

FIELD THEORY OF THE SPINNING ELECTRON: ABOUT THE NEW NON-LINEAR FIELD EQUATIONS.^(*)

Erasmus RECAMI

Facoltà di Ingegneria, Università Statale di Bergamo, 24044-Dalmine (BG), Italy;
INFN, Sezione di Milano, Milan, Italy; and
Dept. of Applied Math., State University at Campinas, Campinas, S.P., Brazil.

and

Giovanni SALESI

Dipart. di Fisica, Università Statale di Catania, 57 Corsitalia, 95129-Catania, Italy.

Abstract – One of the most satisfactory picture of spinning particles is the Barut-Zanghi (BZ) classical theory for the relativistic electron, that relates the electron spin to the so-called *zitterbewegung* (zbw). The BZ motion equations constituted the starting point for two recent works about spin and electron structure, co-authored by us, which adopted the Clifford algebra language. Here, employing on the contrary the tensorial language, more common in the (first quantization) field theories, we “quantize” the BZ theory and derive for the electron field a non-linear Dirac equation (NDE), of which the ordinary Dirac equation represents a particular case.

We then find out the general solution of the NDE. Our NDE does imply *a new probability current* J^μ , that is shown to be a *conserved* quantity, endowed (in the center-of-mass frame) with the zbw frequency $\omega = 2m$, where m is the electron mass. Because of the conservation of J^μ , we are able to adopt the ordinary *probabilistic interpretation* for the fields entering the NDE.

At last we propose a natural generalization of our approach, for the case in which an external electromagnetic potential A^μ is present; it happens to be based on a new system of *five* first-order differential field equations.

^(*) Work partially supported by INFN, CNR, MURST, and by CNPq, FAPESP.

1. – On Barut–Zanghi’s theory for the spinning (free) electron.

Classical models of spin and classical electron theories have been investigated for about seventy years⁽¹⁾. For instance, Schrödinger’s suggestion⁽²⁾ that the electron spin was related to *zitterbewegung* did originate a large amount of subsequent work, including Pauli’s. Let us quote, among the others, ref.⁽³⁾, where one meets even the proposal of models with clockwise and anti-clockwise “internal motions”, as classical analogues of quantum relativistic spinning particles and antiparticles, respectively. The use of Grassmann variables in a classical lagrangian formulation for spinning particles was proposed by Berezin and Marinov⁽⁴⁾ and by Casalbuoni⁽⁴⁾. A recent approach, based on a generalization of Dirac non-linear electrodynamics, where the electron spin is identified with the momentum in such a theory, can be found in ref.⁽⁵⁾. In the Barut–Zanghi (BZ) theory,⁽⁶⁾ the classical electron was actually characterized, besides by the usual pair of conjugate variables (x^μ, p^μ) , by a second pair of conjugate classical *spinorial* variables $(\psi, \bar{\psi})$, representing internal degrees of freedom, which were functions of the (proper) time τ measured in the electron global center-of-mass (CM) system; the CM frame (CMF) being the one in which $\mathbf{p} = 0$ at any instant of time. Barut and Zanghi, then, introduced a classical lagrangian that in the free case (i.e., when the *external* electromagnetic potential is $A^\mu = 0$) writes [$c = 1$]

$$\mathcal{L} = \frac{1}{2}i\lambda(\dot{\bar{\psi}}\psi - \bar{\psi}\dot{\psi}) + p_\mu(\dot{x}^\mu - \bar{\psi}\gamma^\mu\psi) , \quad (1)$$

where λ has the dimension of an action, and ψ and $\bar{\psi} \equiv \psi^\dagger\gamma^0$ are ordinary \mathbb{C}^4 -bispinors, the dot meaning derivation with respect to τ . The four Euler–Lagrange equations, with $-\lambda = \hbar = 1$, yield the following motion equations:

$$\begin{cases} \dot{\psi} + i\mathbf{p}_\mu\gamma^\mu\psi = 0 & (2a) \\ \dot{x}^\mu = \bar{\psi}\gamma^\mu\psi & (2b) \\ \dot{p}^\mu = 0 , & (2c) \end{cases}$$

