

## Deriving Spin within a Discrete-Time Theory

Erasmus Recami<sup>1,2,3</sup> and Giovanni Salesi<sup>1,2</sup>

Received August 27, 2006; revised November 2, 2006; Published online January 19, 2007

Communicated by Alwyn van der Merwe

---

*We prove that the classical theory with a discrete time (chronon) is a particular case of a more general theory in which spinning particles are associated with generalized Lagrangians containing time-derivatives of any order (a theory that has been called “Non-Newtonian Mechanics”). As a consequence, we get, for instance, a classical kinematical derivation of Hamiltonian and spin vector for the mentioned chronon theory (e.g., in Caldirola et al.’s formulation). Namely, we show that the extension of classical mechanics obtained by the introduction of an elementary time-interval does actually entail the arising of an intrinsic angular momentum; so that it may constitute a possible alternative to string theory in order to account for the internal degrees of freedom of the microsystems.*

---

**KEY WORDS:** chronon; discrete time; classical spin.

### 1. INTRODUCTION: AN ELEMENTARY QUANTUM OF TIME IN PARTICLE PHYSICS

The concept of an elementary *time-duration* (a “quantum of time”) has recently returned into fashion in GUTs,<sup>(1)</sup> in String Theories,<sup>(2)</sup> in Quantum Gravity,<sup>(3)</sup> and in the approaches regarding spacetime either as a sort of quantum ether, or as a spacetime foam,<sup>(4)</sup> or as endowed with a non-commutative geometry (like in Deformed or Double Special

---

<sup>1</sup> Università Statale di Bergamo, Facoltà di Ingegneria, viale Marconi 5, 24044 Dalmine (BG), Italy.

<sup>2</sup> Istituto Nazionale di Fisica Nucleare–Sezione di Milano, Italy via Celoria 16, 20133 Milan, Italy.

<sup>3</sup> To whom correspondence should be addressed. E-mail: recami@mi.infn.it

Relativity<sup>(5)</sup>).<sup>4</sup> In particular, the M-theory and the Loop Quantum Gravity (as in its version with semiclassical spin-network structure of spacetime)<sup>(6)</sup> lead even to a discrete *space-time*, in which a fundamental time-scale (or, equivalently, a mass-energy scale) naturally arises, besides  $\hbar$  and  $c$ .

It is known, and we shall see it again below, that the time-discretization implies for an elementary object an “internal” motion associated with microscopic space distances: so that one can expect for such an elementary particle an extended-like, rather than a pointlike, structure. This is a good since, indeed, even the *classical* theory of a pointlike charged particle leads to obvious divergencies, only seemingly overcome by the renormalization techniques (indeed, in QED the infinities do actually persist). Moreover, a fundamental length is known to be directly linked to the existence of the energy cut-off needed for avoiding the so-called ultraviolet catastrophes in any quantum field theories.

One of the first, and simplest, theories which assumed *a priori* a minimum time interval was Caldirola’s theory of the electron,<sup>(7)</sup> based on the existence of an elementary proper-time duration: the so-called *chronon*.

Such a finite difference theory possesses rather good characteristics: for instance, it succeeds—already at the classical level—in forwarding a solution for the motion of a particle endowed with a non-negligible charge in an external electromagnetic field, overcoming all the known difficulties met by Abraham–Lorentz’s and Dirac’s approaches (and even allowing a clear answer to the question whether a free falling charged particle does or does not emit radiation); while—at the quantum level—it yields a remarkable mass spectrum for leptons.

## 2. ABOUT THE “CHRONON” THEORY

Let us recall that Caldirola’s theory seems, moreover, to explain the origin of the “classical (Schwinger’s) part,”  $e\hbar/2mc \cdot \alpha/2\pi = e^3/4\pi mc^2$ , of the anomalous magnetic momentum of the electron.

In the classical version of his theory, however, Caldirola excluded *a priori* the existence of spin contributions in his chronon approach. In contrast, we are going to demonstrate *the rising, even within the classical chronon theory, of an intrinsic angular momentum*.

<sup>4</sup>In the first of ref. 5 it is stated that “the special role of the time coordinate in the structure of  $k$ -Minkowski spacetime forces one to introduce an element of discretization in the time direction: the time derivative of time-to-the-right-ordered functions is indeed standard (just like the  $x$ -derivative of time-to-the-right-ordered functions is standard), but it is a standard  $\lambda$ -discretized derivative (whereas the  $x$ -derivative of time-to-the-right-ordered functions is a standard continuous derivative).”

First of all, let us describe, by starting, e.g., from ref. 7, how the chronon theory allows overcoming—as we were saying—well-known problems like the so-called “pre-accelerations” and “run-away solutions” of the Abraham–Lorentz–Dirac equation for the electron, and furnishing a clear solution to the ambiguities associated with the “hyperbolic motion.”

If  $\rho$  is the charge density of a particle on which an external electromagnetic field acts, the famous Lorentz’s force law

$$\mathbf{F} = q \left( \mathbf{E} + \frac{1}{c} \mathbf{v} \wedge \mathbf{B} \right) \quad (1)$$

is valid only when the particle charge  $q$  is negligible with respect to the external field sources. Otherwise, the classical problem of the motion of a (non-negligible) charge in an electromagnetic field is still an open question.<sup>(8)</sup> For instance, after the known attempts by Abraham and Lorentz, ref. 9 obtained and proposed his famous classical equation

$$m \frac{du_\mu}{ds} = F_\mu + \Gamma_\mu, \quad (2)$$

where  $\Gamma_\mu$  is the Abraham 4-vector

$$\Gamma_\mu = \frac{2}{3} \frac{e^2}{c} \left( \frac{d^2 u_\mu}{ds^2} + \frac{u_\mu u^\nu}{c^2} \frac{d^2 u_\nu}{ds^2} \right), \quad (3)$$

that is, the (Abraham) reaction force acting on the electron itself; and  $F_\mu$  is the 4-vector which represents the external field acting on the particle:

$$F_\mu = \frac{e}{c} F_{\mu\nu} u^\nu. \quad (4)$$

At the non-relativistic limit, Dirac’s equation goes formally into the one previously obtained by Abraham–Lorentz:

$$m_0 \frac{d\mathbf{v}}{dt} - \frac{2}{3} \frac{e^2}{c^3} \frac{d^2 \mathbf{v}}{dt^2} = e \left( \mathbf{E} + \frac{1}{c} \mathbf{v} \wedge \mathbf{B} \right). \quad (5)$$

The last equation shows that the reaction force equals  $\frac{2}{3} \frac{e^2}{c^3} \frac{d^2 \mathbf{v}}{dt^2}$ .

