

# A Generalization of Dirac Non–Linear Electrodynamics, and Spinning Charged Particles<sup>(\*)</sup>

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**ABSTRACT:** In this note —dedicated to Prof. Asim O. Barut— we generalize the Dirac *non-linear electrodynamics*, by introducing two potentials (namely, the vector potential  $A$  and the pseudo-vector potential  $\gamma^5 B$  of the electromagnetic theory *with charges and magnetic monopoles*) and by imposing the pseudoscalar part of the product  $\omega\omega^*$  to be zero, with  $\omega \equiv A + \gamma^5 B$ .

We show that the field equations of such a theory possess a soliton–like solution which can represent a priori a “charged particle”, since it is endowed with a Coulomb field plus the field of a magnetic *dipole*. The rest energy of the soliton is finite, and the angular momentum stored in its electromagnetic field can be identified —for suitable choices of the parameters— with the spin of the charged particle. Thus this approach seems to yield a classical model for the charged (spinning) particle, which does not meet the problems met by earlier attempts in the same direction.

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## 1. INTRODUCTION

This note is dedicated to Asim O. Barut, who —with his inquiring work about the structure of elementary particles<sup>[1]</sup>— stimulated so much researches about the electron structure. As is well-known, classical electrodynamics is based on Maxwell equations for the electromagnetic field and the Lorentz equations of motion for the electric charges. This scheme, based on the assumption that all electromagnetic phenomena be due to existence and motion of individual electric charges, was sometimes referred to as “theory of electrons”.<sup>[2,3]</sup> As pointed out by Whittaker<sup>[2]</sup>, however, that name has little to do with Thomson’s discovery (that for all electrons the ratio  $e/m$  is the same); in fact, the *atomicity* of the electric charge has little importance in electromagnetism, when compared with the question of the very *structure* of fundamental charged particles, as the electron.

This question was discussed by several authors. The models of Abraham<sup>[4]</sup> and Lorentz<sup>[5]</sup> assumed the electron to be a spherical, rigid object. Their aim was to establish a purely electromagnetic model for the electron. Although much effort has been made in that direction, those models met difficulties and problems, which are well discussed in the literature.<sup>[2,3,6]</sup> In 1938 Dirac<sup>[7]</sup> abandoned the idea of looking for the electron structure, and tried to develop on the contrary a classical theory for a *point-like* electron. The point-like electron, however, exhibited an infinite self-energy: as expected. Therefore, no conclusive and satisfactory solution was given —nor seems to have been given— to the problem of the structure of charged particles (unless one abandons differential equations for finite-difference equations, following Caldirola<sup>[8]</sup>).

In order to overcome the difficulties associated with point charges, Dirac<sup>[9]</sup> suggested in 1951 a trick, which allows describing electric currents when starting from a theory “without charges”. Namely, Dirac *exploited* the presence in electromagnetism of a vector potential  $A$  (*i.e.*, of those extra variables entering the electromagnetic theory because of its gauge invariance); imposed a non-linear gauge: *i.e.*, the condition that the potential 1-form  $A$  were timelike [ $A^2 = k^2 = \text{positive constant}$ ]; so that  $A$  could be regarded as proportional to a velocity field<sup>[10]</sup>; and finally identified it with a current,<sup>[11]</sup> getting the motion of a continuous stream of electricity (rather than the motion of point charges). This theory is known as Dirac non-linear electrodynamics.<sup>#1</sup>

Recently, Righi and Venturi have shown in an interesting paper<sup>[12]</sup> that the field equations of Dirac’s new theory admits an extended-type, spherically symmetric, static solution which can be considered as a charged particle. In their approach, however, a magnetic dipole moment (and a spin) can arise only as a first-order *quantum* effect; but this would imply the electromagnetic potential to be no longer (at the second order) a 1-form with constant square: contrarily to the initial assumption. Moreover, the classical radius  $r_0$  is given in ref.<sup>[12]</sup> by an equation of the form  $f(r_0) = 0$ , with the condition

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<sup>0#1</sup> It is noteworthy that already Schroedinger<sup>[11]</sup> mentioned two cases in which the potential did actually acquire the role of source of the very field that was derived from it: (i) the case of the scalar potential in Debye’s theory of electrolytes; and (ii) of the vector potential in London’s theory of superconductivity.

$\cosh f(r) \leq 1$ : which means that  $A$  has to assume imaginary values.

In this paper we shall present an alternative approach to the question of the structure of charged particles that, although similar to Dirac's and Righi–Venturi's, does not meet the kind of problems discussed above. In fact, instead of the usual Maxwell equations, let us introduce the *generalized* Maxwell equations, including magnetic monopoles. We shall be able, then, to produce a new formulation of Dirac's non-linear electrodynamics, in which the first (*vector*) potential  $A$  is orthogonal to the second (*pseudo-vector*) potential  $\star B \equiv -\gamma^5 B$ , symbol  $\star$  representing—in the language of differential forms—the Hodge dual. Thus, we shall obtain the Righi–Venturi results in a purely classical way (i.e., without any need of introducing spin as a quantum effect).

