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General formulation for the analysis of scalar diffraction-free beams using angular modulation: Mathieu and Bessel beams

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Abstract

In this paper, we start from the well known Durnin's experimental setup, finding general analytical formulae to investigate generation and propagation of nondiffracting beams. Our general formula makes possible considering any kind of angular modulation. As an example we discuss the Mathieu beams. Moreover, in the study of Bessel beams we consider the width of the slit to compare with the ideal case represented by a Dirac δ transmittance function.

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1. Introduction

Since the Durnin's fundamental work on Bessel beams [1,2] in 1987, many studies have been done about diffraction-free beams in the optical domain. Most of the published works consider basically Bessel, Bessel–Gauss and Gaussian beams [3,4]. Recently, Gutiérrez-Vega et al. [5,6] also demonstrated experimentally the existence of Mathieu beams. These waves have attracted much attention because of their possible use in different applications like wireless communications [7–9], metrology, laser surgery [6,7], nonlinear optics [9], conduits in atom optics [10,11] and so on.

The nondiffracting beams are adequate superposition of plane waves whose propagation vectors create a conical surface. The amplitudes and phases of these plane waves are arbitrarily chosen so there exist an infinite number of different intensity

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profiles. In previous works it was suggested the use of angular modulation for the generation of partially coherent nondiffracting beams [12,13] and most recently it was reported in [14] the generation of predetermined intensity profiles by using arbitrary angular modulation. We have used here the scalar diffraction theory to obtain general formulae on considering arbitrary angular modulation of a slit in Durnin's experimental setup. Bessel and Mathieu beams have been considered to discuss practical aspects of such waves related to their generation and propagation.

The reminder of this paper is described as follows. The coming section describes the Durnin's experimental setup. In Sections 3 and 4, we analyze the generation and properties of the Mathieu and the Bessel beams, respectively. For the Bessel beams we take into account the width of the slit. Finally, some concluding remarks are given in Section 5.

2. Durnin's experimental setup

The Durnin's experimental setup [1] is illustrated in Fig. 1. In this setup, there is a thin lens of radius R separated by one focal length, f, from an annular slit of diameter $2a$ and thickness Δa. The apparatus is illuminated by a collimated laser beam. For the schematic shown in Fig. 1 the maximum invariance range $Z_{\max}$ can be evaluated by the simple way following from the geometry:

$$Z_{\max} = \frac{R}{\tan\theta}; \quad \tan\theta = \frac{a}{f}. \tag{1}$$

Next, we will develop the analytical description of a pseudo-nondiffracting beam propagating in an apparatus similar to that one presented at Fig. 1 using the scalar diffraction theory. The paraxial approximation of diffraction's integral expressed in terms of Bessel function is

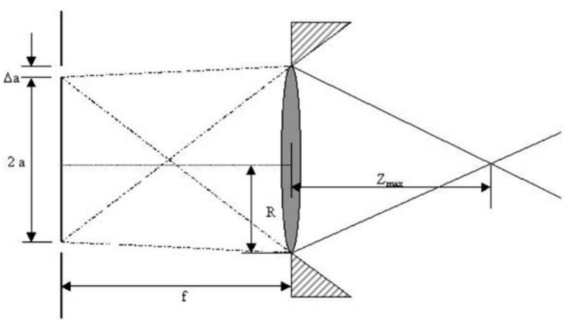

Fig. 1. Experimental setup to create nondiffracting beams.

$$\psi(\rho,\varphi,z) = \frac{k\exp(\mathrm{i}k(z+\rho^2/2z))}{2\pi\mathrm{i}z}\int_0^\infty \rho'\,\mathrm{d}\rho' \times \int_0^{2\pi}\mathrm{d}\varphi'\tau(\rho',\varphi')\psi(\rho',\varphi',0) \times \exp\left(\mathrm{i}\frac{k\rho'^2}{2z}\right)\Lambda\left(\frac{k\rho\rho'}{z},\varphi'-\varphi\right), \tag{2}$$

where k is the vacuum's wave number, $\psi(\rho',\varphi',0)$ is the incident wave, $\tau(\rho',\varphi')$ is the transmittance function (TF) of the aperture, (ρ',φ',z') and (ρ,φ,z) are the diffractive aperture and observation point coordinates and we have defined

$$\Lambda(x,y) = J_0(x) + 2\sum_{n=0}^{\infty}\left[J_{2n+2}(x)\cos\left[(2n+2)\left(y+\frac{\pi}{2}\right)\right] - \mathrm{i}J_{2n+1}(x)\sin\left[(2n+1)\left(y+\frac{\pi}{2}\right)\right]\right]. \tag{3}$$