Dirac’s dynamical Eq. (2) is known to present, however, many troubles, related with the infinite many non-physical solutions that it possesses. Actually, it is a third-order differential equation, requiring three initial conditions for singling out a solutions of its. In the description of a *free*

electron, e.g., it yields “self-accelerating” solutions (*run-away solutions*), for which velocity and acceleration increase spontaneously and indefinitely. Moreover, for an electron submitted to an electromagnetic pulse, further non-physical solutions appear, related this time to *pre-accelerations*: If the electron comes from infinity with a uniform velocity  $v_0$  and, at a certain instant of time  $t_0$ , it will be submitted to an electromagnetic pulse, then it starts accelerating *before*  $t_0$ . Drawbacks like these motivated further attempts to find out a coherent (not pointlike) model for the classical electron.

Considering elementary particles as points is probably the sin plaguing modern physics (a plague that, unsolved in classical physics, was transferred to quantum physics). One of the simplest way for associating a discreteness with elementary particles (let us consider, initially, an electron) is just by the introduction (not of a “time-lattice,” but merely, following<sup>(7)</sup>) of a chronon. We are going to see that, like Dirac’s, Caldirola’s approach is also Lorentz invariant (“continuity,” in fact, is not an assumption required by Lorentz invariance). Let us postulate the existence of a universal interval  $\tau_0$  of *proper* time, even if time flows continuously as in the ordinary theories. When an external force acts on the electron, however, the reaction of the particle to the applied force is not continuous: the value of the electron velocity  $u_\mu$  is supposed to jump from  $u_\mu(\tau - \tau_0)$  to  $u_\mu(\tau)$  *only at certain positions*  $s_n$  along its world line; *these “discrete positions” being such that the electron takes a time  $\tau_0$  for traveling from one position  $s_{n-1}$  to the next one  $s_n$* . The electron, in principle, is still considered as pointlike, but the Dirac relativistic equations for the classical radiating electron are replaced: (i) by a corresponding *finite-difference* (retarded) equation in the velocity  $u^\mu(\tau)$

$$\begin{aligned} \frac{m_0}{\tau_0} \left\{ u_\mu(\tau) - u_\mu(\tau - \tau_0) + \frac{u_\mu(\tau) u_\nu(\tau)}{c^2} [u_\nu(\tau) - u_\nu(\tau - \tau_0)] \right\} \\ = \frac{e}{c} F_{\mu\nu}(\tau) u_\nu(\tau), \end{aligned} \quad (6)$$

which reduces to the Dirac Eq. (2) when  $\tau_0$  is small w.r.t.  $\tau$ ; and (ii) by a second equation [the *transmission law*], connecting this time the discrete positions  $x^\mu(\tau)$  along the world line of the particle among themselves:

$$x_\mu(n\tau_0) - x_\mu[(n-1)\tau_0] = \frac{\tau_0}{2} \{ u_\mu(n\tau_0) - u_\mu[(n-1)\tau_0] \}, \quad (6')$$

which is valid inside each discrete interval  $\tau_0$  and describes the *internal* motion of the electron. In such equations,  $u^\mu(\tau)$  is the ordinary 4-vector velocity, satisfying the condition  $u_\mu(\tau)u^\mu(\tau) = -c^2$  for  $\tau = n\tau_0$ , where

$n = 0, 1, 2, \dots$  and  $\mu, \nu = 0, 1, 2, 3$ ; while  $F^{\mu\nu}$  is the external (retarded) electromagnetic field tensor, while the chronon associated with the electron (by comparison with Dirac's equation) results to be, *in the simple case* of an electron interacting with an external field [with  $k \equiv (4\pi\epsilon_0)^{-1}$ ],

$$\frac{\tau_0}{2} \equiv \theta_0 = \frac{2}{3} \frac{ke^2}{m_0c^3} \simeq 6.266 \times 10^{-24} \text{ s}$$

depending, therefore, on the particle (internal) properties (namely, on its charge  $e$  and rest mass  $m_0$ ). Things would become different when considering, e.g., an electron interacting with a macroscopic object, like in the measurement processes.<sup>(10)</sup>

As a result, the electron happens to appear eventually as an extended-like<sup>(11)</sup> particle, with internal structure, rather than as a pointlike object. For instance, one may imagine that the particle does not react instantaneously to the action of an external force because of its finite extension (the numerical value of the chronon, in the above case, is just of order of the time spent by light to travel along an electron classical diameter). As already mentioned, Eq. (6) describes the motion of an object that happens to be pointlike only at discrete positions  $s_n$  along its trajectory; even if both position and velocity are still continuous and well-behaved functions of the parameter  $\tau$ , since they are differentiable functions of  $\tau$ . It is essential to notice that a discrete character is given in this way to the electron without any need of a "model" for the electron. Actually it is well-known that many difficulties are met not only by the strictly pointlike models, but also by the extended-*type* particle models ("spheres," "tops," "gyroscopes," etc.). We deem the answer to stay with a third type of models, the extended-*like* ones, as the present approach; or as the (related) theories in which the center of the pointlike charge is spatially distinct from the particle center-of-mass.<sup>(11)</sup> Let us repeat, anyway, that also the worst troubles met in quantum field theory, like the presence of divergencies, are probably due to the pointlike character still attributed to (spinning) particles; since, as we already remarked, the problem of a suitable model for elementary particles was brought, unsolved, from classical to quantum physics. One might consider that problem to be one of the most important even in modern particle physics.

Equation (6) and the following one provide, together, a full description of the motion of the electron; and they result to be *free* from pre-accelerations, self-accelerating solutions, and problems with the hyperbolic motion.