To introduce our own approach, in Sect. **2** we reformulate Maxwell equations, and their generalized version, in terms of the Clifford bundle formalism, which provides us with a natural and concise formulation of them. In Sect. **3** we present our approach, and in Sect. **4** we look for solutions of our field equations. A relevant solution is constituted by the electric field of a conducting sphere and by the magnetic field of a magnetized sphere; while the angular momentum stored in the total electromagnetic field is equal to  $\hbar/2$ . In Sect. **5** we discuss this solution.

## 2. THE CLIFFORD–BUNDLE APPROACH TO ELECTROMAGNETISM<sup>[13,14,15]</sup>

Let  $M$  denote the Minkowski spacetime and  $T^*M$  [ $TM$ ] the cotangent [tangent] bundle of  $M$ . Cross-sections  $e \in \sec TM$  [ $\gamma \in \sec T^*M$ ] will be called 1-form [1-vector] fields. Let  $g \in \sec(T^*M \times T^*M)$  [ $\hat{g} \in \sec(TM \times TM)$ ] such that in each fiber  $\pi^{-1}(x)$ ,  $x \in M$ , quantity  $g_x$  [ $\hat{g}_x$ ] is a symmetric bilinear form over  $T_x M$  [ $T_x^*M$ ]. Let  $\{e_\mu\} \in \sec TM$  be a basis of  $TM$  and  $\{\gamma^\mu\} \in \sec T^*M$  be the dual basis. Then  $g = \eta_{\nu\mu} \gamma^\mu \otimes \gamma^\nu$  [ $\hat{g} = \eta^{\mu\nu} e_\mu \otimes e_\nu$ ], where  $(\eta^{\mu\nu}) = \text{diag}(1, -1, -1, -1)$ .

Let  $T(V) = (\bigoplus_{p=0}^{\infty} \otimes^p V, \oplus)$  be the tensor algebra over the real field, where  $\otimes$  is the usual tensor product,  $\oplus$  denotes the weak direct sum and  $\otimes^p V$  the cartesian product of  $p$  copies of  $V$ . The Clifford algebra is the quotient algebra  $T(V)/J$ , where  $J$  is the two-sided ideal generated by elements of the form  $a \otimes a - Q(a)$  with  $a \in V$  and  $Q(a) = g(a, a)$ . A vector bundle is called a Clifford bundle if each fiber is a Clifford algebra. For more details see<sup>[13–15]</sup>.

Let  $\mathcal{C}\ell(M, \hat{g})$  be now the Clifford bundle of *differential forms* over Minkowski spacetime. The spacetime algebra  $\mathbb{R}_{1,3}$  is the typical fiber of the Clifford bundles over Lorentzian space-times. The Dirac operator  $\partial$  acting on sections of  $\mathcal{C}\ell(M, \hat{g})$  is  $\partial = \gamma^\mu \nabla_\mu$ , where  $\nabla$  is the Levi-Civita connection of  $g$ . We can choose for simplicity  $\{\gamma^\mu\}$  such that  $\nabla_\mu = \partial_\mu$ ; thus  $\partial = \gamma^\mu \partial_\mu$ . We also have  $\partial = d - \delta$ , where  $d$  is the differential and  $\delta$  the Hodge codifferential operator.

The Maxwell equations are

$$dF = 0 \ ; \ \delta F = -\frac{4\pi}{c} J, \quad (1)$$

where the electromagnetic field  $F \in \sec \Lambda^2(M) \subset \sec \mathcal{C}\ell(M, \hat{g})$  is a 2-form field ( $F \equiv \frac{1}{2} F_{\mu\nu} \gamma^\mu \wedge \gamma^\nu$ ) and the electric current  $J \in \sec \Lambda^1(M) \subset \sec \mathcal{C}\ell(M, \hat{g})$  is a 1-form field ( $J = J_\mu \gamma^\mu$ ). In the Clifford bundle the Maxwell equations (1) can be written, by using the Dirac operator  $\partial = d - \delta$ , as

$$\partial F = \frac{4\pi}{c} J, \quad (2)$$

which was originally due to Riesz.<sup>[16,17]</sup> If we denote by  $\cdot$  the internal product and by  $\wedge$  the external product, we have  $\partial F = \partial \cdot F + \partial \wedge F$ , so that  $\partial \wedge = d$  and  $\partial \cdot = -\delta$ . In terms of the electromagnetic potential  $A \in \sec \Lambda^1(M) \subset \sec \mathcal{C}\ell(M, \hat{g})$  we have

$$F = \partial \wedge A \quad (3)$$

or, in the Lorenz<sup>(\*)</sup> gauge  $\partial \cdot A = 0$ ,

$$F = \partial A, \quad (4)$$

so that  $\partial^2 A = \square A = \frac{4\pi}{c} J$  is the wave equation, where  $\square = \partial^2 = (d - \delta)^2 = -(d\delta + \delta d)$  is the Laplace-Beltrami operator.

