case, the state of the particle is described by a superposition of the eigenstates and can be written as

$$\Psi(\mathbf{x}, t) = \frac{1}{(2\pi\hbar)^{3/2}} \int d^3p c(\mathbf{p}) \exp\left(-i\alpha(|\mathbf{p}|)t + i\frac{(\mathbf{p}\cdot\mathbf{x})}{\hbar}\right).$$

The coefficients $c(\mathbf{p})$ are determined from the initial condition $\Psi(\mathbf{x}, 0) = \Psi_0(\mathbf{x})$. From the expression for α , it can be observed that beyond a certain value of p the expression loses meaning. When $p \geq \sqrt{\frac{2m_0\hbar}{\tau}}$, the complete solution is defined only if $c(\mathbf{p})=0$. From the stationary phase condition, we have that

$$\mathbf{x} = \frac{\mathbf{p}}{m_0} \frac{t}{\sqrt{1 - \left(\frac{\tau}{\hbar}\right)^2 \left(\frac{p^4}{4m_0^2}\right)}},$$

and, supposing that $c(\mathbf{p})$ corresponds to a distribution of probabilities with a peak at $\mathbf{p} = \mathbf{p}_0$, then the wave packet will move in the direction \mathbf{p}_0 with uniform velocity

$$v = \frac{p_0}{m_0} \left[1 - \left(\frac{\tau}{\hbar}\right)^2 \left(\frac{p^4}{4m_0^2}\right)\right]^{-1/2},$$

which coincides with the group velocity of the packet. It can be promptly observed that when p reaches its maximum value permitted, the velocity diverges: $v \rightarrow \infty$. Thus, the introduction of a fundamental interval of time does not impose any restriction on the velocity of the particle, although it results in a limit for the canonical momentum of the eigenfunctions. Starting from the condition of stationary phase it is possible to redefine the momentum associated with the particle, so that this new momentum does not suffer any restrictions. Thus, one can conclude that it is possible to exist free electrons with any energy, differently from what happens with bound electrons.

For $p > p_{\max}$, the frequency $\alpha(|\mathbf{p}|)$ fails to be real; its dependence on p is shown in Figure 3. An analysis of Eq. (79) shows that if $\alpha(|\mathbf{p}|)$ is complex, then, for $p \leq p_{\max}$, the imaginary component is null and the real part is given by Eq. (79). When $p \geq p_{\max}$, then

$$\text{Re}(\alpha(p)) = \frac{\pi}{2\tau},$$

$$\text{Im}(\alpha(p)) = -\frac{1}{\tau} \ln \left[\left| \frac{\tau p^2}{2m_0 \hbar} \right| + \sqrt{\left(\frac{\tau p^2}{2m_0 \hbar} \right)^2 - 1} \right],$$

with the real part being a constant and the imaginary one tending logarithmically to $-\infty$. Using the expressions above, for $p > p_{\max}$ the eigenstates become unstable, with a time-dependent decay term. When we look for an equivalent Hamiltonian \tilde{H} that, for the continuous Schrödinger equation, supplies equivalent outputs, this is possible only if \tilde{H} is a non-Hermitian operator. It is straightforward to see that this is the case

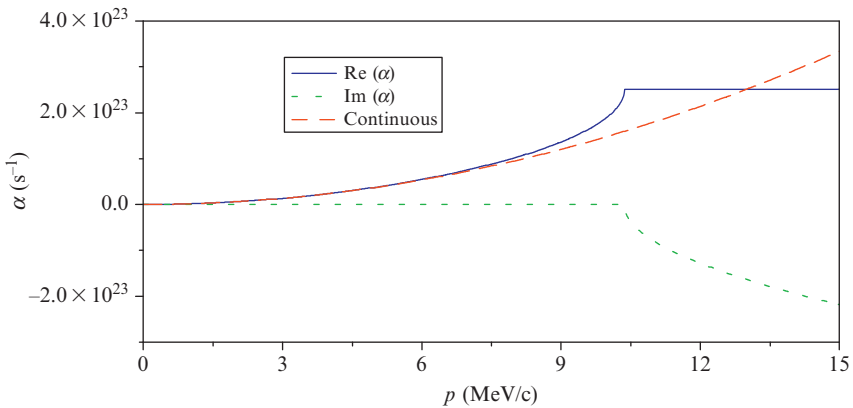


FIGURE 3 Real and imaginary components of $\alpha(|\mathbf{p}|)$ obtained for the symmetric equation compared to the continuous case.

for $\tilde{H} = H_1 + iH_2$, with H_1 and H_2 Hermiteans and such that $H_1 |p\rangle = \hbar \operatorname{Re}(\alpha(p)) |p\rangle$ and $H_2 |p\rangle = \hbar \operatorname{Im}(\alpha(p)) |p\rangle$.

For the retarded equation, using the same ansatz of the symmetric case, the damping factor appears for every value of p . There is no limitation on the values of p , but, when $p \rightarrow \infty$, the real part of $\alpha(|\mathbf{p}|)$ tends to the same limit value observed for the symmetric case. [Figure 4](#) illustrates the behavior of the components of $\alpha(|\mathbf{p}|)$. The general expression for an eigenfunction is found to be

$$\Psi_p(x, t) \propto \exp\left[\frac{ipx}{\hbar} - \frac{it}{\tau} \tan^{-1} \frac{p^2 \tau}{2m\hbar}\right] \exp\left[-\frac{t}{2\tau} \ln\left[1 + \left(\frac{p^2 \tau}{2m\hbar}\right)^2\right]\right].$$

Performing a Taylor expansion and keeping only the terms to the first order in τ we obtain the continuous solution multiplied by a damping factor:

$$\Psi_p(x, t) \propto \exp\left(\frac{ipx}{\hbar} - i\omega t\right) \exp\left(\frac{1}{2}\omega^2 \tau t\right) \quad (81)$$

where $\omega = p^2/2m\hbar$ is the frequency obtained for the continuous case.

The damping term depends only on the Hamiltonian, through the frequency ω , and on the chronon associated with the particle. As the latter is constant for a given particle, that term shows that for very high frequencies the solutions decay quite fast and, as the system evolves, a decay for smaller frequencies also takes place.

The inflection point in [Figure 5](#), delimiting the region of the spectrum where the decay is faster, moves in the direction of smaller frequencies as time passes. The consequence of this decay is the narrowing of the frequency bandwidth, which is relevant for the wave packet describing the particle. This is an echo of the continuous decrease of the energy. As in

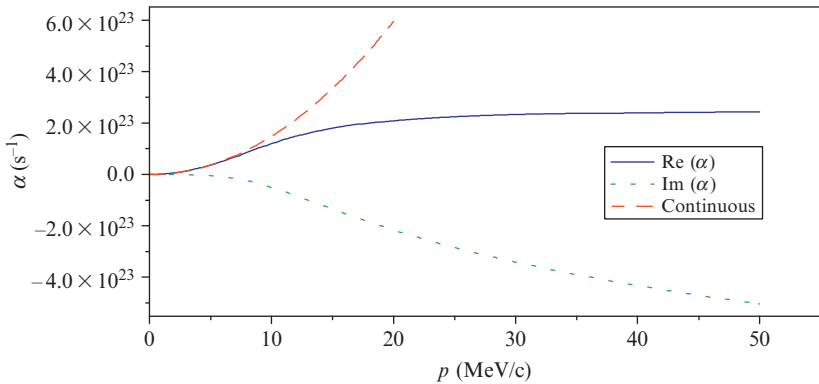


FIGURE 4 Real and imaginary components of $\alpha(|\mathbf{p}|)$ obtained for the retarded equation compared with the continuous case.

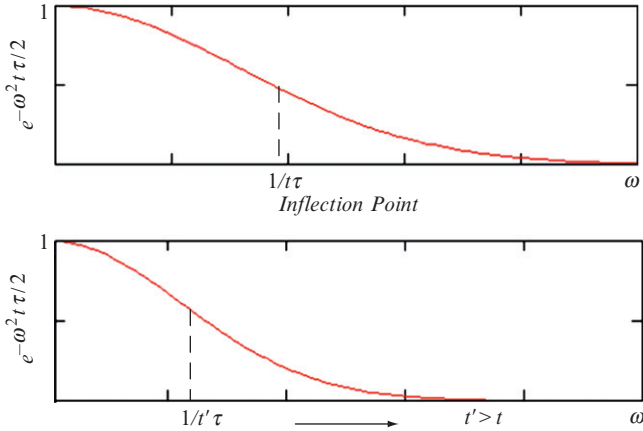


FIGURE 5 Displacement with time of the inflection point.

the symmetric case, obtaining an equivalent Hamiltonian is possible only if non-Hermitean operators are considered.

It is worthwhile to reconsider the question of the physical meaning of the three discretized Schrödinger equations. Apparently, the choice of the equation for a particular situation is determined by the restrictions imposed on the system by the boundary conditions. The symmetric equation is used for special situations for which the system neither emits nor absorbs radiation, or does it in a perfectly “balanced” way. This is the case for the electrons in their atomic orbits. Therefore, the particle is stable until a certain energy limit, beyond which the behavior of the states is similar to that of the retarded solutions. For energies far below that limit, the particle behaves almost identically to the continuous case, except that the new frequencies associated with each wave function differ from the continuous frequencies by a factor of order τ^2 . The probability that a particle is found with energy larger than the limit value decreases exponentially with time. For the bound electron, the limit is that equivalent to the rest mass of the muon. If a parallel with the classical approach is valid, the symmetric equation describes: (1) an isolated system, which does not exchange energy with the surrounding environment, or: (2) a situation of perfect thermodynamic equilibrium, in which a perfect balance between absorbed and dissipated energies is verified. For the classical theory of the electron the symmetric equation is only an approximation that ignores the radiation reaction effects. In QM, however, the existence of nonradiating states is related to the very essence of the theory. The symmetric equation shows that, below the critical limit, the states are physically identical to the outputs from the continuous theory: they are *nonradiating* states.

The retarded equation represents a system that somehow loses energy into the environment. The mechanism of such energy dissipation is related not only to the Hamiltonian of the system but also to some properties of the environment—even the vacuum—as it can be inferred from the description of the free particle. From the solutions obtained it is now observed that time has a well-defined direction of flux and that the frequency composition of the wave packet associated with the particle depends on the instant of time considered. It is clear that it is always possible to normalize the state at a certain instant and consider it as an initial state. This is permitted by the formalism. However, in a strictly rigorous description, the frequency spectrum corresponds to a specific instant of time subsequent to the emission. This aspect can be interesting from the point of view of possible experimental verifications.

4.3. The Discretized Klein-Gordon Equation (for massless particles)

Another interesting application is the description of a free scalar particle—a *scalar* or *zero-spin* “photon”—using a finite-difference form of the Klein-Gordon equation for massless particles.

In the symmetric form, the equation is written as

$$\square^2 A_\mu = 0 \rightarrow \frac{\Psi(t+2\tau) - 2\Psi(t) + \Psi(t-2\tau)}{4c^2\tau^2} - \nabla^2\Psi(t) = 0. \quad (82)$$

Using a convenient ansatz we obtain, for this equation, in the coordinate representation, that

$$\Psi_k(x, t) = A \exp\left(-i \frac{t}{2\tau} \cos^{-1}(1 - 2\tau^2 k^2)\right) \exp(ikx),$$

which can be written as

$$\Psi_p(x, t) = A \exp\left(-i \frac{t}{2\tau} \cos^{-1}(1 - 2c^2\tau^2 E^2/\hbar^2)\right) \exp(ipx/\hbar),$$

since $E = p^2 c^2$ and $p = \hbar k$. Expanding the time exponential in powers of τ , we find, to the second order in τ , a solution that is very similar to the continuous expression:

$$\Psi_p(x, t) = A \exp\left(-\frac{i}{\hbar}(E't - px)\right),$$

with

$$E' \approx E \left(1 + \frac{E^2 \tau^2}{6\hbar^2}\right).$$

A difference of the order of τ^2 is observed between the energy values of the “photons” in the continuous and discrete approaches. The general solution is given by a linear combination of the eigenfunctions found. *A priori*, the value of the chronon for the particle is not known. The time-dependent exponential term in the expressions above leads to an upper limit for the allowed energy, which is given by $E \leq \hbar/\tau$. We could suppose that the value of the chronon for this photon is of about the fundamental time interval of the electromagnetic interactions $\sim 10^{-9}$ s resulting in a critical value of approximately 6.6 keV, which is a very low limit. A smaller chronon should increase this limit but, if there is any generality in the classical expression obtained for the electron, we should expect a larger value for this massless particle.

If instead of a photon we consider a scalar neutrino, taking for the value of the chronon $\tau \sim 10^{-13}$ s—a typical time for the weak decay—the limit for the energy associated with the eigenfunctions is now approximately 0.007 eV. This means that in the composition of the wave packet describing this particle the only contribution comes from eigenfunctions, the energy of which is below that limit.

The eigenfunctions obtained for the Hamiltonian considered are “plane waves” solutions. The dependence of these solutions on energy and time is shown in [Figures 6 and 7](#). For smaller values of τ the decay of the modes with energy above the maximum is faster.

Apparently, it seems possible to determine a limiting value for the chronon starting from the uncertainty relations. This could be obtained, when describing particles, using the expression

$$\tau < \frac{\hbar}{2m_0c^2}$$

that provides for the electron a maximum limit given by 6.4×10^{-22} s. However, this value is two degrees of magnitude larger than the classical value of the chronon for the electron, which is a considerable difference. It is possible to use this relation for a complex system, which is shown later.

We also need to consider the conditions with which a photon must be supplied in order to be described by the symmetric equation. For the electron, it seems clear that not irradiating in a bound state—which is imposed by QM—implies the adoption of the symmetric equation. For the photon (as for a free particle), when the retarded form of the Klein-Gordon equation is used, a solution is also obtained wherein the highest frequencies decay faster than the lowest ones. There is always a tendency in the sense that the lowest frequencies prevail. If we are allowed to assign a physical meaning to such a discretized Klein-Gordon equation, we are also allowed to think that, the farther the light source, the more the spectrum of the emitted light will be shifted for the largest wavelengths, even if the source is at rest with respect to the observer. Thus, we could

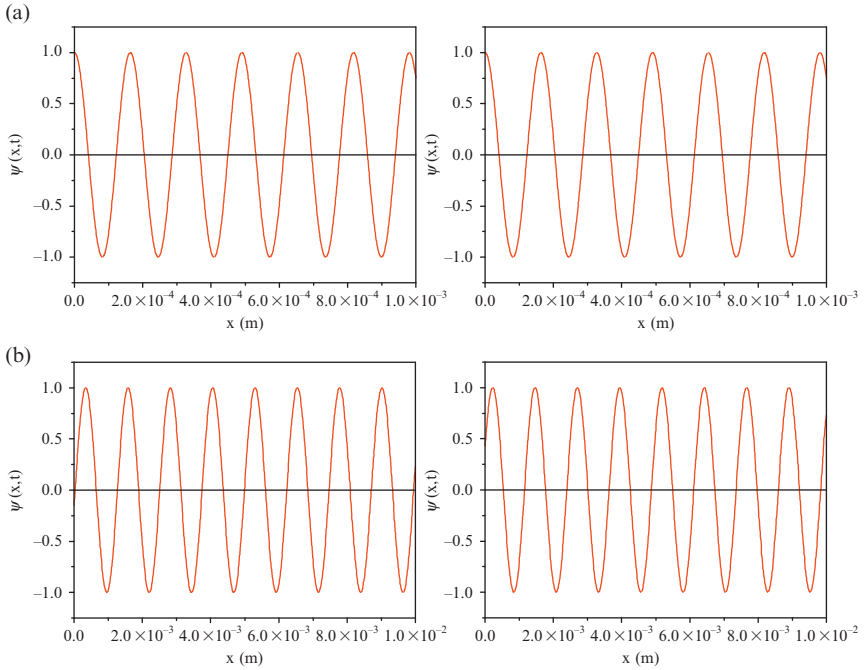


FIGURE 6 Solution of the discretized Klein-Gordon equation, when the energy is smaller than the critical limit, depicted for different values of energy and time.