In the *non-relativistic limit* the previous (retarded) equations get simplified, into the form

$$\frac{m_0}{\tau_0} [\mathbf{v}(t) - \mathbf{v}(t - \tau_0)] = e \left[ \mathbf{E}(t) + \frac{1}{c} \mathbf{v}(t) \wedge \mathbf{B}(t) \right], \quad (7)$$

$$\mathbf{r}(t) - \mathbf{r}(t - \tau_0) = \frac{\tau_0}{2} [\mathbf{v}(t) - \mathbf{v}(t - \tau_0)]. \quad (7')$$

The important point is that Eq. (6), or Eq. (7), bypass the difficulties met by the Dirac classical equation. In fact, the electron *macroscopic* motion gets now completely determined, once velocity and initial position, only, are given. The explicit solutions to the above relativistic equations for the radiating electron—or to the corresponding non-relativistic equations—solve, indeed, the following questions:

(A) *case of exact relativistic solutions*: (1) free electron motion; (2) electron under the action of an electromagnetic pulse; (3) hyperbolic motion;

(B) *case of non-relativistic approximate solutions*: (4) electron under the action of time-dependent forces; (5) electron in a constant, uniform magnetic field; (6) electron moving along a straight line under the action of an elastic restoring force.

An explicit study of the electron radiation properties, as deduced from the finite-difference relativistic Eq. (6), and their series expansions, has been carried out by us in ref. 12, showing in detail the advantages of the present formalism w.r.t. the Abraham–Lorentz–Dirac one.

## 2.1. Three Alternative Formulations

Two more (alternative) formulations are possible of Caldirola's equations, based on different discretization procedures. In fact, Eqs. (6) and (7) describe an intrinsically radiating particle. And, by expanding Eq. (6) in terms of  $\tau_0$ , a radiation reaction term appears: those equations may be called the *retarded* form of the electron equations of the motion.

On the contrary, by rewriting the finite-difference equations in the form

$$\frac{m_0}{\tau_0} \left\{ u_\mu(\tau + \tau_0) - u_\mu(\tau) + \frac{u_\mu(\tau) u_\nu(\tau)}{c^2} [u_\nu(\tau + \tau_0) - u_\nu(\tau)] \right\} = \frac{e}{c} F_{\mu\nu}(\tau) u_\nu(\tau), \quad (8)$$

$$x_\mu[(n+1)\tau_0] - x_\mu(n\tau_0) = \tau_0 u_\mu(n\tau_0) \quad (8')$$

one gets the *advanced* formulation of the electron theory, since the motion is now determined by advanced actions. At variance with the retarded formulation, the advanced one describes an electron which absorbs energy from the external world.

Finally, by adding together retarded and advanced actions, one can write down the *symmetric* formulation of the electron theory:

$$\frac{m_0}{2\tau_0} \left\{ u_\mu(\tau + \tau_0) - u_\mu(\tau - \tau_0) + \frac{u_\mu(\tau) u_\nu(\tau)}{c^2} [u_\nu(\tau + \tau_0) - u_\nu(\tau - \tau_0)] \right\} = \frac{e}{c} F_{\mu\nu}(\tau) u_\nu(\tau), \tag{9}$$

$$x_\mu [(n + 1) \tau_0] - x_\mu ((n - 1) \tau_0) = 2\tau_0 u_\mu (n\tau_0), \tag{9'}$$

which does not include any radiation reactions, and describes a non-radiating electron.

Before ending our introduction to the classical chronon theory, let us mention at least one more result derivable from it. If one considers a free particle and look for the “internal solutions” of Eq. (7'), one gets—for a periodical solution of the type

$$\dot{x} = -\beta_0 c \sin\left(\frac{2\pi\tau}{\tau_0}\right), \quad \dot{y} = -\beta_0 c \cos\left(\frac{2\pi\tau}{\tau_0}\right), \quad \dot{z} = 0$$

(which describes a uniform circular motion) and by imposing the kinetic energy of the internal rotational motion to equal the intrinsic energy  $m_0c^2$  of the particle—that the amplitude of the oscillations is given by  $\beta_0^2 = \frac{3}{4}$ . Thus, the magnetic moment corresponding to this motion is exactly the *anomalous magnetic moment* of the electron, obtained in a purely classical context:  $\mu_a = \frac{1}{4\pi} \frac{e^3}{m_0c^2}$ . This shows, by the way, that the anomalous magnetic moment is an intrinsically classical, and not quantum, result; and the absence of  $\hbar$  in the last expression seems a confirmation of this fact.

As to the three *interesting* formulations that can be analogously constructed in the quantum version of the chronon approach, let us here confine ourselves at quoting.<sup>(10,12)</sup>

### 3. SPIN IN CLASSICAL MECHANICS

In some recent papers of ours,<sup>(11)</sup> it was proposed a classical symplectic theory for extended-like<sup>5</sup> particles, accounting for spin and Zitterbewegung. In particular, in ref. 15, the classical motion of spinning

<sup>5</sup> As already mentioned, the term *extended-like* refers in our language to spinning systems which, even if not “materially” extended, nevertheless are something *half-way* between a point and a rotating body (as, e.g., a top). In fact, let us repeat, in our approaches the center of mass and the center of charge are distinct points, so that velocity and momentum are not parallel vectors, and one meets an internal microscopic motion (the so-called Zitterbewegung<sup>(13,14)</sup>) besides the external one.

particles has been described without recourse to any particular models or formalisms (for instance, without any need of Grassmann variables, Clifford algebras, or classical spinors), but simply by generalizing the standard spinless theory. It was only assumed invariance with respect to the Poincaré group, and, from the conservation of the linear and angular momenta, we derived the Zitterbewegung and the other kinematical properties and motion constraints. One of us, in ref. 15, has called Non-Newtonian Mechanics (NNM) such a classical approach. Indeed, newtonian mechanics is re-obtained as a particular case: namely, for spinless systems with no Zitterbewegung.