Let the main anti-automorphism in  $\mathbb{R}_{1,3}$  (called reversion) be denoted by  $*$ , i.e.,  $(AB)^* = B^* A^*$  and  $A^* = A$  if  $A$  is a scalar or a 1-form. Since  $F$  is a 2-form we have  $F^* = -F$ . Now it is easy to show that

$$\partial_\mu S^\mu = \partial_\mu \left( -\frac{c}{8\pi} F \gamma^\mu F \right) = J \cdot F, \quad (5)$$

where  $J \cdot F = \frac{1}{2}(JF - FJ)$ . We call the quantities  $S^\mu$  the energy momentum 1-forms; then  $E^{\mu\nu} = S^\mu \cdot \gamma^\nu$  are the components of the energy-momentum tensor. In fact, if we project the elements of  $\mathbb{R}_{1,3}$  into the Pauli algebra  $\mathbb{R}_{3,0}$  with  $\sigma^i = \gamma^i \gamma^0$  ( $i = 1, 2, 3$ ), we have that  $F = \mathbf{E} + \hat{i} \mathbf{H}$ , where  $\mathbf{E} = E_i \sigma^i$ ,  $\mathbf{H} = H_i \sigma^i$ ,  $\hat{i} = \sigma^1 \sigma^2 \sigma^3$  and  $E_i = F_{0i}$ ,  $H_1 = F_{23}$ ,  $H_2 = F_{31}$ ,  $H_3 = F_{12}$ , and that  $J \gamma^0 = c\rho + \mathbf{J}$ ,  $S^0 \gamma^0 = c\mathcal{U} + \mathbf{S}^0$ ; and we obtain

$$S^0 \gamma^0 = c\mathcal{U} + \mathbf{S}^0 = \frac{c}{8\pi} (\mathbf{E}^2 + \mathbf{H}^2) + \frac{c}{4\pi} \mathbf{E} \times \mathbf{H} \quad (6)$$

where we recognized

$$\mathcal{U} \equiv \frac{1}{8\pi} (\mathbf{E}^2 + \mathbf{H}^2) \ , \ \mathbf{S}^0 \equiv \frac{c}{4\pi} \mathbf{E} \times \mathbf{H} \quad (7)$$

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<sup>0(\*)</sup> It is perhaps time to recognize that the gauge condition  $\partial \cdot A = 0$  is due to Lorenz and not to Lorentz.

as the energy density and the Poynting vector of the electromagnetic field, respectively (in expression (6) we also used  $\mathbf{E} \times \mathbf{H} = -\hat{i}\mathbf{E} \wedge \mathbf{H}$ ). As well-known

$$\mathbf{p} = \frac{\mathbf{S}^0}{c^2} = \frac{1}{4\pi c} \mathbf{E} \times \mathbf{H} \ ; \ \boldsymbol{\ell} = \mathbf{r} \times \mathbf{p} \quad (8)$$

are the electromagnetic momentum density and the angular momentum density, respectively.

It is important to notice that with the Clifford bundle approach the Lorentz force equation of motion need not to be postulate. In fact, let us write  $-K = J \cdot F$  and project it onto the Pauli algebra, so that

$$K\gamma^0 = \mathbf{J} \cdot \mathbf{E} + c(\rho\mathbf{E} + \mathbf{J} \times \mathbf{H}) . \quad (9)$$

If we define  $\partial_\mu M^{\mu\nu} = K \cdot \gamma^\nu$  we have from eq.(5) that

$$\partial_\mu (E^{\mu\nu} + M^{\mu\nu}) = 0 , \quad (10)$$

where  $M^{\mu\nu}$  plays the role of the symmetric energy-momentum tensor of matter (i.e., of the currents) in the above global conservation equation. After this crucial identification, and following Barut,<sup>[19]</sup> we may write (where  $m$  appears for dimensional reasons):

$$M^{\mu\nu} = -m \int ds \delta(x^\alpha - z^\alpha) \frac{dz^\mu}{ds} \frac{dz^\nu}{ds} , \quad (11)$$

$$J^\mu = e \int ds \delta(x^\alpha - z^\alpha) \frac{dz^\mu}{ds} ,$$

which give the correct equations of motion, i.e.:

$$m\ddot{z}_i = e[\mathbf{E}_i + \frac{1}{c}(\dot{\mathbf{z}} \times \mathbf{H})]_i . \quad (12)$$

Now, the Clifford bundle formalism can be used in order to formulate the Maxwell equations with magnetic monopoles.<sup>[14,15]</sup> Namely, Maxwell equations can be generalized, in order to include monopoles (without string!),<sup>[15]</sup> in the form:

$$dF = -\frac{4\pi}{c} \star J_m \ ; \ \delta F = -\frac{4\pi}{c} J_e \quad (13)$$

where  $J_e$  and  $J_m$  are the electric and magnetic currents (with  $J_e \neq kJ_m$ ), respectively, and  $\star$  denotes the Hodge star operator. Notice that in the present way *electric* and *magnetic* charges have been introduced as objects of similar nature;<sup>[14]</sup> in particular, monopoles are *not* associated to strings, and the topology of Minkowski space-time has not been modified. Then, by using the Dirac operator  $\partial = d - \delta$  and the fact that  $\star J = J^* \gamma^5$ , in the Clifford bundle we have

$$\partial F = \frac{4\pi}{c} J \quad (14)$$

where

$$J = J_e + \gamma^5 J_m . \quad (15)$$

By introducing the electromagnetic pseudo-potential  $\gamma^5 B$  of Cabibbo and Ferrari,<sup>[20]</sup> where  $B \in \sec \Lambda^1(M) \subset \sec \mathcal{C}\ell(M, \hat{g})$ , we can write

$$F = F_e + \gamma^5 F_m \quad (16)$$

with

$$F_e \equiv \partial \wedge A \quad , \quad F_m \equiv -\partial \wedge B . \quad (17)$$