besides the hermitian adjoint of eq.(2a), holding for $\bar{\psi}$. From eq.(1) one can also see that

$$H \equiv p_\mu v^\mu = p_\mu \bar{\psi}\gamma^\mu\psi \quad (3)$$

is a constant of the motion [and precisely is the energy in the CMF].^(7,8) Since H is the BZ hamiltonian in the CMF, we can suitably set $H = m$, where m is the particle rest-mass. The general solution of the equations of motion (2) can be shown to be:

$$\psi(\tau) = [\cos(m\tau) - i\frac{p_\mu\gamma^\mu}{m}\sin(m\tau)]\psi(0) , \quad (4a)$$

$$\bar{\psi}(\tau) = \bar{\psi}(0)[\cos(m\tau) + i\frac{p_\mu\gamma^\mu}{m}\sin(m\tau)] , \quad (4b)$$

with $p^\mu = \text{constant}$; $p^2 = m^2$; and finally:

$$\dot{x}^\mu \equiv v^\mu = \frac{p^\mu}{m} + [\dot{x}^\mu(0) - \frac{p^\mu}{m}] \cos(2m\tau) + \frac{\ddot{x}^\mu}{2m}(0) \sin(2m\tau) . \quad (4c)$$

This general solution exhibits the classical analogue of the phenomenon known as zitterbewegung (zwb): in fact, the velocity v^μ contains the (expected) term p^μ/m plus a term describing an oscillating motion with the characteristic zwb frequency $\omega = 2m$. The velocity of the CM will be given by p^μ/m . Let us explicitly observe that the general solution (4c) represents a helical motion in the ordinary 3-space: a result that has been met also by means of other, alternative approaches.^(9,10) The BZ theory has been recently studied also in the lagrangian and hamiltonian symplectic formulations, both in flat and in curved spacetimes.⁽⁶⁾

2. – About a new non-linear Dirac-like equation (NDE) for the free electron.

At this point we want to introduce the spinorial fields $\psi(x), \bar{\psi}(x)$ and the velocity 4-vector field $V(x)$, starting from the spinor variables $\psi(\tau), \bar{\psi}(\tau)$ and from the 4-velocity $v(\tau)$ along the helical paths. Let us indeed consider a spinorial field $\psi(x)$ such that its *restriction* $\psi(x)|_\sigma$ to the world-line σ (along which the “particle” moves) coincides with $\psi(\tau)$. Consider at the same time a velocity field $V(x)$ together with its integral lines (or stream-lines). Then the velocity distribution $V(x)$ is required to be such that its restriction $V(x)|_\sigma$ to the world-line σ coincide with the ordinary 4-velocity $v(\tau)$ of the considered “particle”. Therefore, for the tangent vector along any line σ the relevant relation holds:

$$\frac{d}{d\tau} \equiv \frac{dx^\mu}{d\tau} \frac{\partial}{\partial x^\mu} \equiv \dot{x}^\mu \partial_\mu . \quad (5)$$

Inserting eq.(5) into eq.(2a) one gets the non-linear equation:

$$i\dot{x}^\mu \partial_\mu \psi = p_\mu \gamma^\mu \psi ,$$

and, since $\dot{x}^\mu = \bar{\psi} \gamma^\mu \psi$ because of eq.(2b), one arrives at the following interesting equation:^(7,8,11)

$$i\bar{\psi} \gamma^\mu \psi \partial_\mu \psi = p_\mu \gamma^\mu \psi . \quad (6)$$

Let us notice that, differently from eqs.(1)–(2), equation (6) can be valid a priori even for massless spin $\frac{1}{2}$ particles, since the CMF proper time does not enter it any longer.

The *non-linear* equation (6) corresponds to the whole system of eqs.(2): quantizing the BZ theory, therefore, does *not* lead to the Dirac equation, but rather to our non-linear, Dirac-like equation (6), that we shall call the NDE. Let us add, furthermore, that the analogous of eq.(3), now holding for our field $\psi(x)$, is the following noticeable normalization constraint:^{#1}

$$p_\mu \bar{\psi} \gamma^\mu \psi = m . \quad (7)$$

This non-linear equation is very probably the simplest⁽¹²⁾ non-linear *Dirac-like* equation.