Let us start from a Poincaré-invariant Lagrangian, which generalizes the newtonian Lagrangian  $\mathcal{L}^{(0)} = \frac{1}{2}mv^2$  (where  $v^2 \equiv v_\mu v^\mu$ ) by means of proper-time derivatives of the velocity up to the  $N$ th-order (when the scalar potential is  $U = 0$ , and only free particles are considered):

$$\mathcal{L}^{(N)} \equiv \frac{1}{2}Mv^2 + \frac{1}{2}k_1\dot{v}^2 + \frac{1}{2}k_2\ddot{v}^2 + \dots - \equiv \sum_{n=0}^N \frac{1}{2}k_n^N v^{(n)2}, \quad (10)$$

where the notation<sup>(n)</sup> indicates the  $n$ th derivative with respect to  $\tau$  and the  $k_n^N$  are scalar coefficients (endowed with alternate signs). The Euler-Lagrange equation of the motion

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{d\tau} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) - \frac{d^2}{d\tau^2} \left( \frac{\partial \mathcal{L}}{\partial \ddot{x}} \right) + \frac{d^3}{d\tau^3} \left( \frac{\partial \mathcal{L}}{\partial \ddot{\ddot{x}}} \right) - \dots \quad (11)$$

yields a constant-coefficient  $N$ th-order differential equation, which can be regarded as a generalization of Newton law ( $F^\mu = Ma^\mu$ ), in which non-newtonian *Zitterbewegung terms* appear:

$$0 = Ma^\mu + \sum_{n=1}^N (-1)^n k_n^N a^{(2i)\mu}, \quad (12)$$

where we have put here  $F = 0$  since we assumed  $U = 0$ . Incidentally, alternate signs for the coefficients of the terms appearing in the Lagrangian are requested if we want stationary solutions and finite oscillatory motions only.

The zeroth-order canonical momentum

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_\mu} - \frac{d}{d\tau} \left( \frac{\partial \mathcal{L}}{\partial \ddot{x}_\mu} \right) + \frac{d^2}{d\tau^2} \left( \frac{\partial \mathcal{L}}{\partial \ddot{\ddot{x}}_\mu} \right) - \dots$$

conjugate to  $x^\mu$ , writes:

$$p_{[0]}^\mu = Mv^\mu + \sum_{n=1}^N (-1)^n k_n^N v^{(2n)\mu} = \sum_{n=0}^N (-1)^n k_n^N v^{(2n)\mu}. \quad (13)$$

By imposing the symmetry of the Lagrangian under 4-rotations, one gets the conservation of the total angular momentum, which results to be composed of the usual orbital angular momentum tensor and of a classical spin tensor, defined by employing classical kinematical quantities only. Indeed, via the Noether theorem, the spin tensor and the spin vector can be written as follows:

$$S_{\mu\nu} = \sum_{n=1}^N k_n^N \sum_{l=0}^{n-1} (-1)^{n-l-1} \left( v_{\mu}^{(l)} v_{\nu}^{(2n-l-1)} - v_{\nu}^{(l)} v_{\mu}^{(2n-l-1)} \right), \quad (14)$$

$$\mathbf{s} = \sum_{n=1}^N k_n^N \sum_{l=0}^{n-1} (-1)^{n-l-1} \mathbf{v}^{(l)} \times \mathbf{v}^{(2n-l-1)}, \quad (15)$$

respectively.

As an example let us take  $N = 4$ . One has

$$\mathbf{p} = M\mathbf{v} - k_1\dot{\mathbf{a}} + k_2\ddot{\mathbf{a}} - k_3\mathbf{a}^{(V)} + k_4\mathbf{a}^{(VII)}. \quad (16)$$

The spin is

$$\begin{aligned} \mathbf{s} = & k_1(\mathbf{v} \times \mathbf{a}) + k_2(\mathbf{a} \times \dot{\mathbf{a}} - \mathbf{v} \times \ddot{\mathbf{a}}) + k_3 \left( \dot{\mathbf{a}} \times \ddot{\mathbf{a}} - \mathbf{a} \times \ddot{\mathbf{a}} + \mathbf{v} \times \mathbf{a}^{(IV)} \right) \\ & + k_4 \left( \ddot{\mathbf{a}} \times \ddot{\mathbf{a}} - \dot{\mathbf{a}} \times \mathbf{a}^{(IV)} + \mathbf{a} \times \mathbf{a}^{(V)} - \mathbf{v} \times \mathbf{a}^{(VI)} \right) \end{aligned}$$

thus, after differentiating and simplifying, we obtain

$$\dot{\mathbf{s}} = \mathbf{v} \times \left( k_1\dot{\mathbf{a}} - k_2\ddot{\mathbf{a}} + k_3\mathbf{a}^{(V)} - k_4\mathbf{a}^{(VII)} \right) = \mathbf{v} \times (M\mathbf{v} - \mathbf{p}) = \mathbf{p} \times \mathbf{v} \quad (17)$$

as expected.<sup>6</sup>

The hamiltonian representation of the theory is obtained by introducing, besides the (constant) zeroth-order momentum  $p_{[0]}^{\mu}$  given by Eq. (13), the other (non-constant)  $l$ th-order momenta  $p_{[l]}^{\mu}$  canonically conjugate to  $x_{[l]} \equiv x^{(l)}$

<sup>6</sup>From the conservation of the total angular momentum  $\mathbf{j}$ , the sum of the orbital and spin angular momentum, that is, from relation

$$\mathbf{j} = \dot{\mathbf{l}} + \dot{\mathbf{s}} = 0$$

we actually get, by taking into account also the momentum conservation,  $\dot{\mathbf{p}} = 0$ , that

$$\dot{\mathbf{s}} = -\dot{\mathbf{l}} = -\frac{d}{d\tau}(\mathbf{x} \times \mathbf{p}) = -\mathbf{v} \times \mathbf{p}.$$

$$p_{[l]}^\mu \equiv \sum_{n=l}^N (-1)^{n-l} \frac{d^{n-l}}{d\tau^{n-l}} \left( \frac{\partial \mathcal{L}}{\partial x^{(n+1)}} \right) = \sum_{n=l}^N (-1)^{n-l} k_n^N v^{(2n-l)} \quad (18)$$

[a definition which includes also the  $l = 0$  case, Eq.(13)]. On employing the high-order momenta, the spin vector (15) can be put in the canonical form

$$\mathbf{s} = \sum_{l=1}^N \mathbf{x}_{[l]} \times \mathbf{p}_{[l]} \quad (19)$$

analogous to that of the orbital angular momentum,  $\mathbf{l} = \mathbf{x} \times \mathbf{p}_{[0]}$ .