And, in the Lorenz gauge,  $\partial \cdot A = 0$ ,  $\partial \cdot B = 0$ , we have:

$$F = \partial A - \gamma^5 \partial B = \partial(A + \gamma^5 B) = \partial \omega \quad (18)$$

where  $\omega \equiv A + \gamma^5 B$  is the generalized electromagnetic potential.<sup>[14]</sup>

From the generalized Maxwell equations it follows that

$$\partial_\mu S^\mu = \partial_\mu \left( -\frac{c}{8\pi} F \gamma^\mu F \right) = J_e \cdot F + J_m \cdot (\gamma^5 F) , \quad (19)$$

where  $S^\mu$  are the energy-momentum 1-forms, such that  $S^0 \gamma^0 = c \mathcal{U} + \mathbf{S}^0$  with  $\mathcal{U}$  and  $\mathbf{S}^0$  given by eq.(7). In<sup>[14,15]</sup> we showed that  $-K_e = J_e \cdot F$  and  $-K_m = J_m \cdot (\gamma^5 F)$  correspond to the expressions for the Lorentz (electric and magnetic) forces, and by writing  $\partial_\mu M^{\mu\nu} = K_e \cdot \gamma^\nu + K_m \cdot \gamma^\nu$  we succeeded in obtaining the correct equations of motion.

Let us stress that the generalized Maxwell equations can be obtained either from the “non canonical” lagrangian adopted by us in refs.<sup>[14]</sup> (which contained both a scalar and a pseudoscalar part), or from the following lagrangian<sup>[15]</sup>

$$\mathcal{L} = \frac{1}{8\pi} \langle F \hat{F} \rangle_0 - \frac{1}{c} \langle J \omega \rangle_0 \quad (20)$$

where the brackets indicate the scalar part, and

$$\hat{F} \equiv F_e - \gamma^5 F_m . \quad (21)$$

### 3. AN ALTERNATIVE APPROACH TO FREE ELECTROMAGNETISM

We have recalled in Sect. 1 that Dirac’s 1951 theory<sup>[10]</sup> does not seem to be able to describe a (spin  $\frac{1}{2}$ ) electric particle endowed with a magnetic dipole field —besides the electrostatic one—; unless we start, as we are going to see, from Maxwell equations with monopoles.

Let us consider, from now on, the *generalized* electromagnetic field in *free space*, so that  $J = 0$ , i.e.,  $J_e = J_m = 0$ . In this case, the free generalized Maxwell equations

$$\partial F = 0 \quad (22)$$

can be obtained from the lagrangian

$$\mathcal{L} = \frac{1}{8\pi} \langle F \widehat{F} \rangle_0. \quad (23)$$

We get the *usual* free Maxwell equations by taking the second potential equal to zero,  $B = 0$ , so that  $F_m = -\partial \wedge B = 0$  and

$$\partial F_e \equiv 0. \quad (24)$$

The following scheme summarizes this approach:

$$\mathcal{L} = \frac{1}{8\pi} \langle F \widehat{F} \rangle_0 \xrightarrow[\text{equation}]{\text{Euler-Lagrange}} \partial F = 0 \xrightarrow{B=0} \partial F_e = 0. \quad (25)$$

Clearly the condition  $B = 0$  can be imposed from the beginning, but in this case the lagrangian has to be modified. Let us introduce a Lagrange multiplier  $\lambda = \lambda(x)$  so that our lagrangian turns into

$$\mathcal{L} = \frac{1}{8\pi} \langle F \widehat{F} \rangle_0 + \frac{1}{2c} \lambda(x) B^2. \quad (26)$$

Now  $\lambda(x)$  must be considered as a field quantity, so that the Euler-Lagrange equations for  $A(x)$ ,  $B(x)$  and  $\lambda(x)$  are

$$\partial F_e = 0 \quad (27)$$

$$\partial F_m = -\frac{4\pi}{c} \lambda B \quad (28)$$

$$B^2 = 0. \quad (29)$$

The new equation (28) implies that  $\partial \wedge F_m = 0$  and  $\partial \cdot F_m = -\frac{4\pi}{c} \lambda B$ , so that from the latter we have  $\partial \cdot (\lambda B) = 0$ , which is just the continuity equation  $\partial \cdot J = 0$  with  $J = -\lambda B$ . But eq.(28) tells us that  $\partial \cdot (\lambda B) = (B \cdot \partial) \lambda + \lambda (\partial \cdot B) = 0$  and, since in general  $\partial_\mu \lambda \neq 0$ , if from eq.(29) we have as solution  $B \neq 0$  we must have  $\partial \cdot B \neq 0$ . However, it is always possible to find a gauge transformation such that  $\partial \cdot B = 0$  (Lorenz gauge), and then we must have  $B = 0$  (provided that in general  $\partial_\mu \lambda \neq 0$ ) and  $F_m = -\partial \wedge B = 0$ . In other words, it seems a priori that in eq.(29) one may have  $B \neq 0$ ; but eq.(28) implies that  $B = 0$ . Thus the usual free Maxwell equations are recovered according to the following scheme:

$$\mathcal{L} = \frac{1}{8\pi} \langle F \widehat{F} \rangle_0 \xrightarrow{B=0} \mathcal{L} = \frac{1}{8\pi} \langle F \widehat{F} \rangle_0 + \frac{1}{2c} \lambda B^2 \xrightarrow[\text{equation}]{\text{Euler-Lagrange}} \partial F_e = 0. \quad (30)$$

Now let us consider the generalized potential  $\omega$  given by eq.(18),  $\omega \equiv A + \gamma^5 B$ . Let us calculate  $\omega \omega^*$ :

$$\omega \omega^* = (A + \gamma^5 B)(A - \gamma^5 B) = (A^2 - B^2) + 2\gamma^5 (A \cdot B), \quad (31)$$

so that its scalar and pseudo-scalar parts are

$$\langle \omega\omega^* \rangle_0 = A^2 - B^2, \quad (32)$$

$$\langle \omega\omega^* \rangle_4 = 2\gamma^5(A \cdot B), \quad (33)$$

respectively. If  $B = 0$ , we have  $\langle \omega\omega^* \rangle_0 = A^2$  and  $\langle \omega\omega^* \rangle_4 = 0$ , i.e. the pseudo-scalar part of  $\omega\omega^*$  vanishes. Thus

$$B = 0 \implies \langle \omega\omega^* \rangle_4 = 0 \quad (34)$$

and

$$B = 0 \implies \omega\omega^* = s \quad (35)$$

where the quantity  $s$  is a *scalar*, not necessarily a constant; i.e.:

$$s = s(x) \in \sec \Lambda^0(M) \subset \sec \mathcal{C}\ell(M, \hat{g}). \quad (36)$$

Clearly the condition  $\langle \omega\omega^* \rangle_4 = 0$  is *more general* than  $B = 0$ , so that  $B = 0$  follows as a particular case of  $\langle \omega\omega^* \rangle_4 = 0$  or  $\omega\omega^* = s$ . What we shall propose here is to *replace the condition  $B = 0$  by the more general condition  $\omega\omega^* = s$* , thus extending the Dirac original, non-linear gauge condition  $A^2 = k^2$ . By eq.(31) the condition  $\omega\omega^* = s$  implies the two following constraints:

$$A^2 - B^2 = s \quad (37)$$

$$A \cdot B = 0. \quad (38)$$

Notice that we do not want *here* to modify the lagrangian  $\mathcal{L} = \frac{1}{8\pi} \langle F\hat{F} \rangle_0$  by adding a term of the type  $\lambda(\omega\omega^* - s)$  because it contains both a scalar and a pseudo-scalar, and this may be regarded as unconventional. A possible approach consists in using two Lagrange multipliers,  $\lambda_1(x)$  and  $\lambda_2(x)$ , one for each condition (37) and (38). Thus our lagrangian will be

$$\mathcal{L} = \frac{1}{8\pi} \langle F\hat{F} \rangle_0 + \frac{\lambda_1}{2c} (A^2 - B^2 - s) + \frac{\lambda_2}{c} (A \cdot B) \quad (39)$$

or, since  $\langle F\hat{F} \rangle_0 = \langle F_e^2 + F_m^2 \rangle_0$ , we shall have, in terms of components:

$$\begin{aligned} \mathcal{L} = & -\frac{1}{16\pi} [(F_e)_{\mu\nu} (F_e)^{\mu\nu} + (F_m)_{\mu\nu} (F_m)^{\mu\nu}] + \\ & + \frac{\lambda_1}{2c} [A_\mu A^\mu - B_\mu B^\mu - s] + \frac{\lambda_2}{c} (A_\mu B^\mu). \end{aligned} \quad (40)$$

Our new field equations (which generalize the Dirac non-linear theory of electromagnetism) are, in components, the following:

$$\partial_\mu (F_e)^{\mu\nu} = -\frac{4\pi}{c} \lambda_1 A^\mu - \frac{4\pi}{c} \lambda_2 B^\mu \quad (41)$$

$$\partial_\mu (F_m)^{\mu\nu} = \frac{4\pi}{c} \lambda_1 B^\mu - \frac{4\pi}{c} \lambda_2 A^\mu \quad (42)$$

$$A_\mu A^\mu - B_\mu B^\mu = s \quad (43)$$

$$A_\mu B^\mu = 0 \quad (44)$$

or, in intrinsic form:

$$\partial F_e = \frac{4\pi}{c} \lambda_1 A + \frac{4\pi}{c} \lambda_2 B \quad (45)$$

$$\partial F_m = -\frac{4\pi}{c} \lambda_1 B + \frac{4\pi}{c} \lambda_2 A \quad (46)$$

$$A^2 - B^2 = s \quad (47)$$

$$A \cdot B = 0. \quad (48)$$

From eq.(45) and eq.(46) it follows that our new field equations can be simply written down as

$$\partial F = \frac{4\pi}{c} \lambda_1 \omega - \frac{4\pi}{c} \lambda_2 \gamma^5 \omega = \frac{4\pi}{c} (\lambda_1 - \lambda_2 \gamma^5) \omega ,$$

and therefore:

$$\boxed{\partial F = \frac{4\pi}{c} \lambda \omega} , \quad (49)$$

with  $\lambda = \lambda_1 - \lambda_2 \gamma^5$ .