For an ideal annular slit one should have

$$\tau(\rho',\varphi') = A(\varphi')\delta(\rho'-a), \tag{4}$$

being $A(\varphi')$ any angular function and $\delta(.)$ the Dirac's function, provided that the following condition is satisfied:

$$\Delta a \ll \frac{\lambda f}{R}. \tag{5}$$

Considering the collimated laser beam to be an approximately plane wave of amplitude $\psi(\rho',\varphi',0) = A_0$ incoming into the slit, which has its TF expressed by (4) and using (2) we have found that the wave propagated between the slit and the lens by the distance $z = f$ will be given by

$$\psi(\rho,\varphi,z=f) = \frac{k\exp(\mathrm{i}k(f+\rho^2/2f))}{2\pi\mathrm{i}f} \times \int_0^{2\pi}\mathrm{d}\varphi' A(\varphi')A_0\exp\left(\mathrm{i}\frac{ka^2}{2f}\right) \times \Lambda\left(\frac{ka\rho}{f},\varphi'-\varphi\right). \tag{6}$$

To obtain the behavior of the wave after the lens, we have again applied (2). In addition, we have set up the plane of the lens at $z''=0$ and supposed a convergent thin lens having the TF [15]

$$\tau_{\rm lens}(\rho'',\varphi'') = \exp\left(-\mathrm{i}\frac{k\rho''^2}{2f}\right). \tag{7}$$

Inserting (6) and (7) into (2) the resulting expression is

$$\psi(\rho,\varphi,z) = \frac{B\exp(\mathrm{i}k(z+\rho^2/2z))}{z}\int_0^{2\pi}\mathrm{d}\varphi' \times \int_0^{2\pi}\mathrm{d}\varphi''\int_0^R \rho''\,\mathrm{d}\rho'' A(\varphi') \times \exp\left(\mathrm{i}\frac{k\rho''^2}{2z}\right)\Lambda\left(\frac{ka\rho''}{f},\varphi'-\varphi''\right) \times \Lambda\left(\frac{k\rho\rho''}{z},\varphi''-\varphi\right), \tag{8}$$

being B a new constant factor. Looking at (8), it must be pointed out that the indices $'$ and $''$ are related to the variables at the slit and the lens, respectively.

The angular modulation $A(\varphi')$ must be periodic then it can be represented by a Fourier series, and for sake of convenience we have separated the even and odd terms

$$A(\varphi') = \sum_{r=0}^{\infty}\big(A_{2r}\cos(2r\varphi') + A_{2r+1}\cos((2r+1)\varphi') + B_{2r+1}\sin((2r+1)\varphi') + B_{2r+2}\sin((2r+2)\varphi')\big). \tag{9}$$

The substitution of (9) into (8) and the integration over φ' and φ'' gives rise to

$$\psi(\rho,\varphi,z) = \frac{2\pi B\exp(\mathrm{i}k(z+\rho^2/2z))}{z}\int_0^R \rho''\,\mathrm{d}\rho'' \times \exp\left(\mathrm{i}\frac{k\rho''^2}{2z}\right)\sum_{r=0}^{\infty}\Big\{(A_{2r}\cos(2r\varphi) + B_{2r}\sin(2r\varphi))J_{2r}\left(\frac{ka\rho''}{f}\right)J_{2r}\left(\frac{k\rho\rho''}{z}\right) + (A_{2r+1}\sin((2r+1)\varphi) - B_{2r+1}\cos((2r+1)\varphi))J_{2r+1}\left(\frac{ka\rho''}{f}\right) \times J_{2r+1}\left(\frac{k\rho\rho''}{z}\right)\Big\}, \tag{10}$$

being A_{2r}, A_{2r+1}, B_{2r} and B_{2r+1} the coefficients of the angular modulation.

Finally, (10) is our general expression for an apparatus similar to Fig. 1. Using such expression one has the freedom to choose an annular slit modulated by any angular function $A(\varphi')$. The only condition that must be satisfied is that one given by (5). Some examples are illustrated on the next sections.