The *conserved* scalar Hamiltonian, obtained by imposing the  $\tau$ -reparametrization invariance of the Lagrangian, is

$$\mathcal{H} = \sum_{l=0}^N p_{[l]}^\mu \dot{x}_{[l]\mu} - \mathcal{L} = \frac{1}{2} M v^2 + \sum_{n=1}^N k_n^N \left[ \frac{1}{2} v^{(n)2} + \sum_{l=0}^{n-1} (-1)^{n-l} v^{(l)\mu} v_\mu^{(2n-l)} \right]. \quad (20)$$

It can be also shown that a couple of Hamilton equations

$$\dot{x}_{[l]}^\mu = \frac{\partial \mathcal{H}}{\partial p_{[l]\mu}}, \quad \dot{p}_{[l]}^\mu = -\frac{\partial \mathcal{H}}{\partial x_{[l]\mu}} \quad (21)$$

holds for any couple of canonical variables  $(x_{[l]}^\mu; p_{[l]}^\mu)$ , and that the set of the Hamilton equations is globally equivalent to the Euler–Lagrange Eq. (12). The Poisson brackets are here defined as follows

$$\{f, g\} \equiv \sum_{l=0}^N \left( \frac{\partial f}{\partial x_{[l]}^\mu} \frac{\partial g}{\partial p_{[l]\mu}} - \frac{\partial f}{\partial p_{[l]}^\mu} \frac{\partial g}{\partial x_{[l]\mu}} \right), \quad (22)$$

while the time evolution of a generic quantity is given by

$$\dot{G} = \frac{\partial G}{\partial t} + \{\mathcal{H}, G\}. \quad (23)$$

The action  $\mathcal{S} = \int \mathcal{L} d\tau$  can be written in the characteristic form

$$\mathcal{S} = \sum_{l=0}^N \int p_{[l]}^\mu dx_{[l]\mu} - \int \mathcal{H} d\tau \quad (24)$$

from which one has:

$$p_{[l]}^\mu = \frac{\partial \mathcal{S}}{\partial x_{[l]}^\mu}, \quad \mathcal{H} = - \frac{\partial \mathcal{S}}{\partial \tau}. \tag{25}$$

#### 4. SPIN AND HAMILTONIAN IN CHRONON THEORY

In the symmetric formulation of the chronon theory, Caldirola’s Lagrangian<sup>(16)</sup> writes (in the case of free particles) as

$$\mathcal{L} = \sum_{n=0}^{\infty} \frac{1}{2} \frac{(-1)^n M \tau_0^{2n}}{(2n + 1)!} v^{(n)2}. \tag{26}$$

It does coincide, therefore, with the infinite-order ( $N \rightarrow \infty$ ) non-newtonian Lagrangian (10), provided that one assumes

$$k_n \equiv \frac{(-1)^n M \tau_0^{2n}}{(2n + 1)!}.$$

We can say that such a *symmetric* chronon theory can be regarded just as a particular case of NNM, entailing a periodic motion, endowed *a priori* with all the infinite harmonics of the ground frequency  $\omega_0 = 2\pi/2\tau_0 = \pi/\tau_0$ .

We shall prove that, in the absence of external fields, the (total) velocity  $v^\mu$  can be expressed by a generic periodic function expanded in Fourier series (in the following,  $E_m^\mu$  and  $H_m^\mu$  are arbitrary constant space-like 4-vectors fixing the “internal” initial conditions, whilst  $p^\mu$  fixes the “external” one<sup>(11,14,15,17)</sup>)<sup>7</sup>

$$v^\mu = \frac{p^\mu}{M} + \sum_{m=1}^{\infty} E_m^\mu \cos(m\omega_0\tau) + H_m^\mu \sin(m\omega_0\tau). \tag{27}$$

Let us recall that the most general representations of the Lorentz group imply that the space–time rotation operator  $J^{\mu\nu}$  is (already classically) decomposed into an orbital part and a spin part,  $J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}$ . Incidentally, the phase-space doubles, and the 4-rotations parameters are  $6 + 6 = 12$ . As a consequence, the motion consists in two parts, the translational (external, newtonian) and the rotational (internal) term; and correspondingly the total velocity  $v$  consists in the two parts  $v = w + V$ . The former, i.e., the drift velocity  $w \equiv p^\mu/M$ , corresponds to the

<sup>7</sup> Solution (27) with  $E_m, H_m \neq 0$  holds only for  $\tau_0 \neq 0$ ; whilst in the “newtonian” limit,  $\tau_0 \rightarrow 0$ , the (free particle) motion equations reduce, of course, to  $a^\mu = 0, v^\mu = p^\mu/M$ , see Eq. (28).

center-of-mass motion, namely to the average motion of the considered particle; by definition, since  $p^2 = M^2$ , one gets that as usual  $w^2 = 1$ . One meets only subluminal speeds for such an external speed, which expresses the 4-momentum propagation speed, and is the only one till now experimentally observed. The latter,  $V = \sum_{m=1}^{\infty} E_m^\mu \cos(m\omega_0\tau) + H_m^\mu \sin(m\omega_0\tau)$ , is directly linked to the discrete time approach, and simultaneously happens to describe the spin motion.

Notice that, on condition that the time parameter be just the time referred to the center-of-mass frame, the total 4-velocity squared  $v^2$  turns out to be *not* constrained to 1, not only in the present theory, but also in many other recent works describing spinning “rigid” particles<sup>(18–20)</sup> where the the classical action contains additional terms dependent on the so-called “extrinsic curvature” (that is, on the 4-acceleration squared).<sup>8</sup> Let us also recall that the kinematical explicit derivation of the spin vector from a classical Lagrangian, presented in this paper for the first time within a discrete-time theory, is a very recent result obtained by us and very few others.<sup>(11,15,18–20)</sup> The previous authors engaged with the chronon approaches (see, e.g., ref. 16) did not notice the natural emergence of spin from time discreteness.