(a) $E < E_M$; $E = 0.0001$ eV; $t = 0$. Discrete and continuous solution are identical.

(b) $E < E_M$; $E = 0.0001$ eV; $t 1 \times 10^{-10}$ s. Discrete and continuous solutions differ in phase.

obtain a *red shift* effect as a consequence of the introduction of the chronon that could be used in the construction of a tired-light theory. . .

Finally, we need to point out that the discretization considered for the Klein-Gordon equation does not follow exactly the same procedure that led to the discretized Schrödinger equation, since it is a relativistic invariant equation. We did not change the proper time, but the time coordinate itself into the discretized form. We considered a discretized version of the Hamiltonian operator by applying the transformations

$$p \rightarrow \frac{\hbar}{i} \nabla,$$

$$H \rightarrow i\hbar\Delta,$$

with Δ as defined in [subsection 3.4](#), on the Hamiltonian of a relativistic free particle,

$$H = \sqrt{p^2c^2 + m^2c^4}$$

as usual in the continuous case.

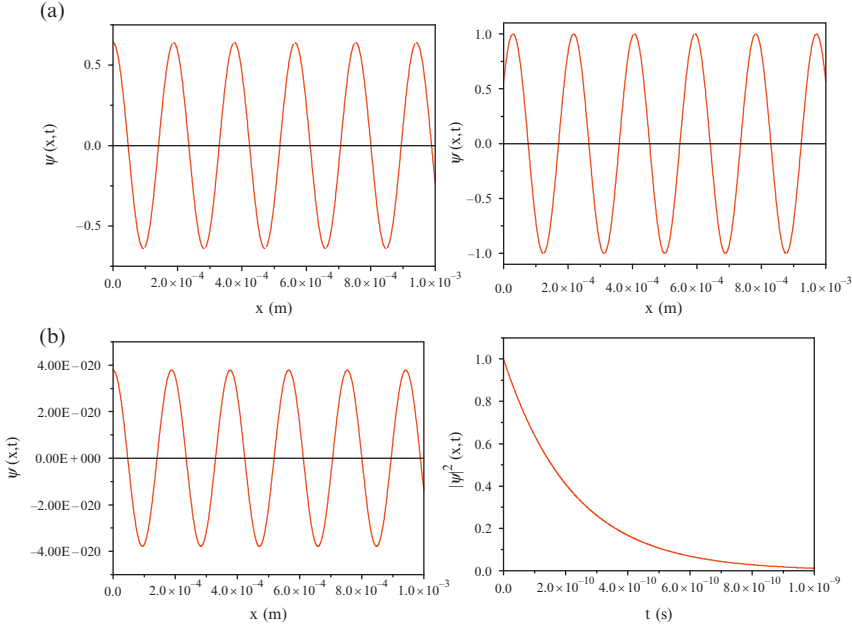


FIGURE 7 Solution of the discretized Klein-Gordon equation when the energy is larger than the critical limit, depicted for different values of energy and time. In this case, the amplitude decay is very fast. (a) For the two insets above, it is $E > E_M$: $E = E_M(1 + 1 \times 10^{-7})$ eV; $t = 1 \times 10^{-10}$ s; discrete and continuous solutions differ in phase and amplitude. (b) For the two insets below, it is $E > E_M$: $E = E_M(1 + 1 \times 10^{-7})$ eV. In the left inset: $t = 1 \times 10^{-8}$ s. The right inset shows the damping of the amplitude with time.

4.4. Time Evolution of the Position and Momentum Operators: The Harmonic Oscillator

It is possible to apply the discretized equations to determine the time evolution of the position and momentum operators, which is rather interesting for the description of the simple harmonic oscillator. To do so, we use the discretized form of the Heisenberg equations which, in the symmetric case, can be obtained by a direct discretization of the continuous equation. Starting from this equation, we determine the coupled Heisenberg equations for the two operators:

$$\frac{\hat{\mathbf{p}}(t + \tau) - \hat{\mathbf{p}}(t - \tau)}{2\tau} = -m\omega^2 \hat{\mathbf{x}}(t), \quad (83)$$

$$\frac{\hat{\mathbf{x}}(t + \tau) - \hat{\mathbf{x}}(t - \tau)}{2\tau} = \frac{1}{m} \hat{\mathbf{p}}(t). \quad (84)$$

Such coupled equations yield two finite-difference equations of second order, the general solutions of which are easily obtained. The most immediate way to determine the evolution of these operators is to use the creation and annihilation operators. Keeping the Heisenberg equation and remembering that for the harmonic oscillator we have $\hat{H} = \omega \left(\hat{A}^\dagger \hat{A} + \frac{1}{2} \right)$, we obtain for the symmetric case:

$$\frac{\hat{A}(t + \tau) - \hat{A}(t - \tau)}{2\tau} = -i\omega\hat{A}(t), \quad (85)$$

$$\frac{\hat{A}^\dagger(t + \tau) - \hat{A}^\dagger(t - \tau)}{2\tau} = i\omega\hat{A}^\dagger(t), \quad (86)$$

such that

$$\hat{A}(t) = \hat{A}(0) \exp\left(-i\frac{t}{\tau} \sin^{-1}(\omega\tau)\right), \quad (87)$$

$$\hat{A}^\dagger(t) = \hat{A}^\dagger(0) \exp\left(i\frac{t}{\tau} \sin^{-1}(\omega\tau)\right), \quad (88)$$

where we used the fact that, for $t = 0$, the Heisenberg and Schrödinger pictures are equivalent: $\hat{A}(t = 0) = \hat{A} = \hat{A}(0)$ and $\hat{A}^\dagger(t = 0) = \hat{A}^\dagger = \hat{A}^\dagger(0)$, with \hat{A} and \hat{A}^\dagger independent of time. To obtain these equations we considered that, for the nonrelativistic case, there is neither creation nor annihilation of particles, such that we can impose restrictions on the frequencies in the phase term of the operators. For the creation operators, for example, the terms with negative frequencies—associated with antiparticles—are discarded.

We can observe that the *Number* and the Hamiltonian operators are not altered:

$$\begin{aligned} \hat{N} &= \hat{A}^\dagger(t)\hat{A}(t) = \hat{A}^\dagger(0)\hat{A}(0), \\ \hat{H} &= \hbar\omega \left(\hat{N} + \frac{1}{2} \right) = \hbar\omega \left(\hat{A}^\dagger(0)\hat{A}(0) + \frac{1}{2} \right). \end{aligned}$$

Thus, starting from these operators, we obtain for the symmetric case:

$$\begin{aligned} \hat{\mathbf{x}}(t) &= \hat{\mathbf{x}}(0) \cos\left[\frac{t}{\tau} \sin^{-1}(\omega\tau)\right] + \frac{\hat{\mathbf{p}}(0)}{m\omega} \sin\left[\frac{t}{\tau} \sin^{-1}(\omega\tau)\right] \\ \hat{\mathbf{p}}(t) &= \hat{\mathbf{p}}(0) \cos\left[\frac{t}{\tau} \sin^{-1}(\omega\tau)\right] - m\omega\hat{\mathbf{x}}(0) \sin\left[\frac{t}{\tau} \sin^{-1}(\omega\tau)\right]. \end{aligned}$$

which differ from the continuous case since the frequency ω here is replaced by a new frequency $\frac{1}{\tau} \sin^{-1}(\omega\tau)$ that, for $\tau \rightarrow 0$, tends to the continuous one. Also, there is now an upper limit for the possible oscillation frequencies given by $\omega \leq 1/\tau$. Above this frequency the motion becomes unstable, as observed in Figure 8.

The existence of a maximum limit for the frequency is equivalent to an upper limit for the energy eigenvalues given by $E_n = \left(n + \frac{1}{2}\right) \hbar\omega \leq \hbar/\tau$, which is equal to the upper limit obtained using Schrödinger's picture. Since

$$\frac{t}{\tau} \sin^{-1}(\omega\tau) \cong \omega + \frac{1}{3!} \omega^2 \tau^2 + O(\tau^4),$$

the difference expected in the behavior of the oscillator with respect to the continuous solution is quite small. For example, if we take the vibration frequency of the hydrogen molecule (H_2), we have that $\omega \sim 10^{14}$ Hz, while the term of the second order in τ is smaller than 10^{-3} Hz (if the analogy

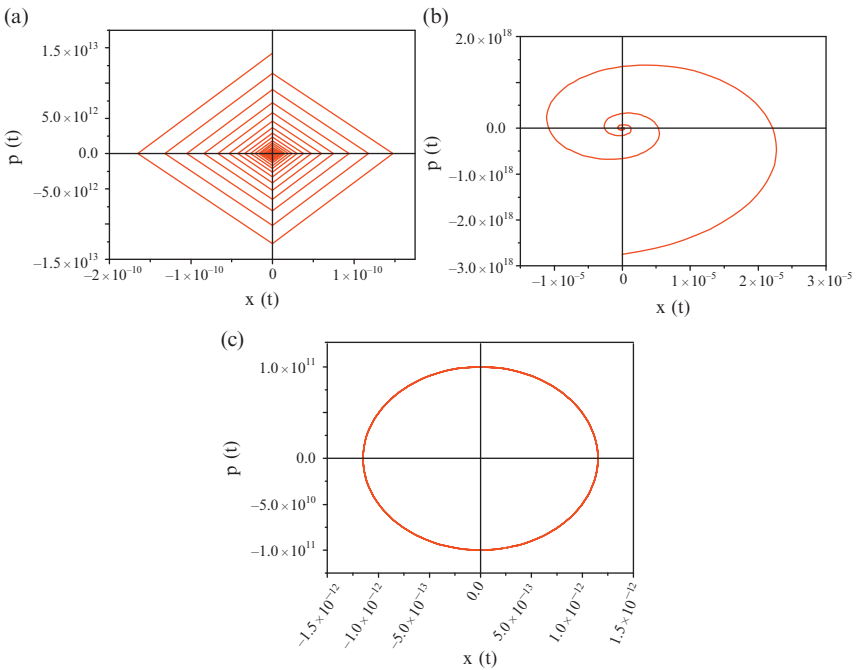


FIGURE 8 Phase space of the harmonic oscillator when $\omega > \frac{1}{\tau}$. In the discrete case, with time intervals multiples of τ : in inset (a) time is regarded as *intrinsically* discrete, so that in the picture only the points where the lines touch one another are meaningful. (b) If time is regarded as *intrinsically* continuous, inset (b) shows the behavior of the oscillator described by the discrete equations. In the actually continuous case, (c), no modification is expected with respect to the ordinary case, under the present hypothesis.

with the classical theory is valid, the chronon is expected to be smaller for more massive systems). In terms of average values we have, for the position operator

$$\langle \hat{\mathbf{x}}(t) \rangle = \langle \hat{\mathbf{x}}(t) \rangle_{\text{cont}} + \frac{\omega^2 \tau^2}{3!m} t \langle \hat{\mathbf{p}}(t) \rangle,$$

that the term of order τ^2 is expected to be considerably smaller than the mean value for the continuous case. At this point, the mean values are determined taking for the system a state composed of a superposition of stationary states. For the stationary states $|u_n\rangle$ themselves the mean values of $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$ are zero.

For the retarded case the solutions can be obtained using the time evolution operators for the Heisenberg equation (Appendix A). As expected, decaying terms appear in the resulting expressions. The creation and annihilation operators obtained for this case are then given by

$$\begin{aligned} \hat{A}(t) &= \hat{A}(0) \left[1 + i\omega\tau + \tau^2\omega^2\xi \right]^{-\frac{t}{\tau}} \approx \hat{A}(0) \exp(-i\omega\tau) \exp \left[- \left(\xi + \frac{1}{2} \right) \omega^2\tau t \right], \\ \hat{A}^\dagger(t) &= \hat{A}^\dagger(0) \left[1 - i\omega\tau + \tau^2\omega^2\xi \right]^{-\frac{t}{\tau}} \approx \hat{A}^\dagger(0) \exp(i\omega\tau) \exp \left[- \left(\xi + \frac{1}{2} \right) \omega^2\tau t \right], \end{aligned}$$

with ξ being a real positive factor. The relation $(\hat{A}^\dagger)^\dagger = \hat{A}$ continues to be valid but the *Number* operator and, consequently, the Hamiltonian, are no longer constant:

$$\begin{aligned} \hat{H}(t) &= \hat{A}^\dagger(0)\hat{A}(0) \left[(1 + \omega^2\tau^2\xi)^2 + \omega^2\tau^2 \right]^{-\frac{t}{\tau}} = \hat{H}(0) \exp \left[-2 \left(\xi + \frac{1}{2} \right) \omega^2\tau t \right], \\ \hat{N}(t) &= \hbar\omega \left\{ \hat{A}^\dagger(0)\hat{A}(0) \left[(1 + \omega^2\tau^2\xi)^2 + \omega^2\tau^2 \right]^{-\frac{t}{\tau}} + \frac{1}{2} \right\}. \end{aligned}$$

Taking into account the terms to the second order in τ , we derive that the oscillation frequencies also decay with time. These results are consistent with the fact that the system is emitting radiation, with the consequent reduction of its total energy. However, it is remarkable that the energy of the quanta associated with the creation and annihilation operators is not constant, even with a tiny variation rate. In the same way, when we calculate the position and momentum operators a damping factor is obtained.

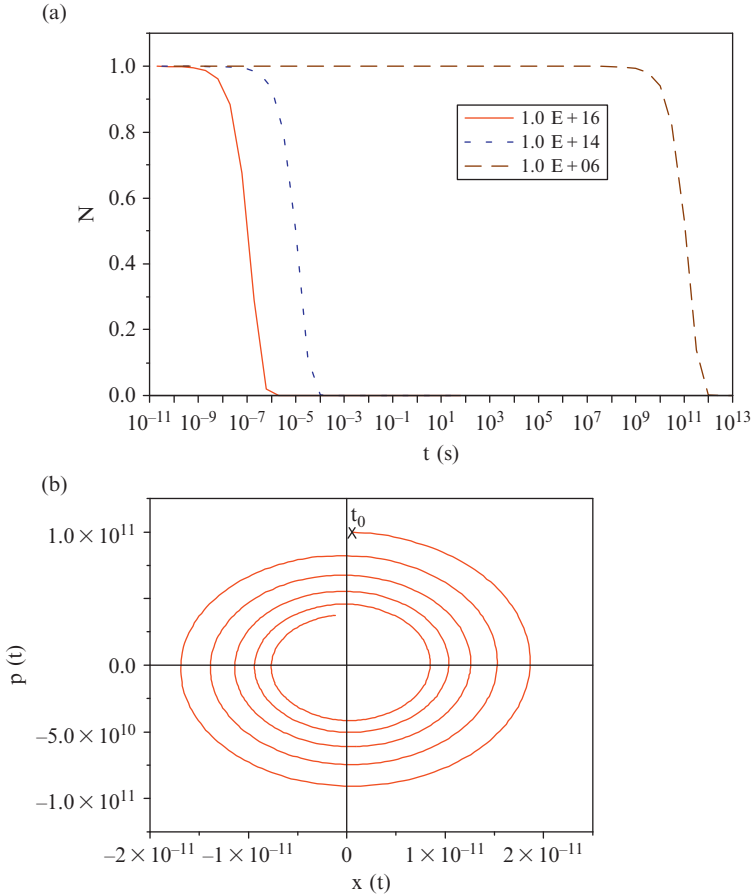


FIGURE 9 (a) Damping factors associated with the *Number* operator calculated for a few frequencies. (b) Damping of the oscillations for the harmonic oscillator described by the retarded equation.

Figure 9a shows the strange damping factor associated with the *Number* operator. This damping occurs within a period of time that is characteristic for each frequency, being slower and postponed for lower frequencies. Figure 9b shows the dampening of the oscillations as described by the retarded equation. Once the expressions for the position and momentum operators are determined, we obtain that, to first order in τ ,

$$\langle \hat{x}(t) \rangle = \left\{ \hat{x}(0) \cos(\omega\tau) + \frac{\hat{p}(0)}{m\omega} \sin(\omega\tau) \right\} \exp \left[- \left(\zeta + \frac{1}{2} \right) \omega^2 \tau t \right],$$

$$\langle \hat{x}(t) \rangle = \langle \hat{x}(t) \rangle_{\text{cont}} \exp \left[- \left(\zeta + \frac{1}{2} \right) \omega^2 \tau t \right].$$

Taking into account the higher-order terms, we can observe a small variation in the oscillation frequency just as observed in the symmetric case. The introduction of time-independent perturbations does not cause any additional variations aside from those found even in the continuous case. We note that the results obtained with this procedure are in agreement with those obtained following Schrödinger's picture.