In a generic frame, the general solution of NDE can be easily shown to be the following [$\not{p} \equiv p_\mu \gamma^\mu$]:

$$\psi(x) = \left[\frac{m - \not{p}}{2m} e^{ip_\mu x^\mu} + \frac{m + \not{p}}{2m} e^{-ip_\mu x^\mu} \right] \psi(0) ; \quad (8a)$$

which, in the CMF, reduces to

$$\psi(\tau) = \left[\frac{1 - \gamma^0}{2} e^{im\tau} + \frac{1 + \gamma^0}{2} e^{-im\tau} \right] \psi(0) \quad (8b)$$

(or, in simpler form, to eq.(4a)).

Let us explicitly observe that, by inserting eq.(8a) or eq.(8b) into eq.(7), one obtains that every solution of the NDE does correspond to the CMF field hamiltonian $H = m$, which is always *positive* even if it appears (as expected) to be a suitable superposition of plane waves endowed with *positive and negative* frequencies.⁽¹³⁾

It can be also noticed that superposition (8a) is a solution of eq.(6), due to its non-linearity, *only* for suitable pairs of plane waves with weights

$$\frac{m \pm \not{p}}{2m} = \Lambda_\pm ,$$

respectively; which are nothing but the usual projectors Λ_+ (Λ_-) onto the positive (negative) energy states of the standard Dirac equation. In other words, the plane wave solution (for a fixed value of p) of the Dirac eigenvalue equation $\not{p}\psi = m\psi$ is a particular case of the general solution of eq.(6): corresponding, namely, to

$$\text{either } \Lambda_+ \psi(0) = 0 \quad \text{or} \quad \Lambda_- \psi(0) = 0 . \quad (9)$$

^{#1} In refs.⁽⁹⁾ it was moreover assumed $p_\mu \dot{x}^\mu = m$, which actually does imply the very general relation $p_\mu \bar{\psi} \gamma^\mu \psi = m$, but *not* the Dirac equation $p_\mu \gamma^\mu \psi = m\psi$, as claimed therein; these two equations in general are not equivalent.

Therefore, the solutions of the Dirac eigenvalue equation are a *subset* of the set of solutions of our NDE. It is worthwhile to repeat that, for each fixed p , the wave function $\psi(x)$ describes both particles and antiparticles: all corresponding however to positive *energies*, in agreement with the reinterpretation forwarded in refs.⁽¹³⁾, as well as with the already mentioned fact that we can always choose $H = m > 0$.

3. – The conserved current J^μ .

We want now to study the probability current J^μ corresponding to the wave functions (8a,b) and (4a). Let us define it as follows:⁽⁷⁾

$$J^\mu \equiv \frac{m}{\mathcal{E}} \bar{\psi} \gamma^\mu \psi , \quad (10)$$

where the normalization factor m/\mathcal{E} (the 3-volume V being assumed to be equal to 1, as usual; so that $\mathcal{E}V \equiv \mathcal{E}$) is required to be such that the classical limit of J^μ , that is $(m/\mathcal{E})v^\mu$, equals $(1; \mathbf{v})$, like for the ordinary quantum probability currents. Notice also that $J^0 \equiv 1$, which means that we have one particle inside the unitary 3-volume $V = 1$. This normalization allows us to recover, in particular, the Dirac current $J_D^\mu = p^\mu/\mathcal{E}$ when considering the (trivial) solutions, without zbw, corresponding to relations (9).

Actually, if we insert quantity $\psi(x)$ given by eq.(8a) into eq.(10), we get

$$J^\mu = \frac{p^\mu}{\mathcal{E}} + E^\mu \cos(2p_\mu x^\mu) + H^\mu \sin(2p_\mu x^\mu) , \quad (10')$$

where

$$E^\mu \equiv J^\mu(0) - p^\mu/\mathcal{E} ; \quad H^\mu \equiv \dot{J}^\mu(0)/2m . \quad (10'')$$

If we now impose conditions (9), we have $E^\mu = H^\mu = 0$ and get therefore the Dirac current $J^\mu = J_D^\mu = p^\mu/\mathcal{E} = \text{constant}$. Let us observe that the normalization factor $\sqrt{m/\mathcal{E}}$ cannot be included into the expressions of ψ and $\bar{\psi}$, as it would seem convenient, because of the non-linearity of eq.(6) and/or of constraint (7).