Furthermore, let us impose the gauge invariance of our theory. For  $A \mapsto A' = A + \partial \chi_1$  and  $B \mapsto B' = B + \partial \chi_2$ , and from eqs.(45) and (46), in order for equations (49) to be gauge invariant we must have the following constraints on the Lagrange multipliers:

$$\lambda_1 \partial \chi_1 + \lambda_2 \partial \chi_2 = 0 \quad (50)$$

$$\lambda_2 \partial \chi_1 + \lambda_1 \partial \chi_2 = 0. \quad (51)$$

#### 4. A SOLUTION OF THE NEW FIELD EQUATIONS

Now let us look for solutions of our new field equations, corresponding to a charged particle at rest. In other words, let us here consider only *stationary* solutions. First we note that the condition  $A \cdot B = 0$  is trivially satisfied by  $B = 0$ . Since we want this condition to be gauge invariant, i.e.  $\langle \omega' \omega'^* \rangle_4 = \langle \omega \omega^* \rangle_4 = 0$ , we must have

$$\langle \omega' \omega'^* \rangle_4 = 2\gamma^5 (A' \cdot B') = 0 \quad (52)$$

and, since  $B' = \partial \chi$  (because  $B = 0$ ),

$$A' \cdot \partial \chi_2 = 0. \quad (53)$$

But  $B = 0$  implies, in eq.(46), that

$$\lambda_2 A = 0 \quad (54)$$

and (since we supposed  $A \neq 0$ ) then

$$\lambda_2 = 0. \quad (55)$$

The gauge invariance of our theory is guaranteed, from eq.(50) and eq.(51), once

$$\lambda_1 \partial \chi_1 = 0 \quad (56)$$

$$\lambda_1 \partial \chi_2 = 0. \quad (57)$$

We now look for a  $\chi_2 = \chi_2(x)$  that does not depend on  $x^0 = ct$  and is spherically symmetric, i.e.:

$$\chi_2 = \chi_2(r) \quad (58)$$

where  $r = |\mathbf{r}|$ . Thus we have

$$\partial_0 \chi_2 = 0, \quad \partial_i \chi_2 = \frac{d\chi_2(r)}{dr} \frac{x_i}{r} \equiv \varphi(r) \frac{x_i}{r} \quad (59)$$

and, for  $A' = A_0 \gamma^0 + A_i \gamma^i$ , the condition (53) gives:

$$A' \cdot \partial \chi_2 = -A_i \varphi(r) \frac{x_i}{r} = -\mathbf{A} \cdot \left[ \varphi(r) \frac{\mathbf{r}}{r} \right] = 0, \quad (60)$$

where  $\mathbf{A} = A_i \sigma^i$ ,  $\mathbf{r} = x_i \sigma^i$  and  $\sigma^i = \gamma^i \gamma^0$ ,  $i = 1, 2, 3$ . Note that, since  $\partial_0 \chi_2 = 0$ , the choice of  $A_0$  is arbitrary; we shall suppose that  $A_0(x)$  does not depend on  $x^0$  and is spherically symmetric, i.e.:

$$A_0(x) = A_0(r). \quad (61)$$

From eq.(60),  $\mathbf{A}$  must be orthogonal to  $\mathbf{r}$ . Since  $(\mathbf{d} \times \mathbf{r}) \cdot \mathbf{r} = 0$  for any vector  $\mathbf{d}$ , we take

$$\mathbf{A} \equiv \phi(r) (\mathbf{d} \times \mathbf{r}), \quad (62)$$

where  $\mathbf{d}$  is a vector to be defined, and  $\phi(r)$  a function to be calculated (just defined via eq.(62)). In summary, our potentials  $A$  and  $B$  are<sup>(\*)</sup>

$$A = A_0 \gamma^0 + A_i \gamma^i \quad \text{with} \quad \begin{cases} A_0 = A_0(r) \\ A_i = \phi(r) (\mathbf{d} \times \mathbf{r})_i \end{cases} \quad (64)$$

$$(65)$$

and

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<sup>0(\*)</sup> From now onward, we shall omit the prime and write  $A$  and  $B$  (since there is no danger of confusion).

$$B = B_0\gamma^0 + B_i\gamma^i \quad \text{with} \quad \begin{cases} B_0 = 0 \\ B_i = \varphi(r)\frac{x_i}{r} \end{cases} \quad \begin{matrix} (66) \\ (67) \end{matrix}$$