3. Mathieu beams

As a next step, an even Mathieu function of order $2m$ modulating a slit with $\tau(\rho',\varphi') = A(\varphi')\delta(\rho'-a)$ has been considered.

$$A(\varphi') = \mathrm{ce}_{2m}(q,\varphi') = \sum_{r=0}^{\infty}A_{2r}^{(2m)}(q)\cos(2r\varphi'). \tag{11}$$

In such case (10) is given by

$$\psi(\rho,\varphi,z) = \frac{2\pi B\exp(\mathrm{i}k(z+\rho^2/2z))}{z}\int_0^R \rho''\,\mathrm{d}\rho'' \times \exp\left(\mathrm{i}\frac{k\rho''^2}{2z}\right)\sum_{r=0}^{\infty}A_{2r}^{(2m)}(q)J_{2r}\left(\frac{ka\rho''}{f}\right) \times J_{2r}\left(\frac{k\rho\rho''}{z}\right)\cos(2r\varphi), \tag{12}$$

being $A_{2r}^{(2m)}(q)$ the Mathieu function coefficients.

We have called (12) as truncated Mathieu beams, being the explanation given later. Fig. 2 shows the three-dimensional intensity pattern evolution of a truncated Mathieu beam. For this figure, we have set $q=25$ in (11), and $\varphi=0$ in (12), which leads to the dependence on the x and z axis, instead of ρ, φ and z. The slit and the lens parameters are an incident plane wave of $\lambda = 632.8$ nm illuminating a slit with $a = 1.25$ mm and $\Delta a = 0.01$ mm and using a lens of $f = 30.5$ cm and $R = 3.5$ mm.

In Figs. 3 and 4, it has appeared the evolution of these Mathieu beams on the planes (x,z) and (y,z), respectively. As can be observed from Fig. 3, this beam has a well defined pattern in x-direction, with almost no oscillations in the transverse direction. On the other side, Fig. 4 has transverse oscillations on the y-direction, being this kind of behavior already expected for the Mathieu beams in the conic region.

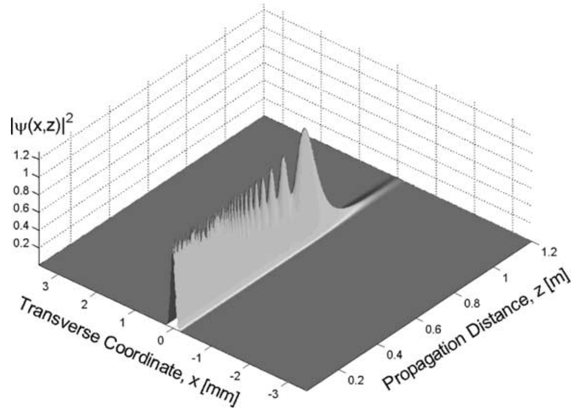

Fig. 2. The 3D evolution of a zero-order Mathieu beam in the plane $x - z$, $\varphi = 0$. $\lambda = 632.8$ nm, $a = 1.25$ mm, $f = 305$ mm, $R = 3.5$ mm and $q = 25$.

To assure the previous beams are truncated Mathieu beams, if one has a lens having infinite radius the integral can be evaluated analytically being the final result [16]

$$\psi(\rho,\varphi,z) = B_1 \exp\left(\mathrm{i}k\left(1 - \frac{a^2}{2f^2}\right)z\right) \times \sum_{r=0}^{\infty} A_{2r}^{(2m)}(q)(-1)^r J_{2r}\left(\frac{ka\rho}{f}\right)\cos(2r\varphi), \tag{13}$$

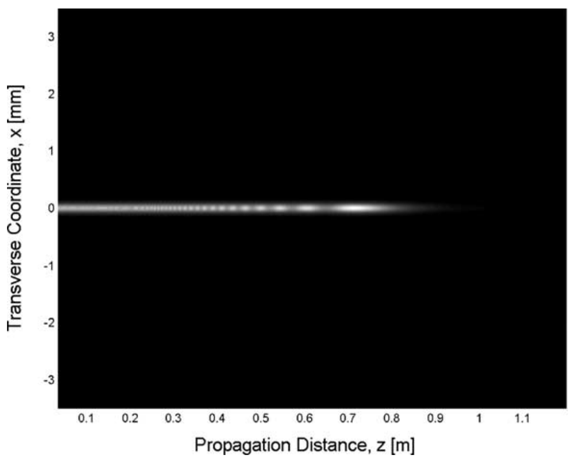

Fig. 3. Evolution of a zero-order Mathieu beam in the plane $x - z$. $\varphi = 0$, and all the slit and lens parameters keep the same of Fig. 2.