From the fact that $p_\mu E^\mu \equiv p_\mu J^\mu(0) - p_\mu p^\mu/\mathcal{E} = m^2/\mathcal{E} - m^2/\mathcal{E} = 0$ (where we used eq.(7) for $x = 0$), that $p_\mu H^\mu \equiv p_\mu \dot{J}^\mu(0)/2m = 0$, obtained deriving both members of eq.(7), and that both E^μ and H^μ are orthogonal to p^μ , it follows that

$$\partial_\mu J^\mu = 2p_\mu H^\mu \cos(2px) - 2p_\mu E^\mu \sin(2px) = 0 . \quad (10''')$$

We may conclude, with reference to the NDE, that our current J^μ is *conserved*: we are therefore allowed to adopt the ordinary probabilistic interpretation for the fields $\psi, \bar{\psi}$.

Equation (10') does clearly show that our conserved current J^μ , as well as its classical limit mv^μ/\mathcal{E} [see eq.(4c)], are endowed with a zitterbewegung-type motion: precisely, with an oscillating motion having the CMF frequency $\Omega = 2m \simeq 10^{21} \text{ s}^{-1}$ and period $T = \pi/m \simeq 10^{-20} \text{ s}$ (we may call Ω and T the zbw frequency and period, respectively).

From eq.(10') one can immediately verify that in general

$$J^\mu \neq p^\mu/\mathcal{E}, \quad J^\mu \equiv J^\mu(x);$$

whilst the Dirac current J_D^μ for the free electron with fixed p , as already mentioned, is constant:

$$J_D^\mu = p^\mu/\mathcal{E} = \text{constant},$$

which correspond to *no* zbw. In other words, our current behaves differently from Dirac's, even if both of them obey^{#2} the constraint [cf. eq.(7)]:

$$p_\mu J^\mu = p_\mu J_D^\mu = m^2/\mathcal{E}.$$

It is noticeable, moreover, that our current J^μ goes into the Dirac one, not only in the (no-zbw) case of eq.(9), but also when considering its time-average over a zbw period:

$$\langle J^\mu \rangle_{\text{zbw}} = \frac{p^\mu}{\mathcal{E}} \equiv J_D^\mu. \quad (11)$$

4 – Generalization of the NDE for the non-free cases.

Let us now pass to consider the presence of *external* electromagnetic fields: $A^\mu \neq 0$. For the non-free case, Barut and Zanghi⁽⁶⁾ proposed the lagrangian

$$\mathcal{L} = \frac{1}{2}i(\dot{\bar{\psi}}\psi - \bar{\psi}\dot{\psi}) + p_\mu(\dot{x}^\mu - \bar{\psi}\gamma^\mu\psi) + eA_\mu\bar{\psi}\gamma^\mu\psi \quad (12)$$

which in our opinion should be better rewritten in the following form, obtained directly from the free lagrangian *via* the minimal prescription procedure:

$$\mathcal{L} = \frac{1}{2}i(\dot{\bar{\psi}}\psi - \bar{\psi}\dot{\psi}) + (p_\mu - eA_\mu)(\dot{x}^\mu - \bar{\psi}\gamma^\mu\psi), \quad (13)$$

^{#2} In the Dirac case, this is obtained by getting, from the ordinary Dirac equation $p_\mu\gamma^\mu\psi_D = m\psi_D$, the non-linear constraint $p_\mu\bar{\psi}_D\gamma^\mu\psi_D = m\bar{\psi}_D\psi_D$, and therefore replacing $\bar{\psi}_D\psi_D$ by m/\mathcal{E} , consistently with the ordinary normalization $\psi_D = e^{-ipx}u_p/\sqrt{2\mathcal{E}}$, where $\bar{u}_p u_p = 2m$.

all quantities being expressed as functions of the (CMF) proper time τ , the generalized impulse being now $p^\mu - eA^\mu$.