Finally we notice that condition (57) for gauge invariance yields  $\lambda_1\partial_\mu\chi = 0$ , and from eq.(59) we must have ( $r \neq 0$ ):

$$\lambda_1\varphi(r) = 0. \quad (68)$$

For  $r$  such that  $\varphi(r) \neq 0$  we must then have  $\lambda_1 = 0$ ; if we suppose that there exists a value of  $r$  (namely  $r_0$ ) such that

$$\varphi(r_0) = 0 \quad (69)$$

we get that  $\lambda_1$  must be of the form

$$\lambda_1 = -\frac{\delta(r-r_0)}{r_0} \frac{c}{4\pi}, \quad (70)$$

where  $r_0$  in the denominator has been introduced for dimensional reasons, and the other constants (and the minus sign) for convenience. The other condition (56) implies that  $\chi_1 = \chi_1(x)$  is an arbitrary function, except at  $r = r_0$  where  $\partial\chi_1|_{r_0} = 0$ . Thus our theory is gauge invariant except at  $r = r_0$ .<sup>(\*)</sup>

At this point, we can claim that eqs.(46)-(48) are satisfied by the above choices of the potentials and Lagrange multipliers. The remaining eq.(45) will specify  $A_0$  and  $A_i$ . Thus we have

$$\partial_\mu F_e^{\mu\nu} = -\frac{\delta(r-r_0)}{r_0} A^\nu. \quad (71)$$

Remembering that  $F^{i0} = -F_{i0}$ ,  $F^{ij} = F_{ji}$  ( $i, j = 1, 2, 3$ ,  $i \neq j$ ) and using eq.(63) and eq.(64), we obtain for  $F_e^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$  ( $i \neq j$ ) that:

$$F_e^{i0} = -\frac{dA_0(r)}{dr} \frac{x_i}{r} \quad (72)$$

$$F_e^{ij} = \frac{d\phi(r)}{dr} \frac{x_i}{r} (\mathbf{d} \times \mathbf{r})_j - \frac{d\phi(r)}{dr} \frac{x_j}{r} (\mathbf{d} \times \mathbf{r})_i + \varepsilon_{ijk} 2\phi(r) d_k. \quad (73)$$

Then, for  $\nu = 0$ , eq.(71) gives:

$$\frac{d^2 A_0(r)}{dr^2} + \frac{2}{r} \frac{dA_0(r)}{dr} = \frac{\delta(r-r_0)}{r_0} A_0(r) \quad (74)$$

and, for  $\nu = 1, 2, 3$ , the same eq.(71) gives the equation

$$\frac{d^2 \phi(r)}{dr^2} + \frac{4}{r} \frac{d\phi(r)}{dr} = -\frac{\delta(r-r_0)}{r_0} \phi(r). \quad (75)$$

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<sup>0(\*)</sup> Notice also that  $\varphi(r_0) = 0$  is an extremum condition since from eq.(60) we have  $\varphi(r_0) = \frac{d\chi_2(r)}{dr}|_{r_0} = 0$ .

A solution of eq.(74) is of the form

$$A_0(r) = \frac{c_1}{r} \Theta(r - r_0) \quad (76)$$

where  $c_1$  is a constant and  $\Theta$  is the step function; that is:

$$A_0(r) = \begin{cases} \frac{c_1}{r} , & r \geq r_0 \\ 0 , & r < r_0 ; \end{cases} \quad (77)$$

while a solution of eq.(75) is of the form

$$\phi(r) = \begin{cases} \frac{c_2}{r^3} , & r > r_0 \\ 0 , & r \leq r_0 , \end{cases} \quad (78)$$

which we can simply write (by using the step-function  $\Theta$ ):

$$\phi(r) = \frac{c_2}{r^3} \Theta(r - r_0) , \quad (79)$$

with, moreover,  $\phi(r_0) = 0$  for  $r = r_0$ . From eq.(65) we get therefore:

$$A_i = \frac{c_2}{r^3} (\mathbf{d} \times \mathbf{r})_i \Theta(r - r_0) . \quad (80)$$

Now, using the last results in eq.(72) and eq.(73), we obtain

$$F_e^{i0} = \begin{cases} 0 , & r < r_0 \\ c_1 \frac{x_i}{r^3} , & r \geq r_0 \end{cases} \quad (81)$$

and

$$F_e^{ij} = \begin{cases} 0 , & r \leq r_0 \\ \frac{3c_2}{r^5} [x_j (\mathbf{d} \times \mathbf{r})_i - x_i (\mathbf{d} \times \mathbf{r})_j] + 2\varepsilon_{ijk} \frac{c_2}{r^3} d_k , & r > r_0 . \end{cases} \quad (82)$$

Remembering that  $F_e^{i0} = E^i$  and  $F_e^{ij} = H^k$  ( $i, j, k$  cyclic); choosing  $\mathbf{d}$  parallel to  $\sigma^3$ ; and using spherical coordinates, we eventually get the soliton-like solution

$$\mathbf{E} = \begin{cases} 0 , & r < r_0 \\ c_1 \frac{\hat{\mathbf{r}}}{r^3} , & r \geq r_0 ; \end{cases} \quad (83)$$

$$\mathbf{H} = \begin{cases} 0, & r \leq r_0 \\ c_2 \frac{1}{r^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\theta}), & r > r_0. \end{cases} \quad (84)$$

where  $\theta$  is the angle between  $\hat{\mathbf{r}}$  and  $\sigma^3$ . The electric field  $\mathbf{E}$  is like the one outside a conducting, charged sphere; and the magnetic field  $\mathbf{H}$  looks like the one outside a magnetized sphere (except that inside the sphere we have *no* magnetic field). Notice that in the interior of the sphere both  $\mathbf{E}$  and  $\mathbf{H}$  are zero, according to this solution.