4.5. Hydrogen Atom

The hydrogen atom is basically a system made up of two particles attracting each other through Coulombian force, which is therefore inversely proportional to the square of the distance between them. The basic Hamiltonian is denoted by

$$\hat{H}_0 = \frac{\hat{\mathbf{P}}^2}{2\mu} - \frac{e^2}{R}, \quad (89)$$

and is composed of the kinetic energy of the atom in the center-of-mass frame, and of the Coulombian electrostatic potential (μ is the reduced mass of the system electron-proton). A more complete description is obtained by adding correction terms (fine structure) to the Hamiltonian, including relativistic effects such as the variation of the electron mass with velocity and the coupling of the intrinsic magnetic moment of the electron with the magnetic field due to its orbit (spin-orbit coupling). There are also the hyperfine corrections that appear as a result of the interaction of the electron with the intrinsic magnetic moment of the proton and, finally, the *Lamb shift*, due to the interaction of the electron with the fluctuations of the quantized electromagnetic field. The Hamiltonian can finally be written as (Cohen-Tannoudji *et al.*, 1977)

$$\hat{H}_1 = m_e c^2 + \hat{H}_0 - \frac{\hat{\mathbf{P}}^4}{m_e^2 c^2 R^3} \hat{\mathbf{L}} \cdot \hat{\mathbf{S}} + \hat{H}_{\text{hf}} + \hat{H}_{\text{Lamb}}. \quad (90)$$

The introduction of the magnetic moment of the nucleus through the hyperfine correction causes the total angular momentum to be $\mathbf{F} = \mathbf{J} + \mathbf{I}$. The Hamiltonian does not depend explicitly on time such that, for the symmetric Schrödinger equation

$$i \frac{\hbar}{2\tau} [\Psi(\mathbf{x}, t + \tau) - \Psi(\mathbf{x}, t - \tau)] = \hat{H}_1 \Psi(\mathbf{x}, t), \quad (91)$$

we obtain, using the separation of variables, the following uncoupled equations:

$$\begin{aligned}\hat{H}_1\Phi(\mathbf{x}) &= E\Phi(\mathbf{x}) \\ i\frac{\hbar}{2\tau}[T(t+\tau) - T(t-\tau)] &= \hat{H}_1T(t),\end{aligned}$$

with the general solution

$$\Psi(\mathbf{x}, t) = \Phi(\mathbf{x}) \exp\left[-i\frac{t}{\tau} \sin^{-1}\left(\frac{\tau E}{\hbar}\right)\right]. \quad (92)$$

The difference related to the continuous case appears only in those aspects involving the time evolution of the states. Since the Hamiltonian is time independent, its eigenvalues are exactly the same as those obtained in the continuous case (Cohen-Tannoudji *et al.*, 1977):

$$E_{(n,j)} \approx m_0c^2 - \frac{1}{2n^2}m_e c^2 \alpha^2 - \frac{m_e c^2}{2n^4} \left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right) \alpha^4 + E_{\text{hf}} + E_{\text{Lamb}}.$$

A situation in which a difference between the two cases can appear is in taking into account the probabilities of transition between the eigenstates for an atom submitted to a time-dependent potential. In the discrete approach, it is possible to use the method of the equivalent Hamiltonian to obtain the transition probabilities. As mentioned previously (subsection 3.5), the problem is treated using the conventional approximate methods for time-dependent perturbations.

If we consider, for example, the nonrelativistic interaction of an atom with an electromagnetic field described by the vector potential $\mathbf{A}(\mathbf{x}, t)$, we have for the low-intensity limit, in the Coulomb gauge, the Hamiltonian

$$\hat{H}(t) = \hat{H}_1 - \hat{V}(t) = \hat{H}_1 - \frac{e}{m_e c} \hat{\mathbf{A}}(\hat{\mathbf{R}}, t) \cdot \hat{\mathbf{P}}, \quad (93)$$

where the potential term is taken as the perturbation. If we consider that the potential describes a monochromatic field of a plane wave, then

$$\mathbf{A}(\mathbf{x}, t) = A_0 \hat{\mathbf{e}} \left[\exp\left(i\omega \frac{\hat{\mathbf{n}} \cdot \mathbf{x}}{c} - i\omega t\right) + \exp\left(-i\omega \frac{\hat{\mathbf{n}} \cdot \mathbf{x}}{c} - i\omega t\right) \right], \quad (94)$$

where $\hat{\mathbf{e}}$ is the linear polarization of the field and $\hat{\mathbf{n}}$ is the propagation direction. The term depending on $(-i\omega t)$ corresponds to the absorption of a quantum of radiation $\hbar\omega$ and the $(i\omega t)$ term to stimulated emission. Let us assume that the system is initially in an eigenstate $|\Phi_i\rangle$ of the time-independent Hamiltonian. Keeping only the perturbations to the first order in $\hat{V}(t)$, we obtain that

$$c_n^1(t) = -\frac{i}{\hbar} \int_0^t \exp(i\omega_{ni}t') V_{ni}(t') dt',$$

where ω_{ni} in the discrete case is given by

$$\omega_{ni} = \frac{1}{\tau} \left[\sin^{-1} \left(\frac{\tau E_n}{\hbar} \right) - \sin^{-1} \left(\frac{\tau E_i}{\hbar} \right) \right].$$

Working with the absorption term, we get by contrast that

$$c_n^{(1)}(t) = \frac{ieA_0}{m_e^2 c \hbar} \langle \Phi_n | e^{i\omega \hat{\mathbf{n}} \cdot \mathbf{x}/c} (\hat{\mathbf{e}} \cdot \mathbf{p}) | \Phi_i \rangle \int_0^t \exp[i(\omega_{ni} - \omega)t'] dt'.$$

Thus, the probability of transition from the initial state $|\Phi_i\rangle$ to the final state $|\Phi_f\rangle$ is given by

$$P_{fi}(t) = |c_f^{(1)}(t)|^2 = \frac{e^2 |A_0|^2}{m_e^2 c^2 \hbar^2} \left| \langle \Phi_f | e^{i\omega \hat{\mathbf{n}} \cdot \mathbf{x}/c} (\hat{\mathbf{e}} \cdot \mathbf{p}) | \Phi_i \rangle \right|^2 \left| \int_0^t \exp[i(\omega_{fi} - \omega)t'] dt' \right|^2,$$

or

$$P_{fi}(t) = \frac{4e^2 |A_0|^2}{m_e^2 c^2 \hbar^2} \left| \langle \Phi_f | e^{i\omega \hat{\mathbf{n}} \cdot \mathbf{x}/c} (\hat{\mathbf{e}} \cdot \mathbf{p}) | \Phi_i \rangle \right|^2 \frac{\sin^2[(\omega_{fi} - \omega)t/2]}{(\omega_{fi} - \omega)^2},$$

so that the determination of the matrix elements of the spacial term, using the electric dipole approximation, provides the selection rules for the transitions. What is remarkable in this expression is the presence of a *resonance* showing a larger probability for the transition when

$$\omega = \omega_{fi} = \frac{1}{\tau} \left[\sin^{-1} \left(\frac{\tau E_f}{\hbar} \right) - \sin^{-1} \left(\frac{\tau E_i}{\hbar} \right) \right]. \quad (95)$$

This expression is formally different from the one obtained for the continuous approach. When we expand this expression in powers of τ , we obtain

$$\omega \approx \frac{E_f - E_i}{\hbar} + \frac{1}{6} \frac{E_f^3 - E_i^3}{\hbar^3} \tau^2. \quad (96)$$

The first term supplies the Bohr frequencies as in the continuous case; the second, the deviation in the frequencies caused by the introduction of the time discretization:

$$\Delta\omega_{fi} = \frac{1}{6} \frac{E_f^3 - E_i^3}{\hbar^3} \tau^2.$$

If we consider the chronon of the classical electron, $\tau \approx 6.26 \times 10^{-24}$ s, it is possible to estimate the deviation in the frequency due to the time discretization. Then, for the hydrogen atom,

$$\Delta\omega_{fi} \approx 2.289 \times 10^{-2} (E_f^3 - E_i^3).$$

If we take into account, for example, the transitions corresponding to the first lines of the series of Lyman and Balmer, that is, of the nondisturbed states $n = n_i \rightarrow n = n_f$ we have

n_i	n_f	ΔE (eV)	ν (Hz)	$\Delta\nu_D$ (Hz)
1	2	10.2	2.465×10^{15}	~ 10
1	3	12.1	2.922×10^{14}	~ 10
1	4	12.75	3.082×10^{14}	~ 10
2	3	1.89	4.566×10^{14}	< 1

where ΔE is the difference of energy between the states, ν is the frequency of the photon emitted in the transition, and $\Delta\nu_D$ is the frequency deviation due to the discretization. Such deviation is always considerably small. We must remember that the hyperfine corrections and those due to the Lamb shift are of the order of one GHz. For the transition $n = 1 \rightarrow n = 2$, for example, the correction due to the Lamb shift is approximately 1.06 GHz.

Larger deviations caused by the discretization occur for mono-electronic atoms with larger atomic numbers. For the first transition the deviation is approximately 90 Hz for the ${}^2\text{He}$, 1.1 kHz for the ${}^3\text{Li}$, and 420 kHz for the ${}^6\text{C}$. However, these deviations are still much smaller than the one due to the Lamb shift; that is also the case for the *muonic* atoms. For a muonic atom with a proton as the nucleus, using for the chronon a value derived from the classical expression for the electron ($\tau_\mu = 3.03 \times 10^{-26}$ s) the deviation is ~ 1.4 kHz for the transition $n = 1 \rightarrow n = 2$. For that transition the frequency of the emitted radiation is $\sim 4.58 \times 10^{17}$ Hz.

For the retarded equation, a difference with respect to the symmetric case is present in the time evolution of the states. The procedure is identical to the one used above, and the general solution is now given by

$$\Psi(\mathbf{x}, t) = \Phi(\mathbf{x}) \left[1 + i \frac{\tau E}{\hbar} \right]^{-t/\tau},$$

so that the transitions now occur with frequencies given by

$$\omega = \omega_{fi} = \frac{-i\hbar}{\tau} \left[\ln \left(1 + \frac{i\tau E_f}{\hbar} \right) - \ln \left(1 + \frac{i\tau E_i}{\hbar} \right) \right]. \quad (97)$$

As results from the characteristics of the retarded equation, this is a complex frequency. The real component of such frequency can be approximated by

$$\text{Re}(\omega_{fi}) \approx \frac{E_f - E_i}{\hbar} + \frac{1}{3} \frac{E_f^3 - E_i^3}{\hbar^3} \tau^2,$$

where the first term is the expression for the continuous case. For the particular transition $n = 1 \rightarrow n = 2$, the deviation due to the discretization is ~ 18 Hz.

The imaginary component, on the other hand, can be approximated by

$$\text{Im}(\omega_{fi}) \approx -\frac{i}{2} \frac{E_f^2 - E_i^2}{\hbar^2} \tau.$$

In the expression for the probability of transition, we have the magnitude of an integral involving the time dependency of the general solution. In this case, the characteristic damping causes the probability to tend to a fixed, nonzero value. An example of such behavior is shown in [Figure 10](#), which shows the variation of the time-dependent term between an initial instant $t_0 = 0$ and some hundred chronons later. To observe the decay of the amplitude factor we have used a larger value for the chronon, $\sim 10^{-18}$ s. The decay is slower when the chronon is of the order of the one we have been considering for the electron.

As can be observed, the effect of the time discretization on the emission spectrum of the hydrogen is extremely small. Using the expressions obtained above we can estimate that, in order for the effect of the time discretization to be of the same order of the Lamb shift, the chronon associated with the electron should be $\sim 10^{-18}$ s, far larger than the classical value (but close to the typical interval of the electromagnetic interactions!). In any case, it should be remembered that the Lamb shift measurements do not seem to be in full agreement with quantum electrodynamics (see, for example, [Lundeen and Pipkin, 1981](#)).

As we conclude this subsection, it is noteworthy that, for a time-independent Hamiltonian, the outputs obtained in the discrete formalism using the symmetric equation are very similar to those from the

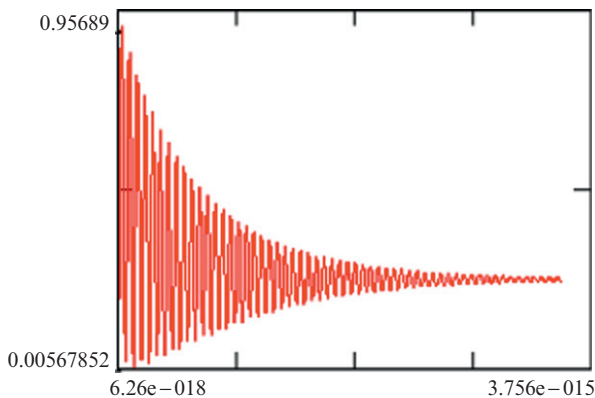


FIGURE 10 Behavior of the time-dependent component of the transition probability as a function of time (in seconds).

continuous case. For such Hamiltonians, as we know, the effect of the discretization appears basically in the frequencies associated with the time-dependent term of the wave function. As already observed, the difference in the time dependency is of the type

$$\exp[-iE_n(t - t_0)/\hbar] \rightarrow \exp\left[-i \sin^{-1}\left(\frac{\tau E_n}{\hbar}\right)(t - t_0)/\tau\right].$$

The discretization causes a change in the phase of the eigenstate, which can be quite large. The eigenfunctions individually describe stationary states, so that the time evolution appears when we have a linear combination of such functions, to describe the state of the system. This state evolves according to

$$|\Psi(t)\rangle = \sum_n c_n(0) \exp\left[-i \sin^{-1}\left(\frac{\tau E_n}{\hbar}\right)(t - t_0)/\tau\right] |\phi_n\rangle,$$

considering that $H|\phi_n\rangle = E_n|\phi_n\rangle$ is the eigenvalue equation associated with the Hamiltonian.

When the stationary states of a particle under, for example, 1D squared potentials, are studied, the same reflection and transmission coefficients and the same tunnel effect are obtained, since they are calculated starting from the stationary states. When we consider a linear superposition of these stationary states, building a wave packet, the time-dependent terms must be taken into account, resulting in some differences with respect to the continuous case. Some attempts have been made to determine significant measurable differences between the two formalisms (compare [Wolf, 1987a,b](#) and [Wolf, 1989a,b](#)), but no encouraging case has been found yet.

5. DENSITY OPERATORS AND THE COARSE-GRAINING HYPOTHESIS

5.1. The “Coarse-Graining” Hypothesis

First, it is convenient to present a brief review of some topics related to the introduction of the *coarse-grained* description of a physical system. This hypothesis will then be used to obtain a discretized form of the Liouville equation, which represents the evolution law of the density operators in the usual QM.

An important remark is that the introduction of a fundamental interval of time is perfectly compatible with a coarse-grained description. The basic premise of such a description, in statistical physics, is the impossibility of a precise determination of the position and momentum of each

particle forming the system, at a certain instant of time. Let us consider, for the sake of simplicity, a system composed of N similar pointlike particles, each with three degrees of freedom described by the coordinates (q_1, q_2, q_3) . We can associate with this ensemble of particles an individual phase space (named μ -space) defined by the six coordinates $(q_1, q_2, q_3; p_1, p_2, p_3)$ so that the system as a whole is represented by a crowd of points in this space.

Since the macroscopic observation is unable to precisely determine the six coordinates for each particle, let us assume that it is possible to know only if a given particle has its coordinates inside the intervals $(q_i + d_{qi})$ and $(p_i + d_{pi})$, with $i = 1, 2, 3$. In order to describe the state of the system in the μ -space, we divide it into cells corresponding to the macroscopic uncertainties δ_{qi} and δ_{pi} , each one occupying in the μ -space a volume defined as

$$w_i = \delta_{q_1} \delta_{q_2} \delta_{q_3} \delta_{p_1} \delta_{p_2} \delta_{p_3}. \quad (98)$$

These cells must be sufficiently small related to the macroscopically measurable dimensions but also sufficiently large to contain a great number of particles.