Lagrangian (13) does yield, in this case, the following system of differential equations:

$$\begin{cases} \dot{\psi} + i(p_\mu - eA_\mu)\gamma^\mu\psi = 0 & (14a) \\ \dot{x}^\mu = \bar{\psi}\gamma^\mu\psi & (14b) \\ \dot{p}^\mu - e\dot{A}^\mu = 0. & (14c) \end{cases}$$

As performed at the beginning of Sect.2, we can insert the identity (5) into eqs.(14a), (14b), and exploit the definition of the velocity field, eq.(14c). We easily get the following *five* first-order *differential equations* (one scalar plus one vector equation) containing the five (independent) unknown functions $\psi(x)$ and $p^\mu(x)$:

$$\begin{cases} i(\bar{\psi}\gamma^\mu\psi)\partial_\mu\psi = (p_\mu - eA_\mu)\gamma^\mu\psi & (15a) \\ (\bar{\psi}\gamma^\mu\psi)\partial_\mu(p^\nu - eA^\nu) = 0, & (15b) \end{cases}$$

which are now *field* equations (quantities $\psi, \bar{\psi}, p$ and A being all functions of x^μ).

The solutions $\psi(x)$ of system (15) may be now regarded as the classical spinorial fields for relativistic spin- $\frac{1}{2}$ fermions, in presence of an electromagnetic potential $A^\mu \neq 0$. We can obtain from eqs.(15) well-defined time evolutions, both for the velocity p^μ/m of the CMF and for the ‘‘particle’’ velocity v^μ . It is rather likely that, by imposing the condition of finite motions, i.e., that $v(\tau)$ and $p(\tau)$ be periodic in time (and ψ vanish at spatial infinity), one will be able to find a discrete spectrum, out from the continuum set of solutions of eqs.(15). Therefore, without solving any eigenvalue equations, within our field theory we may *expect* to be able to single out discrete energy level spectra for the stationary states, in analogy with what we already found⁽⁷⁾ in the free case (in which the *uniform* motion condition *implied* the z-componens s_z of spin s to be discrete).⁽⁷⁾ We shall expand on this point elsewhere: paying attention to applications, especially to ‘‘classical’’ problems so as hydrogen atom, Zeeman effect, and tunnelling through a barrier.

Acknowledgements

This paper is dedicated to the memory of Asim O. Barut. The authors wish to acknowledge stimulating discussions with J. Keller, M. Pavšič, S. Sambataro, D. Wisnivesky, and particularly W.A. Rodrigues Jr. and J. Vaz Jr. Thanks for useful discussions and kind collaboration are also due to G. Andronico, G.G.N. Angilella, A. Bugini, M. Bor-

rometi, S. Cherubini, G. Lamagna, R. Maltese, R. Milana, R.L. Monaco, E.C. Oliveiras, M. Pignanelli, E. Previtali, G.M. Prospero, M. Sambataro, E. Tonti, R.Turrisi and M.T. Vasconcelos.