The solution (27) satisfies the Euler–Lagrange equation for Lagrangian (26), that is, to say

$$Ma + M \frac{\tau_0^2}{3!} \ddot{a} + M \frac{\tau_0^4}{5!} a^{(IV)} + \dots = \sum_{n=0}^{\infty} \frac{M \tau_0^{2n}}{(2n+1)!} a^{(2n)} = 0 \quad (28)$$

as well as the (relativistic and non-relativistic) Caldirola equations in the non-symmetrical (advanced or retarded) formulations, in the absence of electromagnetic fields. This can be easily proved as follows. Solution (27) is periodic with period  $2\tau_0$ ; then we have for any  $\tau$

$$v_\mu(\tau + \tau_0) - v_\mu(\tau - \tau_0) = 0.$$

On expanding in Taylor series, one gets

<sup>8</sup> Classical equations of the motion for a rigid particle or for a rigid  $n$ -dimensional worldsheet, either in flat or curved background spacetimes, have been derived from Lagrangians containing also terms dependent on higher derivatives of the 4-velocity (“torsion”-terms, etc.). In ref. 21 the equations of motion are reformulated in terms of the principal worldline curvatures which result to be motion integrals, namely mass and spin. As it occurs in NNM, in the mentioned approaches the total velocity squared is in general a function of mass and spin as well.

$$\sum_{n=0}^{\infty} \frac{v_{\mu}^{(n)}(\tau)}{n!} \tau_0^n - \sum_{n=0}^{\infty} (-1)^n \frac{v_{\mu}^{(n)}(\tau)}{n!} \tau_0^n = 0$$

and then

$$\sum_{n=0}^{\infty} \frac{v_{\mu}^{(2n+1)}(\tau)}{(2n+1)!} \tau_0^{2n+1} \equiv \sum_{n=0}^{\infty} \frac{a_{\mu}^{(2n)}(\tau)}{(2n+1)!} \tau_0^{2n+1} = 0 \quad (29)$$

from which Eq. (28) follows.

Let us find the explicit expression of the spin vector in the chronon theory, choosing for convenience the center-of-mass (where  $\mathbf{p} = 0$ ) as the reference frame. By inserting Eq. (27) with  $\mathbf{p} = 0$  into equation

$$\mathbf{s} = \sum_{n=1}^{\infty} k_n \sum_{l=0}^{n-1} (-1)^{n-l-1} \mathbf{v}^{(l)} \times \mathbf{v}^{(2n-l-1)}$$

[which is nothing but Eq. (15) for  $N \rightarrow \infty$ ], after some algebra one obtains

$$\mathbf{s} = \sum_{m=1}^{\infty} A_m (\mathbf{E}_m \times \mathbf{H}_m) \quad (30)$$

with

$$A_m \equiv \frac{1}{2} \sum_{n=1}^{\infty} n \bar{k}_n m^{2n}, \quad (31)$$

where the dimensionless coefficients  $\bar{k}_n$  are defined as follows:

$$\bar{k}_n \equiv k_n \frac{\omega_0^{2n}}{M}. \quad (32)$$

By exploiting the explicit expression of  $\bar{k}_n$ , we have

$$\sum_{n=0}^{\infty} \bar{k}_n x^{2n} = \frac{1}{\pi x} \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} x^{2n+1} = \frac{\sin(\pi x)}{\pi x}.$$

Differentiating side-by-side the above equation, we obtain

$$\sum_{n=0}^{\infty} n \bar{k}_n x^{2n-1} = \frac{\cos(\pi x)}{2x} - \frac{\sin(\pi x)}{2\pi x^2},$$

which, after multiplication of both its sides by  $x$ , yields for  $x = m \in \mathbb{N}^+$ :

$$\sum_{n=0}^{\infty} n \bar{k}_n m^{2n} = \frac{\cos(\pi m)}{2} = \frac{(-1)^m}{2}. \quad (33)$$

The property

$$\sum_{n=0}^{\infty} \bar{k}_n m^{2n} = 0 \quad (34)$$

holds also for any positive integer  $m$ . Indeed, this relation can be obtained by putting  $x$  equal to  $m \in \mathbb{N}^+$  into the equation

$$\sum_{n=0}^{\infty} \bar{k}_n x^{2n} = \frac{\sin(\pi x)}{\pi x}.$$

Taking into account Eq. (33), the spin vector can be eventually written as follows:

$$\mathbf{s} = \frac{1}{4} \sum_{m=1}^{\infty} (-1)^m \mathbf{E}_m \times \mathbf{H}_m. \quad (35)$$

We can proceed quite analogously in order to get the Hamiltonian in the chronon theory, in a generic reference-frame. By exploiting Eq. (20), with  $N \rightarrow \infty$ ,

$$\mathcal{H} = \frac{1}{2} M v^2 + \sum_{n=1}^{\infty} k_n \left[ \frac{1}{2} v^{(n)2} + \sum_{l=0}^{n-1} (-1)^{n-l} v^{(l)\mu} v_{\mu}^{(2n-l)} \right]$$

after some calculations we get ( $E^2 \equiv E_{\mu} E^{\mu}$ ;  $H^2 \equiv H_{\mu} H^{\mu}$ ) that

$$\mathcal{H} = \frac{p^2}{2M} + M^3 \sum_{m=1}^{\infty} B_m (E_m^2 + H_m^2) \quad (36)$$

with

$$B_m \equiv 1 + 2 \sum_{n=1}^{\infty} \left( n + \frac{1}{2} \right) \bar{k}_n m^{2n}. \quad (37)$$

Taking into account Eqs. (33) and (34), we finally have:

$$\mathcal{H} = \frac{p^2}{2M} + M^3 \sum_{m=1}^{\infty} [1 + (-1)^m] (E_m^2 + H_m^2), \quad (38)$$

where, besides the ordinary “external” drift term  $\frac{p^2}{2M}$ , it appears an “internal” term, which seem to constitute a signature of the actual “non-newtonian nature” of the chronon theory.

## 5. CONCLUSIONS

In this paper, we have analytically derived the spin vector and the Hamiltonian for a *non-quantum* theory employing a discretized time: the chronon theory. We have shown that such an approach is nothing but a particular case of NNM (“non-newtonian mechanics”), a classical theory where the motion of the spinning, extended-like particles is described in terms of an infinite set of time derivatives: the spin arises just from this underlying non-local structure. In spite of the classical character of the chronon theory, we have obtained an explicit kinematical formulation of the intrinsic angular momentum (usually considered a pure *quantum* quantity), through a sum over all harmonic modes of the solution to the motion equation.