When considering the system as a whole, its macroscopic state is given by a collection of points n_i corresponding to the number of particles inside each cell. Now, if we take into account the $6N$ -dimensional phase space Γ , in which each of the states assumed by the system is represented by a point, to each configuration n_i corresponds in Γ a cell with volume given by

$$(\delta V)_\Gamma = \prod_{n=1}^N (w_i)^{n_i}.$$

Considering that the permutation of the particles inside the cells of the Γ space does not change the macroscopic state of the system, then to each collection of numbers n_i corresponds a volume Ω_n in the Γ space¹⁹ given by

$$W(\Omega_n) = \frac{N!}{\prod_i n_i!} \prod_i (w_i)^{n_i} \quad \left(\sum_i n_i = N \right).$$

The state of the system is determined by the *star* occupied by the representative point of the system in the Γ space. This way, macroscopically, it is only possible to distinguish in which "star" the system is located, such that any point in this star corresponds to the same macroscopic state. When we consider a system that is not in equilibrium, a change in its macroscopic state can be observed only when the point describing the

¹⁹ Jancel (1969) calls it a *star*.

system changes star. The crossing time is small but finite. During this period of time the macroscopic state of the system does not change, notwithstanding the fact that its microscopic state is continuously changing.

Thus, from the point of view of statistical physics, the introduction of a fundamental interval of time appears very naturally. That is still more significant when we remember that the predictions of QM are always obtained as mean values of observables. The uncertainty relations, according to the usual interpretation of QM—the Copenhagen interpretation—are independent of the arguments above. If we accept that they play a fundamental role in the microscopic world—and this is postulated by Copenhagen—then the concept of chronon, as a fundamental interval of time, must be related to them.

5.2. Discretized Liouville Equation and the Time-Energy Uncertainty Relation

An attempt to establish a relationship between the chronon and the time-energy uncertainty relation has been put forward by [Bonifacio \(1983\)](#), extending the *coarse-graining* hypothesis to the time coordinate. In conventional QM the density operator evolves according to the Liouville-von Neumann equation:

$$\frac{\partial \hat{\rho}}{\partial t} = -i\mathcal{L}\hat{\rho}(t) = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}], \quad (99)$$

where \mathcal{L} is the Liouville operator. One can immediately observe that, if H is time-independent, the solution is given by

$$\hat{\rho}(T) = \exp\left(-i\frac{H}{\hbar}T\right)\hat{\rho}(0)\exp\left(i\frac{H}{\hbar}T\right), \quad (100)$$

which gives the time evolution of the density operator starting from an initial time t_0 , such that $T = t - t_0$ is the evolution time.

When we build a coarse-grained description of the time evolution, by introducing a graining of value τ such that the evolution time is now given by $T = k\tau$ ($k = 1, 2, \dots, \infty$), the resulting density operator ρ does not satisfy the continuous [Eq. \(99\)](#) but a discretized form of it given by

$$\frac{\hat{\rho}(t) - \hat{\rho}(t - \tau)}{\tau} = i\mathcal{L}\hat{\rho}(t), \quad (101)$$

with $t = k\tau$, which reduces to the Liouville–von Neumann equation when $\tau \rightarrow 0$. In the energy representation $|n\rangle$, once satisfied certain conditions that ensure that $\rho(k)$ is a density operator, [Eq. \(101\)](#) rules for ρ an evolution that preserves trace, obeys the semigroup law, and is an irreversible evolution toward a stationary diagonal form. In other words, we observe

a reduction of state in the same sense as in the measurement problem of QM. This reduction is not instantaneous and depends on the characteristic value τ :

$$\rho(t) \xrightarrow{t \rightarrow \infty} \sum_n \rho_{nn}(0) |n\rangle \langle n|.$$

It is important to observe that *the nondiagonal terms tend exponentially to zero* according to a factor which, to the first order, is given by

$$\exp \left| \frac{-\omega_{nm}^2 \tau t}{2} \right|.$$

Thus, the reduction to the diagonal form occurs provided we have a finite value for τ , no matter how small, and provided we do not have $\omega_{nm}\tau \ll 1$ for every n and m , where $\omega_{nm} = (E_n - E_m)/\hbar$ are the transition frequencies between the different energy eigenstates. This latter condition is always satisfied for systems not bounded.

These results, together with an analysis of the discrete Heisenberg equation defined in terms of the average values of observables

$$\bar{A}(t) = \text{Tr}(\rho(t)\hat{A})$$

in the coarse-grained description, suggest an interpretation of τ in terms of the uncertainty relation $\Delta E \Delta t \geq \hbar/2$ such that τ is a characteristic interval of time satisfying the inequality

$$\tau \geq \tau_E \equiv \frac{\hbar}{2\Delta E} \quad \text{with} \quad \Delta E = \sqrt{\langle H^2 \rangle - \langle H \rangle^2}, \quad (102)$$

so that the mathematical meaning of the time-energy uncertainty relation is that of fixing a lower limit for the time interval within which the time evolution can be described. Thus, "...the coarse-grained irreversibility would become a necessary consequence of an intrinsic impossibility to give an instantaneous description of time evolution due to the time-energy uncertainty relation" (Bonifacio, 1983).

Since the density operator, in the energy representation, tends to a diagonal form, it is tempting to apply it to the measurement problem. We can also observe that, even without assuming any coarse-graining of time, namely, without using (Recami and Farias, 2009) any statistical approaches as Bonifacio's, the reduction to a diagonal form results straightforwardly from the discrete Liouville equation and some asymptotic conditions regarding the behavior of the solution, once satisfied (Bonifacio and Caldirola, 1983, 1982) the inequality $\omega_{nm} \tau \ll 1$. (See also Ghirardi and Weber, 1984; Ghirardi, Rimini, and Weber, 1985)

The crucial point from which both the decay of the nondiagonal terms of the density operator and the very discrete Liouville equation are

derived, is that the time evolution operator obtained from the coarse-grained description *is not a unitary operator*. This way, the operator

$$\hat{V}(t = k\tau, t = 0) = \frac{1}{\left(1 + \frac{i\tau\hat{L}}{\hbar}\right)^k}, \quad (103)$$

like all the nonunitary operators, does not preserve the probabilities associated with each of the energy eigenstates that make up the expansion of the initial state in that basis of eigenstates. We must recall that the appearance of nonunitary time evolution operators is not associated with the coarse-grained approach only, since the operators also result from the discrete Schrödinger equations.

5.3. Measurement Problem in Quantum Mechanics

Let us apply the discrete formalism introduced in the previous subsection to the measurement problem. Using a quite general formalization, we can describe the measurement process taking advantage of the properties observed for the evolution of the density operator as determined by the discrete Liouville-von Neumann equation.²⁰

When speaking of measurement, we must keep in mind that, in the process, an object \mathcal{O} , of which we want to measure a dynamic variable R , and an apparatus \mathcal{A} , which is used to perform such measurement, are involved. Let us suppose that \hat{R} is the operator associated with the observable R , with an eigenvalue equation given by $\hat{R}|r\rangle = r|r\rangle$ and defines a complete basis of eigenstates. Thus, considered by itself, any possible state of the object can be expanded in this basis:

$$|\Psi\rangle_0 = \sum_r c_r |r\rangle_0. \quad (104)$$

With respect to the apparatus \mathcal{A} , we are interested only in its observable A , whose eigenvalues α represent the possible values indicated by a pointer. In addition, let its various internal quantum numbers be labeled by an index n . These internal quantum numbers are useful to specify a complete basis of eigenvectors associated with the apparatus:

$$\hat{A}|\alpha, n\rangle_A = \alpha|\alpha, n\rangle_A. \quad (105)$$

Now, let us suppose that the apparatus is prepared in an initial state given by $|0, n\rangle_A$, that is, in the initial state the value displayed is zero. The interaction between the two systems is introduced by means of the

²⁰ We follow closely the description exhibited in Recami and Farias (2009) and in Ballentine (1986).

time evolution operator and is such that there is a correlation between the value of r and the measure α_r . We consider a quite general situation to deal with the measurement process itself. First, let us consider the following pure state of the system *object + apparatus* ($O + A$):

$$|\Psi_n^i\rangle = |\Psi\rangle_0|0, n\rangle_A. \quad (106)$$

The evolution of this state, in the continuous description, using the evolution operator, is given by

$$\hat{U}(t, t_0)|\Psi\rangle_0|0, n\rangle_A = \sum_r c_r |\alpha_r; r, n\rangle = |\Psi_n^f\rangle \quad (107)$$

which is a coherent superposition of macroscopically distinct eigenstates, each one corresponding to a different measure α_r . The major problem for the Copenhagen interpretation results from the fact that it considers the state $|\Psi_n^i\rangle$ as associated with a single system. A pure state provides a complete and exhaustive description of an individual system. Thus, the coherent superposition above describes a single system so that, at the end of the interaction that settles the measurement, the display should not show a well-defined output since Eq. (107) describes a system that is a superposition of all its possible states.

However, we know from experience that the apparatus always displays a single value as the output of the measurement. It is this disagreement between observation and the description provided by the formalism, when interpreted according to Copenhagen, that results in the necessity of introducing the postulate of the reduction of the vector state

$$|\Psi_n^f\rangle \longrightarrow |\alpha_{r_0}; r_0, n\rangle,$$

where r_0 is the value displayed by the apparatus.

This fact has been considered by many as a problem for the usual interpretation of QM (Wigner, 1963; Ballentine, 1986, 1970). The attempts to find a solution, in the context of different interpretations, have been numerous, from the *many-worlds interpretation*, proposed by Everett and Wheeler (Everett, 1957; Wheeler, 1957) to the measurement theory by Daneri *et al.* (1962),²¹ in which the reduction of the quantum state is described as a process triggered by the appearance of aleatory phases in the state of the apparatus, simply because of its interaction with the elementary object. The approach introduced here is—by contrast—somewhat simpler.

²¹ See also Caldirola (1974), Lanz *et al.* (1971), Ludwig (1953), and George *et al.* (1972).

As an initial state in the measurement process, let us consider a mixed state for the composite system $\mathcal{O} + \mathcal{A}$,

$$\rho^i = \sum_n C_n |\Psi_n^i\rangle \langle \Psi_n^i|, \quad (108)$$

where C_n is the probability associated with each of the states $|\Psi_n^i\rangle$. Such probability is, as in the classical physics, an ignorance probability, that is, it is not intrinsic to the system. In the continuous case, when we apply the time evolution operator to that density operator, we get a final state given by

$$\rho^f = \hat{U} \rho^i \hat{U}^\dagger = \sum_n C_n |\Psi_n^i\rangle \langle \Psi_n^i| \quad (109)$$

$$\rho^f = \sum_{r_1, r_2} c_{r_1}^* c_{r_2} \sum_n C_n \{|\alpha_{r_1}; r_1, n\rangle \langle \alpha_{r_1}; r_1, n|\}, \quad (110)$$

so that the presence of nondiagonal terms corresponds to a coherent superposition of states. In this case, the postulate of the reduction of the quantum state is connected with the nondiagonal terms of the density operator. It is usually postulated that when a measurement is carried out on the system, the nondiagonal terms tend instantaneously to zero. Since in the continuous case the time evolution of the state results from the application of a unitary operator, which preserves the pure state condition $\hat{\rho}^2 = \hat{\rho}$, it is impossible to obtain the collapse of the pure state from the action of such an operator. In the diagonal form the density operator describes an incoherent mixture of the eigenstates of \hat{A} , and the indetermination regarding the output of the measurement is a sole consequence of our ignorance about the initial state of the system.

In the discrete case (Recami and Farias, 2009), which has the time evolution operator given by Eq. (103), with the interaction between apparatus and object embedded in the Hamiltonian H , the situation is quite different. The main cause of such difference is the fact that the time evolution operator is not unitary. Let us consider the energy representation, describing the eigenvalue equation of the Hamiltonian as $H|n\rangle = E_n|j\rangle$ so that the eigenstates $|n\rangle$ are the states with defined energy. From the formalism of the density matrices we know that when the operator \hat{R} is diagonal in the energy representation, then when calculating the expected value of the observable, we do not obtain the interference terms describing the *quantum beats* typical of a coherent superposition of the states $|n\rangle$.

Because the time evolution operator is a function of the Hamiltonian and, therefore, commutes with it, the basis of the energy eigenstates is also a basis for this operator. We can now use a procedure identical to the one applied by Bonifacio (1983), and consider the evolution of the system in

this representation. Thus, the operator \hat{V} ($t = k\tau$, $t = 0$) takes the initial density operator ρ^i to a final state for which the nondiagonal terms decay exponentially with time:

$$\rho_{rs}^f = \langle r|V(t = k\tau, t = 0)|s\rangle = \frac{\rho_{rs}^i}{(1 + i\omega_{rs}\tau)^{t/\tau}}, \quad (111)$$

with

$$\omega_{rs} = \frac{1}{\hbar}(E_r - E_s) = \frac{1}{\hbar}\Delta E_{rs}. \quad (112)$$

Equation (111) can be written as

$$\rho_{rs}(t) = \rho_{rs}(0)e^{-\gamma_{rs}t}e^{-i\nu_{rs}t} \quad (113)$$

such that

$$\gamma_{rs} = \frac{1}{2\tau} \ln(1 + \omega_{rs}^2 \tau^2), \quad (114)$$

$$\nu_{rs} = \frac{1}{\tau} \tan^{-1}(\omega_{rs}\tau). \quad (115)$$

We can observe directly that the nondiagonal terms tend to zero with time and the decay is faster the larger the value of τ , which here is an interval of time related to the entire system $\mathcal{O} + \mathcal{A}$. If we keep in mind that in the coarse-grained description the value of the time interval τ originates from the impossibility of distinguishing between two different states of the system, we must remember that the system $\mathcal{O} + \mathcal{A}$ is not an absolutely quantum system. That means that τ could be significantly larger, implying a much faster damping of the nondiagonal terms of the density operator (Figure 11). We then arrive at a process like the one of the reduction of the quantum state, even if in a rudimentary formalization. This result seems encouraging for future researches on such important and controversial subject.

Some points must be noted out from this brief approach of the measurement problem. First, this result does *not* occur when we use the time evolution operators obtained directly from the retarded Schrödinger equation. The dissipative character of that equation causes the norm of the state vector to decay with time, also leading to a nonunitary evolution operator. However, this operator is such that, in the definition of the density operator we obtain damping terms that are effective even for the diagonal terms. This point, as well as the question of the compatibility between Schrödinger's picture and the formalism of the density matrix, are analyzed in Appendix A. As the composite system $\mathcal{O} + \mathcal{A}$ is a complex system, it is suitably described by the coarse-grained description, so that

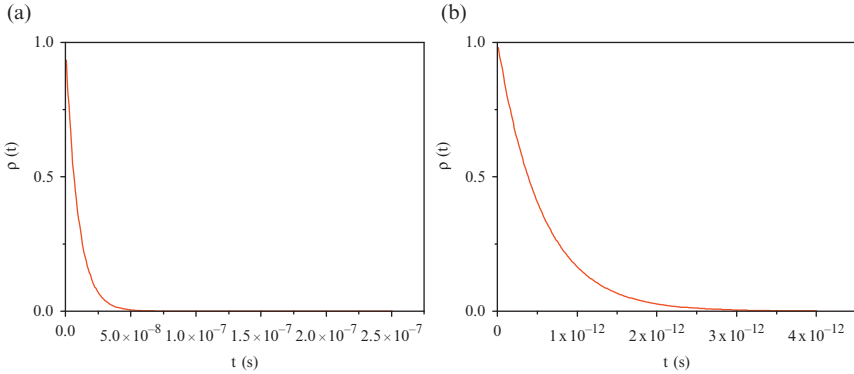


FIGURE 11 Vanishing in time of the nondiagonal terms of the density operator for two different values of τ . For both cases we have used $\Delta E = 4$ eV. (a) Slower damping for $\tau = 6.26 \times 10^{-24}$ s; (b) faster damping for $\tau = 10^{-19}$ s.

the understanding of the relationship between the two pictures can be useful to gain a deeper insight on the processes involved.