The constants  $c_1$  and  $c_2$  are proportional to the electric charge and to magnetic moment, respectively. Indeed from eq.(71), if we identify  $\frac{4\pi}{c} J^0 = -\frac{\delta(r-r_0)}{r_0} A^0$ , where  $J^0 = c\rho$ , we have

$$\begin{aligned} \int d^3x J_0 &= c e = -\frac{c}{4\pi} \int d^3x \frac{\delta(r-r_0)}{r_0} \frac{c_1}{r} = \\ &= -\frac{c}{4\pi} \int dr r \delta(r-r_0) = -c c_1 \implies c_1 = -e \end{aligned} \quad (85)$$

Now, if we choose

$$r_0 \equiv \frac{e^2}{2mc^2} \left[ 1 + \frac{3}{4} \left( \frac{\hbar c}{e^2} \right)^2 \right] \quad (86)$$

$$c_2 \equiv \frac{3}{8} \frac{e\hbar}{mc} \left[ 1 + \frac{3}{4} \left( \frac{\hbar c}{e^2} \right)^2 \right] \quad (87)$$

we obtain the whole mass  $m$  to be electromagnetic in origin:

$$\mathcal{U} = \frac{1}{8\pi} \int (\mathbf{E}^2 + \mathbf{H}^2) dv = mc^2 \quad (88)$$

and the angular momentum of the whole Poynting vector field to get the value

$$\mathbf{L} = \frac{1}{c^2} \int (\mathbf{r} \times \mathbf{S}^0) dv = \frac{\hbar}{2} \sigma^3, \quad (89)$$

so that  $|\mathbf{L}| = L_z = \hbar/2$ . In connection with eqs.(86), (87), one can immediately observe that the magnetic moment  $c_2$  vanishes only if  $\hbar \rightarrow 0$ , in which case  $r_0$  is just equal to the so-called classical radius of the electron (without spin). On the contrary, if our particle *has* spin ( $\hbar \neq 0$ ), then  $r_0$  jumps<sup>(\*)</sup> to the value of the “electromagnetic world”<sup>[21]</sup> radius: *i.e.*, to a value of the order of the hydrogen atom size ( $1 \rho A$ ). It should be stressed, however, that the interior of such a sphere is completely accessible to any other particles (apart from possible electromagnetic repulsions).

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<sup>0(\*)</sup> Unless one attributes a large enough value to  $m$ , or replaces in eq.(86) the coupling-constant  $\alpha \equiv e^2/\hbar c$  by the constant  $g^2/\hbar c$  (quantity  $g$  being the elementary magnetic-pole charge).

## 5. CONCLUSIONS

Let us comment again about the solution we just found. It yields the model for a charged particle as a “sphere” with radius  $r_0 = \frac{e^2}{2mc^2} [1 + \frac{3}{4} (\frac{\hbar c}{e^2})^2]$ , which for  $\hbar \rightarrow 0$  is just the electron classical radius  $r_{cl} = \frac{e^2}{2mc^2}$ . The electric field looks just like the one of a conducting sphere, which is a most natural case if the electron is to be modelled as a sphere. On the other hand, although the magnetic field looks like the one of a magnetized sphere, it is indeed different. For a magnetized sphere the magnetic field is non-vanishing also inside the sphere and at its surface, while for our solution it vanishes both inside the sphere and at its surface. [Actually, even if the magnetic field happened to be different from zero inside the sphere and at its surface, the angular momentum stored in the field would turn out to be the same, i.e.,  $L_z = \hbar/2$ , since the electric field does vanish inside the sphere.] Therefore, the present solution suggests for a charged particle the picture of a hole in an “electromagnetic fluid”.

Finally, let us make some comments about our approach. Clearly it does not meet the same kind of problems discussed in the introduction, concerning the approach of Righi and Venturi. Moreover, it is “classical” in two senses: first, that it is “non-quantistic”; second, that —although we started from the generalized Maxwell equations— we did assume afterwards the pseudo-potential to be zero, i.e.  $\gamma^5 B = 0$ : which is the case for ordinary electromagnetism. Thus, our solution seems to picture a charged particle as a sphere with nothing inside it, and such that an angular momentum  $\hbar/2$  is stored in its field (even if it has nothing to do with any internal degrees of freedom like spin). It may be interesting to compare this result with the discussion in Hestenes<sup>[22]</sup> concerning the possible meaning of spin and the “zero-point” angular momentum in quantum mechanics.

The possible meaning of our solution, and its possible role in the physical situations in which Dirac’s non-linear electrodynamics is an acceptable theory, may need more discussion.

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