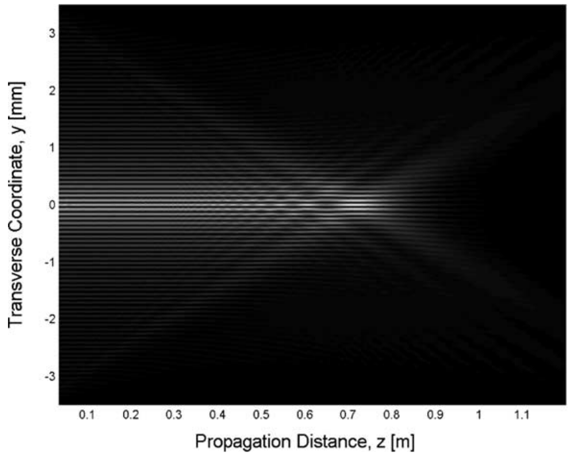

Fig. 4. Evolution of a zero-order Mathieu beam in the plane $y - z$. $\varphi = \pi/2$ rad, and all the slit and lens parameters keep the same of Fig. 2.

where $B_1 = 2\pi \mathrm{i} B/k$. Nevertheless, according to [17], (13) can be rewritten as

$$\psi(u,v,z) = \frac{A_0^{(2m)}(q)B_1}{\mathrm{ce}_{2m}(q,0)\mathrm{ce}_{2m}(q,\pi/2)} \exp(\mathrm{i}k_z z) \times \mathrm{Ce}_{2m}(q,u)\mathrm{ce}_{2m}(q,v), \tag{14}$$

being q defined by

$$q = \frac{h^2 k_\rho^2}{4}; \quad k_\rho \cong \frac{ka}{f},$$

where u, v are the elliptic cylindrical coordinates and the expression (14) being known as an even Mathieu beam order $2m$ and parameter q. This beam propagates indefinitely along z-direction without spreading. For practical purposes the lens has finite dimensions and the beam is truncated.

From this section we have seen that the Mathieu beams can be understood as an adequate superposition of Bessel beams (see (13)). This fact is evident once the Bessel functions form a complete set of orthogonal functions that are solutions of Helmholtz equation in cylindrical coordinates. In the same way, the Bessel beams can be thought as a superposition of Mathieu beams, although it is easier and more natural to think Mathieu beams as the superposition of Bessel beams, as it was recently reported [18].

4. Bessel beams: considering the thickness of the slit

To create a truncated zero-order Bessel beam we have considered a slit with $A(\varphi') = 1$:

$$\psi(\rho,\varphi,z) = \frac{2\pi B \exp(\mathrm{i}k(z+\rho^2/2z))}{z} \int_0^R \rho'' \,\mathrm{d}\rho'' \times \exp\left(\mathrm{i}\frac{k\rho''^2}{2z}\right) J_0\left(\frac{ka\rho''}{f}\right) J_0\left(\frac{k\rho\rho''}{z}\right). \tag{15}$$

Making use of (15), Fig. 5 reproduces a normalized three-dimensional numerical simulation of the experimental setup of Durnin. In this case, it used the focal distance of lens $f = 305$ mm, radius of lens $R = 3.5$ mm and radius of slit $a = 1.25$ mm, with $\Delta a = 10$ μm and $\lambda = 632.8$ nm.

In the Durnin's experiment the predicted value was reported to be 854 mm. In our simulation, the maximum invariance range of the beam, $Z_{\max}$, which could be easily calculated by the expression (1), is in perfect agreement with the Durnin's work (see Fig. 5).

To obtain (10), our only supposition was that (5) should be satisfied. As a point of fact, if it is not the case one should go back to (2) and again do the calculations. In such equation, if one considers a plane wave $(\psi'(\rho',\phi',0) = A_0)$ arriving in the slit with no angular changes $(\tau(\rho',\phi') = 1)$ and evaluates the integral over φ' he will find

$$\psi(\rho,\varphi,z) = \frac{A_0 k \exp(\mathrm{i}k(z+\rho^2/2z))}{\mathrm{i}z} \int_{a-\frac{\Delta a}{2}}^{a+\frac{\Delta a}{2}} \rho' \,\mathrm{d}\rho' \times \exp\left(\mathrm{i}\frac{k\rho'^2}{2z}\right) J_0\left(\frac{k\rho\rho'}{z}\right). \tag{16}$$