Notwithstanding the simplicity of the approach, we could also observe the intrinsic relation between *measurement process* and *irreversibility*. The time evolution operator \hat{V} meets the properties of a semigroup, so that it does not necessarily possess an inverse; and noninvertible operators are related to irreversible processes. In a measurement process, in which the object is lost just after the detection, we have an irreversible process that could very well be described by an operator such as \hat{V} .

Finally, it is noteworthy that the measurement problem is controversial even regarding its mathematical approach. In the simplified formalization introduced previously, we did not include any consideration beyond those common to the quantum formalism, allowing an as clear as possible individualization of the effects of the introduction of a fundamental interval of time in the approach to the problem.

The introduction of a fundamental interval of time in the description of the measurement problem makes possible a simple but effective formalization of the state-reduction process. Such behavior is observed only for the retarded case. When we take into account a symmetric version of the Liouville-von Neumann equation, the solution is given by

$$\rho_{nm}(t) = \rho_{nm}(0) \exp \left\{ -\frac{it}{\tau} \sin^{-1} \left[\frac{\tau}{\hbar} (E_n - E_m) \right] \right\},$$

where the diagonal elements do not change with time and the nondiagonal elements have an oscillatory behavior. This means that the symmetric equation is not suitable to describe a measurement process, and this is an important distinction between the two formulations: actually, only the retarded one describes dissipative systems.

It is important to stress that the retarded case of direct discretization of the Liouville-von Neumann equation results in the same equation obtained via the coarse-grained description. This led us to consider our equation as the basic equation to describe complex systems, which is always the case when a measurement process is involved. Our results, moreover, are independent of any statistical approach.

6. CONCLUSIONS

In this paper we attempted to gain a better insight into the applicability of the various distinct formalisms obtained when performing a discretization of the continuous equations. For example, what kind of physical description is provided by the retarded, advanced, and symmetric versions of the Schrödinger equation? This can be achieved by observing the typical behavior of the solutions obtained for each case and, particularly, attempting to derive these equations from Feynman's approach. We then have an advanced equation that describes a system that absorbs energy from the environment. We can imagine that, in order to evolve from one instant to another, the system must absorb energy, and this could justify the fact that, by using Feynman's approach with the usual direction of time, we can obtain only the advanced equation. The propagator depends only on the Hamiltonian because it is independent of the wave function that describes the initial state. Thus, it describes a transfer of energy to the system.

The retarded equation is obtained by a time reversion, by an inversion of the direction of the propagator, that is, by inverting the flux of energy. The damping factor characteristic of the retarded solutions refers to a system that is continuously releasing energy into the environment. Thus, both the retarded and the advanced equations describe *open systems*.

Finally, the symmetric equation describes a system in an energy equilibrium with the environment. Thus, the only way to obtain stationary states is by using the symmetric equation.

Regarding the nature of such an energy, it can be related to the very evolution of the system. It can be argued that a macroscopic time evolution is possible only if there is some energy flux between the system and the environment. The states described by the symmetric equation are basically equilibrium states, without net dissipation or absorption of energy by the system as a whole. We can also conceive of the symmetric equation as describing a closed system, which does not exchange energy with the external world.

On the other hand, when a comparison is made with the classical approach, we can speculate that the symmetric equation ceases to be valid when the interaction with the environment changes rapidly within a chronon of time. Thus, phenomena such as the collision of highly

energetic particles require the application of the advanced or retarded equations. The decay of the norm associated with the vector states described by the retarded equation would indicate the very decay of the system, a system abandoning its initial "equilibrium state." The behavior of the advanced equation would indicate the *transition* of the system to its final state. This speculation suggests another interpretation, closer to the quantum spirit. We could consider the possible behavior of the system as described by all three equations. However, the ordinary QM works with averages over ensembles, which is a description of an ideal, purely mathematical reality. The point is that if we accept the ergodic hypothesis, such averages over ensembles are equivalent to *averages over time*. The fact is that the quantum formalism always works with average values when dealing with the real world. When the potentials involved vary slowly with respect to the value of the chronon of the system, which means a long interaction time, the contributions due to the transient factors from the retarded and advanced equations compensate for each other and cancel out. Then, on the average, the system behaves according to the symmetric equation. On the contrary, when the potentials vary strongly within intervals of time of the order of the chronon, we do *not* have stationary solutions. *The discrete formalism describes such a situation by making recourse, during the interaction, to the transient solutions, which will yield the state of the system after the interaction. Afterward, the system will be described again by a symmetric solution.*

The most conservative quantum interpretation would be that of believing that only the symmetric equation describes a quantum system. During the interaction process the theory does not provide any description of the system, pointing only to the possible states of the system *after* the transient period. The description of the interaction would demand one more ingredient: the knowledge of the interaction process (which would imply an additional theoretical development, for example, the determination of an interaction model).

In addition to the question of the physical meaning of the discretized equations, that is, to the type of physical description underlying it, there is the question of the time evolution of the quantum states. The Schrödinger equations describe the evolution of a wave function, with which an amplitude of probability is associated. An analogy with the electron theory leads us to the supposition that this wave function does not react instantaneously to the external action, but reacts after an interval of time that is characteristic of the described system. In discrete QM, the justification of the non-instantaneous reaction comes from the fact that the uncertainty principle prevents a reaction arbitrarily close to the action application instant (Wolf, 1990a,b,c,d, 1992a,b, 1994). Such uncertainty could be related to the perturbation caused by the Hamiltonian on the state of the system, resulting in an uncertainty relation like the

Mandelstam-Tamm time-energy correlation (Fock, 1962): a time evolution in which the macroscopic state of the system *leaps discontinuously* from one instant to the other. Therefore, the *quantum jumps* appear not only in the measurement process but are an intrinsic aspect of the time evolution of the quantum system.²² The difference, in our case, is that the jump does not take the system suddenly out of the quantum state with which it was endowed, but only determines the *evolution* of that state.

Another aspect characteristic of the discrete approach is the existence of an upper limit for the eigenvalues of the Hamiltonian of a bounded system. The description of a free particle showed the existence of an upper limit for the energy of the eigenfunctions composing the wave packet that describes the particle, but this limit does not imply an upper value for the energy of the particle. The existence of this limiting value determines the Hamiltonian eigenvalue spectrum within which a normalization condition can hold. Once that value is exceeded, a transition to the internal excited states of the system occurs. As an example, this allowed us to obtain the *muon* as an excited internal state of the electron.

It must be remarked the *nonlinear* character of the relation between energy and oscillation frequency of a state, and the fact that the theory is intrinsically *nonlocal*, as can be inferred from the discretized equations. It must also be stressed that the theory described in this chapter is nonrelativistic.

Finally, it must be emphasized that the symmetric form of the discrete formalism reproduces *grosso modo* the results of the continuous theory. The effects of the introduction of a fundamental interval of time are evident in the evolution of the quantum systems, but they are—in general—extremely tiny. There have been some attempts to find physical situations in which measurable differences between the two formalisms can be observed, but until now with little success.²³ A possibility is that this could be afforded by exploiting the consequences of the phase shifts caused by the discretization (see subsections 4.2 and 4.3). Regarding the justifications for introducing a fundamental interval of time, let us recall Bohr's (1935) reply to the famous 1935 paper by Einstein, Podolski, and Rosen (Einstein *et al.*, 1935): "The extent to which an unambiguous meaning can be attributed to such an expression as *physical reality* cannot of course be deduced from *a priori* philosophical conceptions, but . . . must be founded on a direct appeal to experiments and measurements." Considering time as continuous may be regarded as a criticizable philosophical position since, at the level of experiments and measurements, nature seems to be discrete.

More important is to recall that the new formalism allows not only the description of the stationary states, but also a space-time description of

²² For the controversy involving quantum jumps see, for example, Schrödinger (1952) and Bell (1987).

²³ Several systems have been analyzed in Wolf (1987a,b, 1989a,b, 1990a,b,c,d, 1992a).

transient states. The retarded formulation yields a natural quantum theory for *dissipative* systems. Significantly, it leads to a simple solution of the *measurement problem* in QM. Such interesting problems await future attention.

APPENDICES

A. Evolution Operators in the Schrödinger and Liouville–von Neumann Discrete Pictures

In applying the formalism introduced in the previous sections to the measurement problem, the requirement of the existence of a well-defined evolution operator is evident. By *well-defined* we mean that, as in the continuous case, a unitary operator that satisfies the properties of a group.

In the continuous case, when the Hamiltonian is independent of time, the time evolution operator has the form

$$\hat{U}(t, t_0) = \exp\left(-i(t - t_0)\hat{H}/\hbar\right)$$

and is a unitary operator that satisfies the condition that \hat{H} be Hermitean. In the continuous case, by definition, every observable is represented by a Hermitean operator. An operator is unitary when its Hermitean conjugate is equal to its inverse, such that

$$\hat{A}^\dagger \hat{A} = \hat{A} \hat{A}^\dagger = 1.$$

Another important aspect regarding a unitary operator is related to the probability conservation. In other words, if the initial state is normalized to 1, it will keep its norm for all subsequent times. The evolution operator does not change the norm of the states on which it operates. Thus, we know beforehand that the evolution operators associated with the retarded and advanced discretized Schrödinger equations are not unitary operators.

A.1. Evolution Operators in the Schrödinger Picture

For the discretized Schrödinger equation the discrete analog of the time evolution operator can be obtained easily. Let us initially consider the symmetric equation, which is the closest to the continuous description. After some algebraic manipulation the evolution operator can be written as

$$\hat{U}(t, t_0) = \exp\left[-\frac{i(t - t_0)}{\tau} \sin^{-1}\left(\frac{\tau\hat{H}}{\hbar}\right)\right],$$

so that

$$|\Psi(x, t)\rangle = \hat{U}(t, t_0)|\Psi(x, t_0)\rangle = \exp\left[-\frac{i(t-t_0)}{\tau} \sin^{-1}\left(\frac{\tau\hat{H}}{\hbar}\right)\right]|\Psi(x, t_0)\rangle.$$

Thus, if the eigenvalue equation of the Hamiltonian is given by

$$\hat{H}|\Psi(x, t_0)\rangle = E|\Psi(x, t_0)\rangle,$$

we have that

$$|\Psi(x, t)\rangle = \exp\left[-\frac{i(t-t_0)}{\tau} \sin^{-1}\left(\frac{\tau E}{\hbar}\right)\right]|\Psi(x, t_0)\rangle.$$

Since \hat{H} is a Hermitean operator, the evolution operator for the symmetric equation is also Hermitean. However, the existence of a limit for the possible values of the eigenvalues of \hat{H} implies that, beyond such threshold, *the evolution operator is no longer Hermitean*. In fact, if we consider that beyond the threshold the operator \hat{H} has the form

$$\hat{H} = \hat{v} + i\hat{k},$$

where \hat{v} and \hat{k} are Hermitean operators, we obtain in the continuous approach the same results obtained in the discrete case. One of the characteristics of a non-Hermitean operator is the fact that it does not conserve the norm of the state on which it acts.

For the retarded equation, the evolution operator is given by

$$\hat{U}(t, t_0) = \left[1 + \frac{i}{\hbar} \tau \hat{H}\right]^{-(t-t_0)/\tau}, \quad (116)$$

such that, in the limit $\tau \rightarrow 0$,

$$\lim_{\tau \rightarrow 0} \left[1 + \frac{i}{\hbar} \tau \hat{H}\right]^{-(t-t_0)/\tau} = e^{-\frac{i}{\hbar}(t-t_0)\hat{H}},$$

which is an expression known as the *Trotter equality*. Taking the conjugate Hermitean operator \hat{U}^\dagger we can verify that this operator is not unitary. In the basis of eigenstates of \hat{H} , we can verify that

$$\langle n|\hat{U}^\dagger\hat{U}|n\rangle = \left[1 + \frac{\tau^2 E_n^2}{\hbar^2}\right]^{-(t-t_0)/\tau},$$

is not equal to 1. This explains why the probabilities are not conserved for the solutions of the retarded equation. In addition, as the evolution operator for the advanced equation is given by

$$\hat{U}(t, t_0) = \left[1 - \frac{i}{\hbar} \tau \hat{H} \right]^{(t-t_0)/\tau},$$

it can be verified that the formal equivalence between the two equations is obtained by the inversion of the time direction and of the sign of the energy. In the relativistic case, this is understandable if we remember that, if a transformation changes the sign of the time component of a coordinate four-vector, then it also changes the sign of the energy, which is the corresponding element of the energy-momentum four-vector. Then the retarded equation describes a particle endowed with positive energy traveling forward in time, and the advanced equation describes an object with negative energy traveling backward in time, that is, an *antiparticle* (Recami, 1978; Recami and Rodrigues, 1982; Recami *et al.*, 1983; Pavsic and Recami, 1982).

A.2. Evolution Operator in the Density Matrix Picture

For the sake of simplicity, let $|\psi(t)\rangle$ be a pure state. The density of states operator is defined as

$$\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)|.$$

It can be shown that such operator evolves according to the following dynamic laws. For the retarded case,

$$\Delta_R \hat{\rho}(t) = \frac{1}{i\hbar} \left[\hat{H}(t), \hat{\rho}(t) \right] - \frac{\tau}{\hbar^2} \hat{H}(t) \hat{\rho}(t) \hat{H}(t);$$

for the advanced case,

$$\Delta_A \hat{\rho}(t) = \frac{1}{i\hbar} \left[\hat{H}(t), \hat{\rho}(t) \right] + \frac{\tau}{\hbar^2} \hat{H}(t) \hat{\rho}(t) \hat{H}(t);$$

and, finally, for the symmetric case,

$$\Delta \hat{\rho}(t) = \frac{1}{i\hbar} \left[\hat{H}(t), \hat{\rho}(t) \right].$$

We can thus observe that the retarded and the advanced equations cannot be obtained by a direct discretization of the continuous Liouville-von Neumann equation. Such formal equivalence occurs only for the symmetric case. Taking into account the retarded case, we can obtain the equivalent time evolution operator as

$$\hat{V}(t, t_0) = \frac{1}{\left[1 + \frac{i\tau}{\hbar} \hat{\mathcal{L}} + \frac{\tau^2}{\hbar^2} \hat{H} \dots \hat{H} \right]^{(t-t_0)\tau}}. \quad (117)$$

Of note, this operator is different from the one obtained from the coarse-grained approach,

$$\hat{V}_{\text{CG}}(t, t_0) = \frac{1}{\left[1 + \frac{i\tau}{\hbar} \hat{\mathcal{L}}\right]^{(t-t_0)\tau}}. \quad (118)$$

and it is *not* unitary as well. Quantity \hat{V}_{CG} is defined as having the properties of a semigroup: It does not necessarily have an inverse but possesses the other group properties such as commutativity and existence of an identity (in addition to the translational invariance of the initial condition).

We can conclude from the difference between the two operators that, apparently, the descriptions clash. In the coarse-grained approach the starting point was the continuous Liouville-von Neumann equation and, by introducing the graining of the time coordinate, an evolution operator was obtained satisfying the retarded equation

$$\Delta_R \hat{\rho}(t) = \frac{1}{i\hbar} \left[\hat{H}(t), \hat{\rho}(t) \right].$$

The second path started from the definition of the density operator to determine the dynamical equation it satisfies and then obtained the evolution operator.

For the symmetric case, the evolution operator is given by

$$\hat{V}(t, t_0) = \exp \left[-\frac{i(t-t_0)}{\tau} \sin^{-1} \left(\frac{\tau \hat{\mathcal{L}}}{\hbar} \right) \right], \quad (119)$$

which is similar to the operator obtained for the continuous case.