The present approach to particle dynamics appears as an alternative to the better known string model, for taking account of the internal degrees of freedom which generate the rich variety of the observed particles. As a matter of fact, also the general solution to the string motion equation consists of a sum over all harmonic modes; further, in ref. 17, one of us has shown that even bosonic strings obey the motion equation of the chronon theory, Eq.(28). In a sense, the chronon theory (in particular, Caldirola’s) might be considered as having anticipated, 50 years ago already, the string concept of our days. In a forthcoming paper, we shall try to study more deeply the quantized version<sup>(12)</sup> Caldirola’s theory, in order to obtain the allowed spin states and a mass spectrum.<sup>9</sup>

<sup>9</sup> On this respect, let us anticipate that, however, in a (second-quantization) quantum field theory, the quantization follows from the replacement of the Poisson brackets with the corresponding commutators: so that no explicit time-discretization will be really needed (as, instead, it occurs in the first-quantization of the chronon theory, which has to make recourse to finite-difference Schrödinger-like wave equations.<sup>(12)</sup>)

## ACKNOWLEDGMENTS

Many thanks are due to Ruy H.A. Farias, for his very useful collaboration along many years, and to Rodolfo Bonifacio, Michel Zamboni-Rached for some stimulating discussions or hints. This work has been partially supported by I.N.F.N. and M.I.U.R.

## REFERENCES

1. D. Bailey and A. Love, *Supersymmetric Gauge Field Theory and String Theory*, Chap. 6, and references therein (I.O.P. Publishing, Bristol, 1994).
2. V. A. Kostelecký and S. Samuel, *Phys. Rev. D* **39**, 683 (1989); *Phys. Rev. Lett.* **63**, 224 (1989); **66**, 1811 (1991); V. A. Kostelecký and R. Potting, *Nucl. Phys. B* **359**, 545 (1991); *Phys. Lett. B* **381**, 89 (1996); *Phys. Rev. D* **63**, 046007 (2001); V. A. Kostelecký, M. J. Perry, and R. Potting, *Phys. Rev. Lett.* **84**, 4541 (2000); C. P. Burgess, *JHEP* **0409**, 033 (2004); *JHEP* **0203**, 043 (2002); A. R. Frey, *JHEP* **0304**, 012 (2003); J. Cline and L. Valcárcel, e-print ph/0312245; F. Lizzi and R. J. Szabo, *Commun. Math. Phys.* **197**, 667 (1998); F. Lizzi, G. Mangano and G. Miele, *JHEP* **06**, 049 (2002); M. Kalb and D. Ramond, *Phys. Rev. D* **9**, 2273 (1974); E. Cremmer and J. Scherk, *Nucl. Phys.* **72**, 117 (1974); Y. Nambu, *Phys. Rep. C* **23**, 251 (1976).
3. F. R. Klinkhamer, *Nucl. Phys. B* **578**, 277 (2000); J. Alfaro, H. A. Morales-Técotl and L. F. Urrutia, *Phys. Rev. D* **66**, 124006 (2002); D. Sudarsky, L. Urrutia, and H. Vucetich, *Phys. Rev. Lett.* **89**, 231301 (2002); *Phys. Rev. D* **68**, 024010 (2003); F. R. Klinkhamer and C. Rupp, e-print th/0312032; C. J. Isham, arXiv:gr-qc/9510063; J. Butterfield and C. J. Isham, e-print gr-qc/9901024; C. Rovelli, e-print gr-qc/0006061; *Quantum Gravity* (Cambridge University Press, Cambridge, 2004); G. Amelino-Camelia, *Nature* **408**, 661 (2000); C. P. Burgess, *Living Rev. Rel.* **7**, 5 (2004).
4. L. J. Garay, *Phys. Rev. Lett.* **80**, 2508 (1998); G. Amelino-Camelia, J. R. Ellis, N. E. Mavromatos, and D. V. Nanopoulos, *Nature* **12**, 607 (1997).
5. G. Amelino-Camelia, arXiv:hep-th/0211022; *Phys. Lett. B* **392**, 283 (1997); *Int. J. Mod. Phys. D* **11**, 35 (2002); *Phys. Lett. B* **510**, 255 (2001); G. Amelino-Camelia and T. Piran, *Phys. Rev. D* **64**, 036005 (2001); G. Amelino-Camelia, L. Doplicher, S. Nam, and Y. Seo, *Phys. Rev. D* **67**, 085008 (2003); N. R. Bruno, G. Amelino-Camelia, and J. Kowalski-Glikman, *Phys. Lett. B* **522**, 133 (2001); S. X. Chen and Z. Y. Yang, *Mod. Phys. Lett. A* **18**, 2913 (2003); Z. Y. Yang and S. X. Chen, *J. Phys. A* **35**, 9731 (2002); J. Lukierski, H. Ruegg and W. J. Zakrzewski, *Ann. Phys.* **243**, 90 (1995); Z. Guralnik, R. Jackiw, S. Y. Pi, and A. P. Polychronakos, *Phys. Lett. B* **517**, 450 (2001); V. Nazaryan and C. E. Carlson, *Phys. Rev. D* **71**, 025019 (2005); C. E. Carlson, C. D. Carone, and R. F. Lebed, *Phys. Lett. B* **549**, 337 (2002); M. Hayakawa, *Phys. Lett. B* **478**, 394 (2000); arXiv:hep-th/9912167; S. M. Carroll, J. A. Harvey, V. A. Kostelecký, C. D. Lane, and T. Okamoto, *Phys. Rev. Lett.* **87**, 141601 (2001); A. Anisimov, T. Banks, M. Dine, and M. Graesser, *Phys. Rev. D* **65**, 085032, (2002); T. Kifune, *Astrophys. J. Lett. L* **518**, 21 (1999); W. Kluzniak, arXiv:astro-ph/9905308; R. J. Protheroe and H. Meyer, *Phys. Lett. B* **493**, 1 (2000); D. Bahns, S. Doplicher, K. Fredenhagen, and G. Piacitelli, *Phys. Rev. D* **71**, 025022 (2005); C. K. Zachos, *Mod. Phys. Lett. A* **19**, 1483 (2004).
6. R. Gambini and J. Pullin, *Phys. Rev. D* **59**, 124021 (1999); *Phys. Rev. D* **65**, 103509 (2002); G. 't Hooft, *Class. Quant. Grav.* **13**, 1023 (1996); J. Alfaro, H. A. Morales-Tecotl,