Eq. (16) represents the field after the slit but before the lens. After the lens, we have approximated the exponential $\exp\left(\mathrm{i}\frac{k\rho'^2}{2z}\right)$ as a constant over the interval of integration [16], in such a way that (16) results on

$$\psi(\rho,\varphi,z) = \frac{A_f k \exp(\mathrm{i}k(z+\rho^2/2z))}{\mathrm{i}z} \times \int_0^R \rho'' \,\mathrm{d}\rho'' \exp\left(\mathrm{i}\frac{k\rho''2}{2z}\right) J_0\left(\frac{k\rho\rho''}{z}\right) \times \left\{\left(a-\frac{\Delta a}{2}\right) J_1\left(\frac{k(a-\frac{\Delta a}{2})}{f}\rho''\right) - \left(a+\frac{\Delta a}{2}\right) J_1\left(\frac{k(a+\frac{\Delta a}{2})}{f}\rho''\right)\right\} \Big/ \rho''. \tag{17}$$

As expected, the function between parentheses in (17) tends to a zero-order Bessel function when $\Delta a \to 0$, giving the same result of (15). Eq. (17) is the final expression for the zero-order Bessel beam, when the slit does not obey (5). In this situation, the incoming pattern at the lens' surface is not an exactly zero-order Bessel function, resulting in a behavior similar to the one shown in Fig. 6.

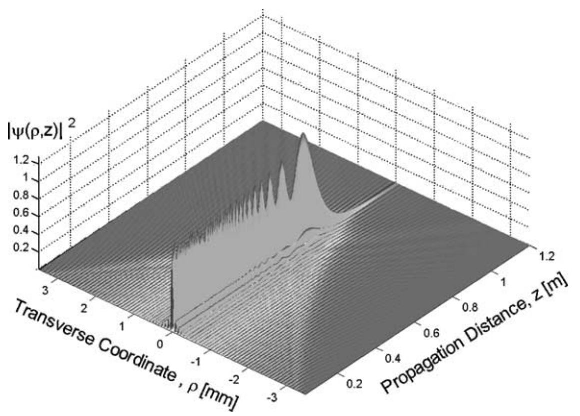

Fig. 5. The 3D evolution of a zero-order Bessel beam when $\Delta a \ll \frac{\lambda f}{R}$. $\lambda = 632.8$ nm, $a = 1.25$ mm, $f = 305$ mm and $R = 3.5$ mm.

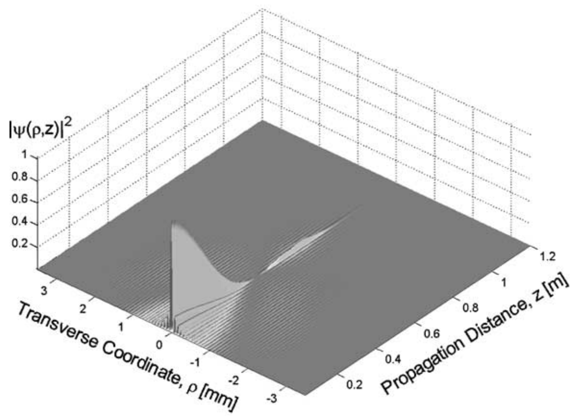

Fig. 6. The 3D evolution of a zero-order Bessel beam when $\Delta a \ll \frac{\lambda f}{R}$ is not satisfied. $\lambda = 632.8$ nm, $a = 1.25$ mm, $\Delta a = 0.1$ mm, $f = 305$ mm and $R = 3.5$ mm.

Fig. 6 was obtained using $f = 305$ mm, $R = 3.5$ mm, $a = 1.25$ mm, $\Delta a = 0.1$ mm and $\lambda = 632.8$ nm. Comparing Figs. 5 and 6, one can see they have very different behaviors. Here, it has become clear the importance of the condition (5), because for the former such condition is contemplated while for the latter it is not.

For practical purposes, one should attain the role of R in (5) and (1). For high R values, the distance Z_{max} increases, which assures a higher invariance range. On the other side, Δa must decrease, which can make difficult the feasibility of the experiment.

5. Remarks and conclusions

In this paper the scalar diffraction theory has been considered to obtain closed analytical expressions that can permit discussing generation and propagation of any kind of nondiffracting beams based on Durnin et al.'s apparatus. To improve our general formula we have studied the Mathieu beam. Finally, we have considered the Bessel beam generated with a real slit with a finite width instead of a Dirac δ transmittance function. This makes possible to infer what conditions related to the dimensions of the lens and thickness of the slit must be satisfied to produce effectively a nondiffracting beam of long range. Complex situations involving any kind of angular modulation can be studied using our general formula, which is actually valid not only for the Durnin's configuration but also for any kind of circular aperture with an adequate transmittance function. It suggests the use of holographic methods to produce the diffraction-free beams, instead of the Durnin's apparatus, which has been till now the mostly used system to generate this kind of waves today.

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