A.3. Compatibility Between the Previous Pictures

We thus have two distinct evolution operators for the retarded Schrödinger and Liouville equations so that, once a connection is established between them, we arrive at the question of the compatibility of the two descriptions. We try to set up a relation between those operators by observing their action on the density operator. So, we expect that both operators satisfy the expression

$$\hat{V}(t, t-\tau) \hat{\rho}_0 = \hat{U}(t, t-\tau) \hat{\rho}_0 \hat{U}^\dagger(t, t-\tau),$$

where the different action of the operators is basically due to the bilinearity (Recami *et al.*, 2010) of the operator \hat{V} given by Eq. (117), while \hat{U} , given by Eq. (116), is linear. This relation is valid in the continuous case, where the evolution operators act on the density operator according to

$$\hat{\rho}(t) = \exp[-i\mathcal{L}(t - t_0)/\hbar]\hat{\rho}_0 = \exp[-iH(t - t_0)/\hbar]\hat{\rho}_0 \exp[iH(t - t_0)/\hbar].$$

Considering the basis of Hamiltonian eigenstates $|n\rangle$, we have

$$\langle n|\hat{\mathcal{L}}\hat{\rho}(0)|m\rangle = (E_n - E_m)\rho_{nm}(0),$$

so that

$$\exp(-i\hat{\mathcal{L}}t)\hat{\rho}(0) = \exp[-it(E_n - E_m)]\rho_{nm}(0), \quad (120)$$

$$\exp(-i\hat{H}t)\hat{\rho}(0)\exp(i\hat{H}t) = \exp[-it(E_n - E_m)]\rho_{nm}(0). \quad (121)$$

The question is knowing whether the same is valid for the discrete case. For the retarded approach, we must check whether the relation

$$\frac{1}{\left[1 + \frac{i\tau}{\hbar}\hat{\mathcal{L}} + \frac{\tau^2}{\hbar^2}\hat{H}\dots\hat{H}\right]^{(t-t_0)/\tau}}\hat{\rho}_0 = \frac{1}{\left[1 + \frac{i}{\hbar}\tau\hat{H}\right]^{(t-t_0)/\tau}}\hat{\rho}_0 \frac{1}{\left[1 - \frac{i}{\hbar}\tau\hat{H}\right]^{(t-t_0)/\tau}}$$

is valid. We see that, if we consider that equations such as (120) and (121) continue to be valid in the discrete case, then the above relation is valid. For a generic element of the operator, we then obtain

$$\frac{1}{\left[1 + \frac{i\tau}{\hbar}(E_n - E_m) + \frac{\tau^2}{\hbar^2}E_n E_m\right]^{t/\tau}}\rho_{nm}(0) = \frac{1}{\left[1 + \frac{i}{\hbar}\tau E_n\right]^{t/\tau}}\rho_{nm}(0) \frac{1}{\left[1 - \frac{i}{\hbar}\tau E_m\right]^{t/\tau}}.$$

Such equivalence also can be observed for the other cases. However, when we consider the evolution operator obtained from the coarse-grained approach, we find an incompatibility with the operator deriving from the Schrödinger one. For the operator [Eq. (116)] we have

$$\langle n|\frac{1}{\left[1 + \frac{i}{\hbar}\tau\hat{\mathcal{L}}\right]^{t/\tau}}\hat{\rho}(0)|m\rangle = \frac{1}{\left[1 + \frac{i}{\hbar}\tau(E_n - E_m)\right]^{t/\tau}}\rho_{nm}(0).$$

The question now is to determine the fundamental difference between the two descriptions: Are both valid, and under what conditions? Some points must be emphasized. First, remember that the coarse-grained description is a semi-classical approach that assumes a system with a certain degree of complexity, whereas the vector state description is a fundamentally quantum approach without any imposition, in principle, on the number of degrees of freedom of the system described. The two approaches differ importantly even in the way they conceive the chronon. In the coarse-grained approach, it is understood as a magnitude inwardly connected to

the experimental limitations or, for an ideal measurement device, to the limitations imposed by the uncertainty relations. For the Schrödinger equation, the value of the chronon is taken as a fundamental interval of time associated with interaction processes among the components of the system, and of the system as a whole with some external potential; that is, it is associated with the internal processes of the system (as it has been conceived for the classical electron). In this way, the absence of the mixed term in the evolution operator obtained with the semi-classical procedure is comprehensible, as is its incompatibility with the purely quantum description provided by the Schrödinger equations. As a semiclassical approach, the range of applicability of the coarse-grained formalism extends to the cases where the system to be studied is not purely microscopic, particularly in the measurement processes. We stress that, in this formalization, only the retarded equation was obtained. Thus, the system as described dissipates energy: It is an open system. This is the characteristic that make it possible for us to have access to the output of a measurement.

In connection with the operator obtained directly in the Schrödinger picture for the retarded case, all the elements of the density matrix, even the diagonal ones, are damped with time. There is also the controversy linked to the non-existence, in QM, of an applicability limit of the theory due to the number of degrees of freedom involved. The formalism does not distinguish between a microscopic and a macroscopic system, so that it should reproduce what is obtained with the coarse-grained formalism. This means that the measurement problem appears in the discrete formalism also through the non-equivalence of the evolution operators in Eqs. (117) and (118).

B. Non-Hermitian Operators in the Discrete Formalism

One feature we have stressed throughout this work is the *non-Hermitian* character of the discrete formalism. In the Schrödinger representation, for example, the continuous equation can reproduce the outputs obtained with the discretized equations once we replace the conventional Hamiltonian by a suitable non-Hermitian Hamiltonian we have called the equivalent Hamiltonian. One characteristic of a non-Hermitian operator is that its eigenvalues are defined over the field of complex numbers. A linear non-Hermitian operator can always be considered as consisting of a Hermitian part, which supplies the real component of the eigenvalues, and an anti-Hermitian part, which gives the complex component (Recami *et al.*, 2010).

In the continuous case, let us take the Hamiltonian as being a non-Hermitian operator given by

$$\tilde{H} = \hat{v} + i\hat{\kappa},$$

where \hat{v} and $\hat{\kappa}$ are Hermitean. Then we have, in the Schrödinger picture, that the time evolution operator is given by

$$\hat{U}_{\text{cont}}(t, t_0) = \exp\left[\frac{1}{\hbar}(\hat{\kappa} - i\hat{v})(t - t_0)\right]. \quad (122)$$

For the discrete case, comes from [Appendix A](#) that the evolution operator for the retarded states is given by [Eq. \(116\)](#)

$$\hat{U}(t, t_0) = \left[1 + \frac{i}{\hbar}\tau\hat{H}\right]^{-(t-t_0)/\tau}, \quad (123)$$

where \hat{H} is the Hermitean operator associated with the conventional Hamiltonian. This evolution operator can be written as

$$\hat{U}_{\text{ret}}(t, t_0) = \exp\left[-\frac{(t-t_0)}{2\tau}\ln\left(1 + \frac{\tau^2\hat{H}^2}{\hbar^2}\right)\right] \exp\left[-\frac{i(t-t_0)}{\tau}\tan^{-1}\left(\frac{\tau\hat{H}}{\hbar}\right)\right]. \quad (124)$$

Comparing [Eqs. \(122\)](#) and [\(124\)](#) we obtain the equivalence of the Hamiltonians once \hat{v} and $\hat{\kappa}$ are given by

$$\begin{aligned} \hat{v} &= \frac{\hbar}{\tau}\tan^{-1}\left(\frac{\tau\hat{H}}{\hbar}\right), \\ \hat{\kappa} &= -\frac{\hbar}{2\tau}\ln\left(1 + \frac{\tau^2\hat{H}^2}{\hbar^2}\right). \end{aligned}$$

For the advanced case we obtain the same expressions except for a minus sign for $\hat{\kappa}$. For the symmetric case, below the critical limit, we have

$$\begin{aligned} \hat{v} &= \frac{\hbar}{\tau}\sin^{-1}\left(\frac{\tau\hat{H}}{\hbar}\right), \\ \hat{\kappa} &= 0. \end{aligned}$$

Above that limit \hat{v} ceases to be Hermitean and, in this case, the evolution operator can be written as

$$\hat{U}_{\text{sym}}(t, t_0) = \exp\left[-\frac{i\pi}{2\tau}(t - t_0)\right] \exp\left\{-\frac{(t-t_0)}{\tau}\ln\left[\left|\frac{\tau\hat{H}}{\hbar}\right| + \sqrt{\left(\frac{\tau\hat{H}}{\hbar}\right)^2 - 1}\right]\right\}$$

so that

$$\hat{v} = \frac{\hbar\pi}{2\tau},$$

$$\hat{\kappa} = -\frac{\hbar}{\tau} \ln \left[\left| \frac{\tau\hat{H}}{\hbar} \right| + \sqrt{\left(\frac{\tau\hat{H}}{\hbar} \right)^2 - 1} \right],$$

with \hat{v} being now independent of the Hamiltonian and $\hat{\kappa}$ ceases to be zero.

The expressions obtained above show the characteristics that \hat{v} and $\hat{\kappa}$ must fulfill, so that the continuous equation reproduces the outputs of the discretized equations. By observing the continuous evolution operator we have that the anti-Hermitian part of \tilde{H} shows a nonstationary behavior, resulting in a damping or amplifying term associated with the evolution of the quantum state. Thus, the stationary solutions appear only for the symmetric case below the critical limit. In all the other cases, the transient term always appears.

In QM, the non-Hermitian operators have been used mainly as mathematical shortcuts, as in the case of the Lippmann-Schwinger equation in the scattering theory. It has already been observed that the introduction of such operators could make possible the description of unstable states, by phenomenologically linking the transient factor to the lifetime of the considered states (Agodi *et al.*, 1973, and Cohen-Tannoudji *et al.*, 1977). If in a certain instant $t_0 = 0$ the system is in one of the eigenstates $|n\rangle$ of the Hamiltonian \hat{H} , then if such state is unstable, the probability of the system to be found in the same state at a later instant t is

$$P_n(t) = |\langle n | \hat{U}^\dagger \hat{U} | n \rangle| = \exp(-t/\tau_{\mathcal{L}}),$$

and that allows us to specify a lifetime $\tau_{\mathcal{L}}$, for the retarded case, as

$$\tau_{\mathcal{L}} = \frac{\tau}{\ln\left(1 + \frac{\tau^2 E_n^2}{\hbar^2}\right)}, \quad (125)$$

and for the symmetric case, above the critical energy, as

$$\tau_{\mathcal{L}} = \frac{\tau}{2 \ln\left(\left| \frac{\tau E_n}{\hbar} \right| + \sqrt{\left(\frac{\tau E_n}{\hbar} \right)^2 - 1}\right)}.$$

Such lifetimes are connected with states that, in the discretized formalism, are intrinsically unstable. Only the retarded equation seems to be associated with quantum states that decay with time. If that is truly valid, we have an expression that could be used for phenomenologically

determining the value of the chronon. Finally, we can conclude that the time discretization brings forth a formalism which, even if only Hermitian Hamiltonians are involved, is equivalent to the introduction of non-Hermitian operators in the continuous QM.

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REFERENCES

- Abraham, M. (1902). Prinzipien der Dynamik des Elektrons. *Annalen Der Physik*, 315(1), 105–179.
- Abraham, M. (1905). *Theorie der Elektrizität*, Vol. II. Leipzig: Springer.
- Agodi, A., Baldo, M., & Recami, E. (1973). Approach to the Theory of Unstable States. *Annals of Physics*, 77, 157–173.
- Ambarzumian, V., & Ivanenko, D. D. (1930). Zur Frage nach Vermeidung der unendlichen Selbststrückwirkung des Elektrons. *Zeitschrift für Physiotherapie*, 64, 563–567.
- Arzeliès, H. (1966). *Rayonnement et Dynamique du Corpuscule Chargé Fortment Accélééré*. Paris: Gauthier-Villars.
- Ashauer, S. (1949). On the classical equations of motion of radiating electrons. *Proceedings Cambridge Philological Society*, 45, 463–475.
- Ballentine, L. E. (1970). The statistical interpretation of quantum mechanics. *Reviews of Modern Physics*, 42(4), 358–381.
- Ballentine, L. E. (1986). What is the point of the quantum theory of measurement? In L. Roth & A. Inomata (Eds.), *Fundamental Questions in Quantum Mechanics*. Gordon & Breach.
- Barut, A. O. (1978a). The creation of a photon: A heuristic derivation of the Planck's constant h or fine structure constant. *Zeitschrift für Naturforschung*, A33, 993–994.
- Barut, A. O. (1978b). The mass of the muon. *Physics Letters*, B73, 310–312.
- Barut, A. O. (1979). Lepton mass formula. *Physical Review Letters*, 42, 1251.
- Barut, A. O. (1991). Brief History and Recent Developments in Electron Theory and Quantumelectrodynamics. In D. Hestenes & A. Weingartshofer (Eds.), *The Electron: New Theory and Experiment*. Dordrecht: Kluwer.
- Bateman, H. (1910). The transformation of the electrodynamic equations. *Proc. London Math. Soc.*, B8, 223.
- Bell, J. S. (1987). Are There Quantum Jumps? In C. W. Kilmister (Ed.), *Schrödinger: Centenary Celebration of a Polymath* (pp. 41–52). Cambridge: Cambridge U. P.
- Belloni, L. (1981). Historical remarks on the 'classical' electron radius. *Lett. Nuovo Cim.*, 31, 131–134.
- Benza, V., & Caldirola, P. (1981). de Sitter microuniverse associated with the electron. *Nuovo Cimento*, A62, 175–185.
- Bhabha, H. I. (1946). On the expansibility of Solutions in Powers of the Interaction Constant. *Physical Review*, 70, 759–760.

- Bhabha, M. J., & Corben, H. C. (1941). General Classical Theory of Spinning Particles in a Maxwell Field. *Proc. Roy. Soc. (London)*, *A178*, 273–314.
- Bohm, D., & Weinstein, M. (1948). The self-oscillations of a charged particle. *Physical Review*, *74*, 1789–1798.
- Bohr, N. (1935). Can quantum-mechanical description of physical reality be considered complete? *Physical Review*, *48*, 696–702.
- Bonifacio, R. (1983). A coarse grained description of time evolution: Irreversible state reduction and time-energy relation. *Lett. Nuovo Cim.*, *37*, 481–489.
- Bonifacio, R., & Caldirola, P. (1983). Finite-difference equations and quasi-diagonal form in quantum statistical mechanics. *Lett. Nuovo Cim.*, *38*, 615–619.
- Bonifacio, R., & Caldirola, P. (1982). Unstable states of a finite-difference Schrödinger equation. *Lett. Nuovo Cim.*, *33*, 197–202.
- Bonnor, W. B. (1974). A New Equation of Motion for a Radiating Charged Particle. *Proc. Roy. Soc. (London)*, *A337*, 591–598.
- Bracci, L., Fiorentini, G., Mezzorani, G., & Quarati, P. (1983). Bounds on a hypothetical fundamental length. *Physics Letters*, *B133*, 231–233.
- Bunge, M. (1955). A picture of the electron. *Nuovo Cimento*, *1*, 977–985.
- Caldirola, P. (1953). Sull'equazione del moto dell'elettrone nell'elettrodinamica classica. *Nuovo Cimento*, *10*, 1747–1752.
- Caldirola, P. (1954). Spiegazione classica del momento magnetico anomalo dell'elettrone. *Nuovo Cimento*, *11*, 108–110.
- Caldirola, P. (1956). A new model of classical electron. *Suppl. Nuovo Cim.*, *3*, 297–343.
- Caldirola, P. (1974). *Dalla Microfisica alla Macrofisica*. Milan: Mondadori.
- Caldirola, P. (1976a). On the introduction of a fundamental interval of time in quantum mechanics. *Lett. Nuovo Cim.*, *16*, 151–155.
- Caldirola, P. (1976b). On the Finite Difference Schrodinger Equation. *Lett. Nuovo Cim.*, *17*, 461–464.
- Caldirola, P. (1977a). On the existence of a heavy lepton. *Lett. Nuov Cim.*, *20*, 519–521.
- Caldirola, P. (1977b). The chronon and uncertainty relations in the theory of the excited-mass states of the electron. *Lett. Nuov Cim.*, *20*, 632–634.
- Caldirola, P. (1977c). Chronon in Quantum Theory. *Lett. Nuovo Cim.*, *18*, 465–468.
- Caldirola, P. (1978a). The chronon in the quantum theory of the electron and the existence of the heavy leptons. *Nuovo Cimento*, *A45*, 549–579.
- Caldirola, P. (1978b). The mass of the neutrinos and the Pontecorvo oscillations. *Lett. Nuovo Cim.*, *21*, 250–252.
- Caldirola, P. (1979a). A relativistic theory of the classical electron. *Rivista Nuovo Cim*, *2*(13), 1–49.
- Caldirola, P. (1979b). On a relativistic model of the electron. *Nuovo Cimento*, *A49*, 497–511.
- Caldirola, P. (1979c). Introduzione del cronone nella teoria relativistica dell'elettrone. In M. Pantaleo & F. de Finis (Eds.), *Centenario di Einstein – Astrofisica e Cosmologia, Gravitazione, Quanti e Relatività* (p. 829). Florence: Giunti-Barbera.
- Caldirola, P. (1979d). Progressi nella teoria dell'elettrone. In *Annuario '79, Enciclopedia EST-Mondadori* (pp. 65–72). Milan: Mondadori Pub.
- Caldirola, P. (1980). The introduction of the Chronon in the electron theory and a charged-lepton mass formula. *Lett. Nuovo Cim.*, *27*, 225–228.
- Caldirola, P. (1984a). Introduction of the chronon in the theory of electron and the wave-particle duality. In S. Diner *et al.* (Ed.), *The Wave-Particle Dualism* (pp. 183–213). Dordrecht: Reidel.
- Caldirola, P. (1984b). A geometrical model of point electron. In *Revista Brasil. de Física* (pp. 228–260). special vol. Mário Schönberg on his 70th Birthday (July).
- Caldirola, P., & Montaldi, E. (1979). A new equation for quantum dissipative systems. *Nuovo Cimento*, *B53*, 291–300.