References

- [1] A.H. Compton: Phys. Rev. **14** (1919) 20, 247, and refs. therein. See also W.H. Bostick: “Hydromagnetic model of an elementary particle”, in *Gravity Res. Found. Essay Contest* (1958 and 1961); J. Frenkel: Z. Phys. **37** (1926) 243; M. Mathisson: Acta Phys. Pol. **6** (1937) 163; H. Hönl and A. Papapetrou: Z. Phys. **112** (1939) 512; **116** (1940) 153; M.J. Bhabha and H.C. Corben: Proc. Roy. Soc. (London) **A178** (1941) 273; K. Huang: Am. J. Phys. **20** (1952) 479; H. Hönl: Ergeb. Exacten Naturwiss. **26** (1952) 29; A. Proca: J. Phys. Radium **15** (1954) 5; M. Bunge: Nuovo Cimento **1** (1955) 977; F. Gursev: Nuovo Cimento **5** (1957) 784; B. Liebowitz: Nuovo Cimento **A63** (1969) 1235; H. Jehle: Phys. Rev. **D3** (1971) 306; F. Riewe: Lett. Nuovo Cim. **1** (1971) 807; G.A. Perkins: Found. Phys. **6** (1976) 237; D. Gutkowski, M. Moles and J.P. Vigiier: Nuovo Cim. **B39** (1977) 193; A.O. Barut: Z. Naturforsch. **A33** (1978) 993; J.A. Lock: Am. J. Phys. **47** (1979) 797; M.H. McGregor: *The enigmatic electron* (Kluwer; Dordrecht, 1992); W.A. Rodrigues, J. Vaz and E. Recami: Found. Phys. **23** (1993) 459.
- [2] E. Schrödinger: Sitzungber. Preuss. Akad. Wiss. Phys. Math. Kl. **24** (1930) 418. See also P.A.M. Dirac: *The principles of quantum mechanics* (Clarendon; Oxford, 1958), 4th edition, p. 262; J. Maddox: “Where Zitterbewegung may lead”, Nature **325** (1987) 306.
- [3] H.C. Corben: Phys. Rev. **121** (1961) 1833; Nuovo Cim. **20** (1961) 529; Phys. Rev. **D30** (1984) 2683; Am. J. Phys. **61** (1993) 551; “Primitive quantization of Zitterbewegung”, preprint (June 1994).
- [4] F.A. Berezin and M.S. Marinov: J.E.T.P. Lett. **21** (1975) 320; R. Casalbuoni: Nuovo Cimento **A33** (1976) 389.
- [5] W.A. Rodrigues Jr., J. Vaz and E. Recami: Found. Phys. **23** (1993) 469.
- [6] A.O. Barut and N. Zanghi: Phys. Rev. Lett. **52** (1984) 2009. See also A.O. Barut and A.J. Bracken: Phys. Rev. **D23** (1981) 2454; **D24** (1981) 3333; A.O. Barut and M. Pavsic: Class. Quantum Grav. **4** (1987) L131; A.O. Barut: Phys. Lett. **B237** (1990) 436.
- [7] G. Salesi and E. Recami: Phys. Lett. **A190** (1994) 137; “On the field theory of the extended-like electron”, Report IC/94/185 ICTP; Trieste, 1994).

- [8] M. Pavsic, E. Recami, W.A. Rodrigues, G.D. Maccarrone, F. Raciti and G. Salesi: Phys. Lett. **B318** (1993) 481.
- [9] M. Pavsic: Phys. Lett. **B205** (1988) 231; **B221** (1989) 264; Class. Quant. Grav. **7** (1990) L187.
- [10] A.O. Barut and M. Pavsic: Phys. Lett. **B216** (1989) 297; F.A. Ikemori: Phys. Lett. **B199** (1987) 239. See also D. Hestenes: Found. Phys. **20** (1990) 1213; S. Gull, A. Lasenby and C. Doran: “Electron paths, tunneling and diffraction in the space-time algebra”, to appear in Found. Phys. (1993); D. Hestenes and A. Weingartshofer (eds.): *The electron* (Kluwer; Dordrecht, 1991), in particular the contributions by H. Krüger, by R. Boudet, and by S. Gull; A. Campolattaro: Int. J. Theor. Phys. **29** (1990) 141; D. Hestenes: Found. Phys. **15** (1985) 63.
- [11] W.A. Rodrigues Jr., J. Vaz, E. Recami and G. Salesi: Phys. Lett. **B318** (1993) 623.
- [12] V.I. Fushchich and R.Z. Zhdanov: Sov. J. Part. Nucl. **19** (1988) 498.
- [13] For the physical interpretation of the negative frequency waves, without any recourse to a “Dirac sea”, see e.g. E. Recami: Found. Phys. **8** (1978) 329; E. Recami and W.A. Rodrigues: Found. Phys. **12** (1982) 709; **13** (1983) 533; M. Pavsic and E. Recami: Lett. Nuovo Cim. **34** (1982) 357. See also R. Mignani and E. Recami: Lett. Nuovo Cim. **18** (1977) 5; A. Garuccio *et al.*: Lett. Nuovo Cim. **27** (1980) 60.