- and L. F. Urrutia, *Phys. Rev. Lett.* **84**, 2318 (2000); C. Rovelli and L. Smolin, *Phys. Rev. D* **52**, 5743 (1995); *Nucl. Phys. B* **442**, 593 (1995); Erratum, *ibid.*, **B 456**, 753 (1995).
7. P. Caldirola, *Nuovo Cimento* **10**, 1747 (1953); *Riv. Nuovo Cimento* **2**(13) (1979), and refs. therein; *Revista Brasil. de Fisica*, special volume (1984, July), p. 228. See also R. Cirelli, *Nuovo Cimento* **1**, 260 (1955); L. Lanz, *Nuovo Cimento* **23**, 195 (1962); F. Casagrande and E. Montaldi, *Nuovo Cimento A* **40**, 369 (1977); *A* **44**, 453 (1978); P. Caldirola and E. Montaldi, *Nuovo Cimento B* **53**, 291 (1979); P. Caldirola, G. Casati, and A. Prosperetti, *Nuovo Cimento A* **43**, 127 (1978); P. Caldirola, *Nuovo Cimento A* **49**, 497 (1979); A. Prosperetti, *Nuovo Cimento B* **57**, 253 (1980); L. Belloni, *Lett. Nuovo Cimento* **31**, 131 (1981); V. Benza and P. Caldirola, *Nuovo Cimento A* **62**, 175 (1981); G. C. Ghirardi and T. Weber, *Lett. Nuovo Cimento* **39**, 157 (1984); and in particular R. Bonifacio and P. Caldirola, *Lett. Nuovo Cimento* **38**, 615 (1983); **33**, 197 (1982). Cf. also T. D. Lee, "Can time be a discrete dynamical variable?," *Phys. Lett. B* **122**, 217 (1983); G. Jaroszkiewicz, "Principles of discrete time mechanics," *J. Phys. A: Math. Gen.* **30**, 3115 (1997); *ibid.*, 3145; **31**, 977 (1998); *ibid.*, 1001.
  8. A. D. Yaghjian, *Relativistic Dynamics of a Charged Sphere* (Springer, New York, 1992).
  9. P. A. M. Dirac, "The classical theory of electron," *Proc. R. Soc. A* **167**, 148 (1938); *Ann. Inst. Poincaré* **9**, 13 (1938). Cf. also M. Schönberg *et al.*, *Phys. Rev.* **69**, 211 (1945); and *Anais Ac. Brasil. Cie.* **19**(3), 46–98 (1947).
  10. E. Recami, "A simple quantum equation for dissipation and decoherence" [report NSF-ITF-02-62 (I.T.P., UCSB, California, 2002)], in *Quantum Computing and Quantum Bits in Mesoscopic Systems*, A. J. Leggett, B. Ruggiero, and P. Silvestrini, ed. (Kluwer/Plenum, New York, 2004), pp. 111–122.
  11. G. Salesi, *Mod. Phys. Lett. A* **11**, 1815 (1996); *Int. J. Mod. Phys. A* **12**, 5103 (1997); G. Salesi and E. Recami, *Phys. Lett. A* **190**, 137 (1994); *A* **195**, E389 (1994); *Found. Phys.* **28**, 763 (1998); E. Recami, and G. Salesi, *Phys. Rev. A* **57**, 98 (1998); *Adv. Appl. Cliff. Alg.* **6**, 27 (1996); in *Gravity, Particles and Space-Time*, P. Pronin and G. Sardanashvily, ed. (World Scientific Singapore, 1996), pp. 345–368; M. Pavšič, E. Recami, W. A. Rodrigues, G. D. Maccarrone, F. Raciti, and G. Salesi, *Phys. Lett. B* **318**, 481 (1993); W. A. Rodrigues, J. Vaz, E. Recami, and G. Salesi, *Phys. Lett. B* **318**, 623 (1993); J. Vaz and W. A. Rodrigues, *Phys. Lett. B* **319**, 203 (1993).
  12. R. H. A. Farias and E. Recami, "Introduction of a quantum of time (chronon), and its consequences for quantum mechanics," Report IC/98/74 (ICTP, Trieste, 1998), appeared in preliminary form as e-print quant-ph/9706059. Cf. also R. H. A. Farias, "Introduction of a 'quantum' of time into the formalism of quantum mechanics," Ph.D Thesis, E. Recami supervisor (UNICAMP, Campinas, S.P., 1994).
  13. P. A. M. Dirac, *The Principles of Quantum Mechanics*, 4th edn. (Clarendon, Oxford, 1958), p. 262; J. Maddox, *Nature* **325**, 306 (1987).
  14. E. Schrödinger, *Sitzunger. Preuss. Akad. Wiss. Phys.-Math. Kl.* **24**, 418 (1930); **25**, 1 (1931).
  15. G. Salesi, *Int. J. Mod. Phys. A* **17**, 347 (2002) e-print: quant-ph/0112052); *A* **20**, 2027 (2005)
  16. P. Caldirola, *Suppl. Nuovo Cimento* **3**, 297 (1956).
  17. G. Salesi, *Found. Phys. Lett.* **19**, 367 (2006).
  18. M. Pavšič, *Phys. Lett. B* **205**, 231 (1988); *B* **221**, 264 (1989); *Class. Quant. Grav.* **L 7**, 187 (1990).
  19. M. S. Plyushchay, "Comment on the relativistic particle with curvature and torsion of world trajectory," arXiv:hep-th/9810101; *Phys. Lett. B* **262**, 71 (1991); *Mod. Phys. Lett. A* **3**, 1299 (1988); *A* **4**, 837 (1989); *A* **4**, 2747 (1989); *Int. J. Mod. Phys. A* **4**, 3851 (1989);

- Phys. Lett. B* **243**, 383 (1990); *Phys. Lett. B* **236**, 291 (1990); *B* **235**, 47 (1990); *B* **253**, 50 (1991).
20. A. M. Polyakov, *Nucl. Phys. B* **268**, 406 (1986); *Mod. Phys. Lett. A* **3**, 325 (1988); Yu. A. Kuznetsov and A. M. Polyakov, *Phys. Lett. B* **297**, 49 (1992).
21. V. V. Nesterenko, A. Feoli, and G. Scarpetta, *J. Math. Phys.* **36**, 5552 (1995); V. V. Nesterenko, *Phys. Lett. B* **327**, 50 (1994).