- Caldirola, P., & Recami, E. (1978). The concept of Time in physics. *Epistemologia*, 1, 263–304.
- Caldirola, P., Casati, G., & Prosperetti, A. (1978). On the Classical Theory of the Electron. *Nuovo Cimento*, A43, 127–142.
- Casagrande, F. (1977). The introduction of the Chronon into the classical and quantum theory of the electron. *Scientia; rivista di scienza*, 112, 417–427.
- Casagrande, F., & Montaldi, E. (1977). Some remarks on the finite-difference equations of physical interest. *Nuovo Cimento*, A40, 369–382.
- Casagrande, F., & Montaldi, E. (1978). On the discrete-time Schrödinger equation for the linear harmonic oscillator. *Nuovo Cimento*, A44, 453–464.
- Cheon, I. T. (1979). Special relativity on the discrete space-time and a fundamental length. *Lett. Nuovo Cim.*, 26, 604–608.
- Cirelli, R. (1955). Sul moto di un elettrone investito da un impulso elettromagnetico istantaneo. *Nuovo Cimento*, 1, 260–262.
- Cohen-Tannoudji, C., Diu, B., & Laloë, F. (1977). *Quantum Mechanics*. New York: Wiley.
- Cole, E. A. B. (1970). Transition from a continuous to a discrete space-time scheme. *Nuovo Cimento*, A66, 645–656.
- Coleman, S. (1960). Classical Electron Theory from a Modern Standpoint. Reprinted in. In D. Teplitz (Ed.), *Electromagnetism: Paths to Research* (p. 1982). New York: Plenum.
- Compton, A. H. (1919). The size and shape of the electron. *Physical Review*, 14, 20–43 247–259.
- Corben, H. C. (1961). Spin in Classical and Quantum Theory. *Physical Review*, 121, 1833–1839.
- Corben, H. C. (1968). *Classical and quantum theories of spinning particles*. San Francisco: Holden-Day.
- Corben, H. C. (1977). Particle spectrum. *American Journal of Physics*, 45, 658–662.
- Corben, H. C. (1984). Quantized relativistic rotator. *Physical Review*, D30, 2683–2689.
- Corben, H. C. (1993). Factors of 2 in magnetic moments, spinorbit coupling, and Thomas precession. *American Journal of Physics*, 61, 551–553.
- Cunningham, E. (1909). The principle of relativity in electrodynamics and an extension thereof. *Proc. London Math. Soc.*, 8, 77–97.
- Cvijanovich, G. B., & Vigier, J. P. (1977). New extended model of the Dirac electron. *Foundations of Physics*, 7, 77–96.
- Daneri, A., Loinger, A., & Prosperi, G. M. (1962). Quantum theory of measurement and ergodicity conditions. *Nucl. Phys.*, 33, 297–319.
- Darling, B. T. (1950). The Irreducible Volume Character of Events. I. A Theory of the Elementary Particles and of Fundamental Length. *Physical Review*, 80, 460–466.
- DerSarkissian, M., & Nelson, T. J. (1969). Development of an S-matrix theory based on the calculus of finite differences. *Nuovo Cimento*, A64, 337–344.
- Dirac, P. A. M. (1938a). The classical theory of electron. *Proc. Royal Soc.*, A167, 148–169.
- Dirac, P. A. M. (1938b). La théorie de l'électron et du champ électromagnétique. *Ann. Inst. Poincaré*, 9, 13–49.
- Dirac, P. A. M. (1962). An extensible model of the electron. *Proc. Roy. Soc.*, A268, 57–67.
- Einstein, A. (1915). Der energiesatz in der allgemeinen relativitätstheorie. *Berl. Ber.*, 349, 448–459.
- Einstein, A., Podolsky, B., & Rosen, N. (1935). Can quantum-mechanical description of physical reality be considered complete? *Physical Review*, 47, 777–780.
- Eliezer, C. J. (1943). The hydrogen atom and the classical theory of radiation. *Proceedings Cambridge Philological Society*, 33, 173–180.
- Eliezer, C. J. (1950). A note on Electron Theory. *Proceedings Cambridge Philological Society*, 46, 199–201.
- Erber, T. (1961). The classical theories of radiation reaction. *Fortschritte für Physik*, 9, 343–392.
- Ehrlich, R. (1976). Possible evidence for the quantization of particle lifetimes. *Physical Review*, D13, 50–55.

- Everett, H. (1957). Relative State Formulation of Quantum Mechanics. *Reviews of Modern Physics*, 29, 454–462.
- Fermi, E. (1922). Über einen Widerspruch zwischen der elektrodynamischen und der relativistischen Theorie der elektromagnetischen Masse. *Physik. Z.*, 23, 340–344.
- Fleming, G. N. (1965). Nonlocal Properties of Stable Particles. *Physical Review*, B139, 963–968.
- Fock, V. A. (1962). Criticism of an attempt to disprove the uncertainty relation between time and energy. *Sov. Phys. JETP*, 15(4), 784–786.
- Ford, K. W. (1968). Electrodynamics with a quantum of length. *Physical Review*, 175, 2048–2053.
- Frenkel, J. (1926). Die Elektrodynamik des Rotierenden Elektrons. *Zeitschr. f. Phys.*, 37, 243–262.
- Frenkel, J. I. (1926–28). *Lehrbuch der Elektrodynamik*, 2. (p. 248). Berlin: Springer.
- Friedberg, R., & Lee, T. D. (1983). Discrete Quantum Mechanics. *Nucl. Phys.*, B225, 1–52.
- Froissart, M., Golberger, M. L., & Watson, K. M. (1963). Spatial Separation of Events in S-Matrix Theory. *Physical Review*, 131, 2820–2826.
- Fryberger, D. (1981). Model for the generation of leptonic mass. II. *Physical Review*, D24, 979–998.
- Fulton, T., & Rohrlich, F. (1960). Classical radiation from a uniformly accelerated charge. *Ann. of Phys.*, 9, 499–517.
- Gallardo, J. C., Kálnay, A. J., & Riesenberger, S. H. (1967). Lorentz-invariant localization for elementary systems. *Physical Review*, 158, 1484–1490.
- George, G., Prigogine, I., & Rosenfeld, L. (1972). The macroscopic level of quantum mechanics. *Dansk. Mat. Fys. Medd.*, 38, 1–44.
- Ghirardi, G. C., Rimini, A., & Weber, T. (1985). Unified Dynamics for Microscopic and Macroscopic Systems. *Physical Review*, D34, 470–491.
- Ghirardi, G. C., & Weber, T. (1984). Finite-difference evolution equations and quantum-dynamical semi-groups. *Lett. Nuovo Cim.*, 39, 157–164.
- Golberger, M. L., & Watson, K. M. (1962). Concerning the Notion of 'Time Interval' in S-Matrix Theory. *Physical Review*, 27, 2284–2286.
- Gursey, F. (1957). Relativistic Kinematics of a Classical Point Particle in Spinor Form. *Nuovo Cimento*, 5, 784–809.
- Gutkowski, D., Moles, M., & Vigier, J. P. (1977). Hidden Parameter Theory Of The Extended Dirac Electron. 1. Classical Theory. *Nuovo Cim.*, B39, 193–225.
- Havas, P. (1948). On the Classical Equations of Motion of Point Charges. *Physical Review*, 74, 456–463.
- Hikasa, K. *et al.* (1992) (Particle Data Group). Review of particle properties. *Physical Review*, D45, S1–S574, D46, 5210 (erratum).
- Hill, E. L. (1945). On Accelerated Coordinate Systems in Classical and Relativistic Mechanics. *Physical Review*, 67, 358–363.
- Hönl, H. (1952). Feldmechanik des elektrons und der elementarteilchen. *Ergeb. Exacten Naturwiss.*, 26, 291–382.
- Hönl, H., & Papapetrou, A. (1939). Über die innere Bewegung des Elektrons – I. *Z. Phys.*, 112, 512–540.
- Hönl, H., & Papapetrou, A. (1940). Über die innere Bewegung des Elektrons – III. *Z. Phys.*, 116, 153–183.
- Huang, K. (1952). On the Zitterbewegung of the Dirac Electron. *American Journal of Physics*, 20, 479–484.
- Hsu, J. P., & Mac, E. (1979). Fundamental length, bubble electrons and nonlocal quantum electrodynamics. *Nuovo Cimento*, B49, 55–67.
- Jackson, J. C. (1977). A quantisation of time. *J. Phys. A*, 10, 2115–2122.
- Jammer, M. (1954). *Concepts of Space*. Cambridge: Harvard Univ. Press.

- Jancel, R. (1969). *Foundations of Classical and Quantum Statistical Mechanics*. New York: Pergamon Press.
- Jannussis, A. (1984a). Connection between master equation and Lie - admissible formulation. *Lett. Nuovo Cim.*, 39, 75–80.
- Jannussis, A. (1984b). Quantum equations of motion in the finite-difference approach. *Lett. Nuovo Cim.*, 40, 250–256.
- Jannussis, A. (1984c). Caldirola–Montaldi theory for a finite-difference Dirac equation. *Nuovo Cimento*, B84, 27–34.
- Jannussis, A. (1985a). Difference equations in the Lie-Admissible formulation. *Lett. Nuovo Cim.*, 42, 129–133.
- Jannussis, A. (1985b). Some remarks on the small - distance derivative model. *Nuovo Cimento*, B90, 58–64.
- Jannussis, A. (1990). A Lie admissible complex time model and relativity theorem. *Hadronic Journal*, 13, 425–434.
- Jannussis, A., Leodaris, A., Fillippakis, P., & Fillippakis, T. (1980a). Some remarks on the discrete-time Schrödinger equation. *Lett. Nuovo Cim.*, 29, 259–262.
- Jannussis, A., Leodaris, A., Fillippakis, P., & Fillippakis, T. (1980b). Quantum dissipative system and the Caldirola - Montaldi equation. *Lett. Nuovo Cim.*, 29, 437–442.
- Jannussis, A., Brodimas, G. N., Papatheou, V., Leodaris, A. D., & Zisis, V. (1981a). Perturbation theory for the solution of the Caldirola-Montaldi equation. *Lett. Nuovo Cim.*, 31, 533–538.
- Jannussis, A., Brodimas, G. N., Leodaris, A. D., Papatheou, V., & Zisis, V. (1981b). Quantum friction and the Caldirola-Montaldi procedure. *Lett. Nuovo Cim.*, 30, 289–292.
- Jannussis, A., Leodaris, A., Fillippakis, P., Fillippakis, T., & Zisis, V. (1982a). Heisenberg equations of motion in the Caldirola–Montaldi procedure. *Nuovo Cimento*, B67, 161–172.
- Jannussis, A., Streclas, A., Leodaris, A., Patargias, N., Papatheou, V., Fillippakis, P., et al. (1982b). Some remarks on the Caldirola-Montaldi equation. *Lett. Nuovo Cim.*, 34, 571–574.
- Jannussis, A., Brodimas, G. N., Papatheou, V., Karayiannis, G., Panagopoulos, P., & Ioannidou, H. (1983a). Lie-admissible unification of dissipative Schrödinger's equations. *Lett. Nuovo Cim.*, 38, 181–188.
- Jannussis, A., Brodimas, G. N., Papatheou, V., Karayiannis, G., Panagopoulos, P., & Ioannidou, H. (1983b). Lie-admissible unification of dissipative Schrödinger's equations. *Hadronic Journal*, 6, 1434–1461.
- Jannussis, A., Skaltsas, D., & Brodimas, G. N. (1990). Equivalence between Caldirola-Montaldi model and Lie admissible complex time model. *Hadronic Journal*, 13(89), 415–423.
- Jehle, H. (1971). Relationship of Flux Quantization to Charge Quantization and the Electromagnetic Coupling Constant. *Physical Review*, D3, 306–345.
- Kálnay, A. J. (1970). Lorentz-invariant localization for elementary systems – II. *Physical Review*, D1, 1092–1104.
- Kálnay, A. J. (1971). Lorentz-Invariant Localization for Elementary Systems – III. Zero-Mass Systems. *Physical Review*, D3, 2357–2363.
- Kálnay, A. J., & Torres, P. L. (1971). Lorentz-Invariant Localization for Elementary Systems – IV. The Nonrelativistic Limit. *Physical Review*, D3, 2977–2981.
- Kitazoe, T. (1972). Structure of leptons. *Lett. Nuovo Cim.*, 4, 196–198.
- Kitazoe, T., Ishihara, M., & Nakatani, H. (1978). Dirac equation on a lattice. *Lett. Nuovo Cim.*, 21, 59–64.
- Lanz, L. (1962). On the energy conservation for the motion of a radiating electron with particular reference to a hyperbolic motion. *Nuovo Cimento*, 23, 195–201.
- Lanz, L., Prosperi, G. M., & Sabbadini, A. (1971). Time scales and the problem of measurement in quantum mechanics. *Nuovo Cimento*, B2, 184–192.
- Lattes, C. M., Schönberg, M., & Schützer, W. (1947). Classical theory of charged point particles with dipole moments. *Anais Ac. Brasil. Cien*, 19(3), 46–98.

- Lee, T. D. (1983). Can Time be a discrete dynamical variable? *Physics Letters*, B122, 217–220.
- Lee, T. D. (1987). Difference Equations and Conservation Laws. *Journal of Statistical Physics*, 46, 843–860.
- Leiter, D. (1970). The paradox of radiation from a uniformly accelerated point electron and a consistent physical framework for its resolution. *Int. J. Theor. Phys.*, 3, 387–393.
- Levi, R. (1926). Théorie de L'action universelle et descontinue. *Comptes Rendus de l'Academie*, 183, 865–867.
- Liebowitz, B. (1969). A model of the electron. *Nuovo Cimento*, A63, 1235–1246.
- Likharev, K. K., & Claeson, T. (1992). Single electronics. *Scient. Am.*, 50–55 (June).
- Lock, J. A. (1979). The Zitterbewegung of a free localized Dirac particle. *American Journal of Physics*, 47, 797–802.
- Loinger, A. (1955). *Rendic. Acc. Lincei*, 18, 395.
- Lorentz, H. A. (1892). La Théorie électromagnétique de Maxwell et son application aux corps mouvants. *Arch. Néer. Sciences Exactes et Naturelles*, 25, 363–552.
- Lorentz, H. A. (1904). Electromagnetische verschijnselen in een stelsel, dat zich met will-ekeurige snelheid, kleiner dan die van het licht, beweegt. *Versl. Kon. Acad. Wetensch Amsterdam*, 12, 986–1009.
- Ludwig, G. (1953). Der Meßprozeß. *Z. Phys.*, 135, 483–511.
- Lundeen, S. R., & Pipkin, F. M. (1981). Measurement of the Lamb shift in Hydrogen, $n = 2$. *Physical Review Letters*, 46, 232–235.
- Marx, E. (1975). Electromagnetic energy and momentum from a charged particle. *Int. J. Theor. Phys.*, 14, 55–65.
- Mathisson, M. (1937). Neue Mechanik materieller Systeme. *Acta Physica Polonica*, 6, 163–200.
- Matore, I. M. (1981). *Model of the electron and muon structure*. Dubna: JINR Report P4-81-81.
- McGoveran, D., & Noyes, P. (1989). *Foundations of a Discrete Physics*. SLAC-PUB-4526.
- McGregor, M. H. (1992). *The enigmatic electron*. Dordrecht: Kluwer.
- Meessen, A. (1970). Quantification de l'espace-temps et généralization de l'équation de Dirac. *Ann. Soc. Sci. Bruxelles*, 84, 267–275.
- Mie, G. (1912). Grundlagen einer Theorie der Materie. *Annalen Der Physik*, 342(3), 511–534.
- Mignani, R. (1983). On the Lie-Admissible structure of the Caldirola equations for dissipative processes. *Lett. Nuovo Cim.*, 38, 169–173.
- Mo, T. C., & Papas, C. H. (1971). New equation of motion for classical charged particles. *Phys. Rev. D*, 4, 3566–3571.
- Montaldi, E., & Zanon, D. (1980). On the green functions for the discrete-time Schrödinger equation. *Lett. Nuovo Cim.*, 27, 215–221.
- Motz, L. (1970). A gravitational theory of the mu-meson and leptons in general. *Nuovo Cimento*, A65, 326–332.
- Nambu, Y. (1952). An empirical mass spectrum of elementary particles. *Prog. Theor. Phys.*, 7, 595–596.
- Nishijima, K., & Sato, H. (1978). Higgs-Kibble Mechanism and the Electron-Muon Mass Ratio. *Prog. Theor. Phys.*, 59, 571–578.
- Ouchi, T., & Ohmae, A. (1977). Models for the Electron and the Muon. *Prog. Theor. Phys.*, 58, 1664–1666.
- Page, L. (1918). Is a moving mass retarded by the reaction of its own radiation? *Physical Review*, 11, 376–400.
- Page, L. (1921). Theory of the motion of electrons between co-axial cylinders taking into account the variation of mass with velocity. *Physical Review*, 18, 58–61.
- Page, L., & Adams, N. I.Jr., (1940). *Electrodynamics*. New York: Von Nostrand.
- Pavšic, M. (1976). Discrete Rest Masses Resulting from Relativistically Covariant Massless Field Equations in Five-Dimensions. *Lett. Nuovo Cim.*, 17, 44–48.
- Pavšic, M., & Recami, E. (1982). Charge conjugation and internal space-time symmetries. *Lett. Nuovo Cim.*, 34, 357–362.

- Pavšic, M., Recami, E., Rodrigues, W. A., Maccarrone, G. D., Raciti, F., & Salesi, G. (1993). Spin and electron structure. *Physics Letters*, *B318*, 481–488.
- Pavšic, M., Recami, E., & Rodrigues, W. A. (1995). Electron structure, zitterbewegung and a new non-linear Dirac-like equation. *Hadronic J*, *18*, 97–118.
- Perkins, G. A. (1976). The direction of the Zitterbewegung: a hidden variable. *Foundations of Physics*, *6*, 237–248.
- Petzold, J., & Sorg, M. (1977). Generalized Caldirola equations for the classical radiating electron. *Z. Physik*, *A283*, 207–215.
- Plass, G. N. (1960). Classical Electrodynamical Equations of Motion with Radiative Reaction. *Physical Review Letters*, *4*, 248–249.
- Plass, G. N. (1961). Classical electrodynamical equations of motion with radiative reaction. *Reviews of Modern Physics*, *33*, 37–62.
- Poincaré, H. (1906). Sur la dynamique de l'électron. *Rendic. Circolo Matem. di Palermo*, *21*, 129–175.
- Poincaré, H. (1913). *Dernière Pensées*. Paris: Flammarion.
- Pokrowski, G. I. (1928). Zur Frage nach eine Struktur der Zeit. *Zeits. für Physik*, *51*, 737–739.
- Proca, A. (1954). Mécanique du point. *J. Phys. Radium*, *15*, 65–72.
- Prosperetti, A. (1980). The Motion of a Charge Particle in a Uniform Magnetic Field. *Nuovo Cimento*, *B57*, 253–268.
- Pryce, M. H. L. (1948). The Mass-Centre in the Restricted Theory of Relativity and Its Connexion with the Quantum Theory of Elementary Particles. *Proc. Royal Soc. (London)*, *A195*, 62–81.
- Recami, E. (1978). How to recover Causality in special relativity. *Foundations of Physics*, *8*, 329–340.
- Recami, E. (2002). Multi-verses, micro-universes and elementary particles. Report NSF-ITP-02-94 (KITP. UCSB, 2002); also appeared in e-print form as arXiv:physics/0505149.
- Recami, E., & Farias, H. A. (2009). A simple quantum (finite-difference) equation for disipation and decoherence. *Nuovo Cimento*, *B124*, 765–776.
- Recami, E., Olkhovsky, V. S., & Maydanyuk, S. P. (2010). On non-selfadjoint operators for observables in quantum mechanics and quantum electrodynamics. *Int. J. Mod. Phys.*, *A25*, 1785–1818.
- Recami, E., & Rodrigues, W. A. (1982). Antiparticles from special relativity with orthochronous and anti-chronous Lorentz transformations. *Foundations of Physics*, *12*, 709–718 Errata: *13* (1983). E553.
- Recami, E., Rodrigues, W. A., & Smrz, P. (1983). Some applications of non-hermitian operators in quantum mechanics and quantum field theory. *Hadronic Journal*, *6*, 1773–1789.
- Recami, E., & Salesi, G. (1994). Field theory of the spinning electron and internal motions. *Physics Letters*, *A190*, *A195*, 137–143; Erratum.
- Recami, E., & Salesi, G. (1995). Field theory of the spinning electron: I and II. In J. Dowling (Ed.), *Electron theory and quantum electrodynamics: 100 years later* (pp. 241–260). New York: Plenum. Proceedings, NATO Advanced Study Institute, Edirne, Turkey, September 5–16 (1994).
- Recami, E., & Salesi, G. (1996). Field Theory of the Spinning Electron: About the New Non-Linear Field Equations. *Adv. Appl. Cliff. Alg.*, *6*(1), 27–36.
- Recami, E., & Salesi, G. (1997a). Kinematics and hydrodynamics of spinning particles. In J. Keller & Z. Oziewicz (Eds.), *The Theory of the Electron* (pp. 253–268). Mexico City: UNAM Press.
- Recami, E., & Salesi, G. (1997b). The velocity operator for spinning quantum particles. In J. Keller & Z. Oziewicz (Eds.), *The Theory of the Electron* (pp. 269–278). Mexico City: UNAM Press.
- Recami, E., & Salesi, G. (1998a). Kinematics and hydrodynamics of spinning particles. *Physical Review*, *A57*, 98–105.

- Recami, E., & Salesi, G. (1998b). A velocity field and operator for spinning particles in (nonrelativistic) Quantum Mechanics. *Foundations of Physics*, 28, 763–776.
- Riewe, F. (1971). Generalized Mechanics of a Spinning Particle. *Lett. Nuovo Cim.*, 1, 807–808.
- Rodrigues, W. A., Vaz, J., & Recami, E. (1993). A generalization of Dirac Nonlinear electrodynamics, and spinning charged particles. *Foundations of Physics*, 23, 469–485.
- Rodrigues, W. A., Vaz, J., Recami, E., & Salesi, G. (1993). About zitterbewegung and electron structure. *Physics Letters*, B318, 623–628.
- Rohrlich, F. (1960). Self-energy and stability of the classical electron. *American Journal of Physics*, 28, 639–643.
- Rohrlich, F. (1965). *Classical Charged Particles*. Reading: Addison-Wesley.
- Rosen, G. (1964). Radiative reaction and a possible theory for the muon. *Nuovo Cimento*, 32, 1037–1045.
- Rosen, G. (1978). Approach to a theory for the masses of the electron, muon, and heavier charged leptons. *Int. J. Theor. Phys.*, 17, 1–4.
- Sachs, M. (1972a). The Electron-Muon Mass Doublet from General Relativity. *Nuovo Cimento*, B7, 247–264.
- Sachs, M. (1972b). On the lifetime of the muon state of the electron-muon mass doublet. *Nuovo Cimento*, B10, 339–347.
- Salesi, G., & Recami, E. (1994). *On the field theory of the extended-like electron*. Trieste: ICTP. Report IC/94/185.
- Salesi, G., & Recami, E. (1995). *About the kinematics of spinning particles*. Frascati: INFN. Report INFN/AE-95/16.
- Salesi, G., & Recami, E. (1996). Field theory of the extended-like electron. In P. I. Pronin & G. A. Sardanashvily (Eds.), *Particles, Gravity and Space-Time* (pp. 345–368). Singapore: World Scientific.
- Salesi, G., & Recami, E. (1997a). The velocity operator for spinning particles in quantum mechanics. *Adv. Appl. Cliff. Alg.*, 7, 269–278.
- Salesi, G., & Recami, E. (1997b). Hydrodynamical reformulation and quantum limit of the Barut–Zanghi theory. *Foundations of Physics Letters*, 10, 533–546.
- Salesi, G., & Recami, E. (1998). A Velocity Field and Operator for Spinning Particles in (Nonrelativistic) Quantum Mechanics. Invited paper for the issue in memory of A. O. Barut. *Foundations of Physics*, 28, 763–776.
- Santilli, R. M. (1979a). Status of the mathematical and physical studies on the Lie-admissible formulation as of July 1979, with particular reference to strong interactions. *Hadronic Journal*, 2, 1460–2019.
- Santilli, R. M. (1979b). Initiation of the representation theory of a Lie-admissible algebra of operators on a bimodular Hilbert space. *Hadronic Journal*, 3, 440–506.
- Santilli, R. M. (1981a). An intriguing legacy of Einstein, Fermi, Jordan, and others: The possible invalidation of quark conjectures. *Foundations of Physics*, 11, 383–472.
- Santilli, R. M. (1981b). Generalization Of Heisenberg Uncertainty Principle For Strong Interactions. *Hadronic Journal*, 4, 642–657.
- Santilli, R. M. (1981c). Experimental, Theoretical, And Mathematical Elements For A Possible Lie Admissible Generalization Of The Notion Of Particle Under Strong Interactions. *Hadronic Journal*, 4, 1166–1257.
- Santilli, R. M. (1983). *Foundations of Theoretical Mechanics – Vol. II: Birkhoffian Generalization of Hamiltonian Mechanics*. New York and Heidelberg: Springer.
- Schenberg, M. (1945). Classical Theory of the Point Electron. *Physical Review*, 69, 211–224.
- Schott, G. A. (1912). *Electromagnetic Radiation*. Cambridge.
- Schrödinger, E. (1930). Über die kräftefreie Bewegung in der relativistischen Quantenmechanik. *Sitzunger. Preuss. Akad. Wiss. Phys. Math. Kl.*, 24, 418–428.
- Schrödinger, E. (1952). Are There Quantum Jumps? Part I. *The British Journal for the Philosophy of Science*, 3, 109–123.

- Snyder, H. S. (1947). Quantized space-time. *Physical Review*, 71, 38–41.
- Sorg, M. (1976). The problem of radiation reaction in classical electrodynamics. *Zeitschrift für Naturforschung*, 31A, 1133–1142.
- Sudarshan, E. C. G. (1961). Theory of Leptons I. *Nuovo Cimento*, 21, 7–28.
- Tati, T. (1964). Concepts of Space-Time in Physical Theories. *Progress of Theoretical Physics Supplement*, 29, 1–96.
- Tennakone, K., & Pakvasa, S. (1971). Discrete Scale Transformations and a Possible Lepton Mass Spectrum. *Physical Review Letters*, 27, 757–760.
- Tennakone, K., & Pakvasa, S. (1972). A Model of Leptons. *Physical Review*, D6, 2494–2512.
- Thomson, J. J. (1881). On the electric and magnetic effects produced by the motion of electrified bodies. *Philosophical Magazine*, 11, 229–249.
- Thomson, J. J. (1925–26). The intermittence of electric force. *Proceedings of the Royal Society of Edinburgh*, 46, 90–115.
- Vasholz, D. P. (1975). Discrete space quantum mechanics. *Journal of Mathematical Physics*, 16, 1739–1745.
- von Laue, M. (1909). Die Wellenstrahlung einer bewegten Punktladung nach dem Relativitätsprinzip. *Annalen Der Physik*, 28, 436–442.
- Welch, L. O. (1976). Quantum mechanics in a discrete space-time. *Nuovo Cimento*, B31, 279–288.
- Weyssenhof, J., & Raabe, A. (1947). Relativistic dynamics of spin-fluids and spin-particles. *Acta Physica Polonica*, 9, 7–25.
- Wheeler, J. A. (1957). Assessment of Everett's 'Relative state' formulation of quantum theory. *Reviews of Modern Physics*, 29, 463–465.
- Wheeler, J. A., & Feynman, R. (1945). Interaction with the absorber as the mechanism of radiation. *Reviews of Modern Physics*, 17, 157–181.
- Wigner, E. P. (1963). The problem of measurement. *American Journal of Physics*, 31, 6–15.
- Wolf, C. (1987a). Possible implications of the discrete nature of time in electron spin polarization measurements. *Nuovo Cimento*, B100, 431–434.
- Wolf, C. (1987b). Electron spin resonance as a probe to measure the possible discrete nature of Time as a dynamical variable. *Physics Letters*, A123, 208–210.
- Wolf, C. (1989a). Discrete time effects and the spectrum of hydrogen. *European Journal of Physics*, 10, 197–199.
- Wolf, C. (1989b). Modification Of The Mass Momentum Energy Relation Due To Non-local Discrete Time Quantum Effects. *Nuovo Cimento*, B103, 649–654.
- Wolf, C. (1990a). Testing discrete quantum mechanics using neutron interferometry and the superposition principle: A gedanken experiment. *Foundations of Physics*, 20, 133–137.
- Wolf, C. (1990b). Wave packet anomalies due to a discrete space, discrete time modified Schrödinger equation. *Annales De La Fondation Louis De Broglie*, 15, 189–194.
- Wolf, C. (1990c). Probing the possible composite structure of leptons through discrete time effects. *Hadronic J*, 13, 22–29.
- Wolf, C. (1990d). Spin flip spectra of a particle with composite dyon structure in discrete time quantum theory. *Annales De La Fondation Louis De Broglie*, 15, 487–495.
- Wolf, C. (1992a). Space, mutated spin correlation induced by a non-local discrete time quantum theory. *Hadronic Journal*, 15, 149–161.
- Wolf, C. (1992b). Countability versus Continuum in the quantum mechanics of a free particle. In H. C. Myung (Ed.), *Proceedings of the Fifth International Conference on Hadronic Mechanics and nonpotentials interactions* (pp. 417–428). Commack: Nova Science.
- Wolf, C. (1994). Upper limit for the mass of an elementary particle due to a discrete-time quantum mechanics. *Nuovo Cimento*, B109, 213–220.
- Yaghjian, A. D. (1989). A classical electro-gravitational model of a point charge with a finite mass. *Proc. URSI symposium on electromagnetic theory*, 322–324.

- Yaghjian, A. D. (1992). *Relativistic Dynamics of a Charged Sphere (Updating the Abraham–Lorentz model)*. Berlin: Springer.
- Yukawa, H. (1966). Atomistics and the divisibility of space and time. *Suppl. Prog. Theor. Phys.*, 37–38, 512–523.
- Zin, G. (1949). Su alcune questioni di elettrodinamica classica relative al moto dell'elettrone. *Nuovo Cimento*, 6, 1–23.