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A Velocity Field and Operator for Spinning Particles in (Nonrelativistic) Quantum Mechanics[1]

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Starting from the formal expressions of the hydrodynamical (or "local") quantities employed in the applications of Clifford algebras to quantum mechanics, we introduce—in terms of the ordinary tensorial language—*a new definition for the field of a generic quantity. By translating from Clifford into tensor algebra, we also propose a new (nonrelativistic) velocity operator for a spin-$\frac{1}{2}$ particle. This operator appears as the sum of the ordinary part* $\mathbf{p}/m$ *describing the* mean *motion (the motion* **of** *the center-of-mass), and of a second part associated with the so-called* Zitterbewegung, *which is the spin "internal" motion observed* **in** *the center-of-mass from (CMF). This spin component of the velocity operator is nonzero not only in the* Pauli *theoretical framework, i.e., in the presence of external electromagnetic fields with a nonconstant spin function, but also in the* Schrödinger *case, when the wavefunction is a spin eigenstate. Thus, one gets even in the latter case a* decomposition *of the velocity field for the Madelung fluid into two distinct parts, which constitutes the nonrelativistic analogue of the* Gordon decomposition *for the Dirac current. Explicit calculations are presented for the velocity field in the particular cases of the hydrogen atom, of a spherical well potential, and of an electron in a uniform magnetic field. We find, furthermore, that the Zitterbewegung motion involves a velocity field which is solenoidal, and that the local angular velocity is parallel to the spin vector. In the presence of a nonuniform spin vector (Pauli case) we have, besides the component of the local velocity normal to the spin (present even in the Schrödinger theory), also a component which is parallel to the curl of the spin vector.*

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1. HYDRODYNAMICAL OBSERVABLES IN QUANTUM THEORY

The *multivector or geometric* algebras are essentially due to the work of great mathematicians of the nineteenth century such as Hamilton (1805–1863), Grassman (1809–1877), and mainly Clifford (1845–1879). More recently, starting from the sixties, some authors, and in particular Hestenes,(1-3) did systematically study various interesting physical applications of such algebras, and especially of the real Dirac algebra $R_{1,3}$ often renamed *space-time algebra* (STA).(4-6) Rather interesting appear, in microphysics, the applications to space-time [O(3), Lorentz] transformations, to gauge [SU(2), SU(5), strong and electroweak isospin] transformations, to chiral [$SU(2)_L$] transformations, to the Maxwell equations, magnetic monopoles,(7) and so on. But the richest and most rigorous application is probably the formal and conceptual analysis of the geometrical, kinematical, and hydrodynamical content of the Pauli and Dirac equations, performed by means of the Real Pauli (RPA) and Real Dirac algebras, respectively. We shall refer ourselves to nonrelativistic physics, and therefore adopt the real Pauli Algebra, which is known to be *isomorphic to the ordinary tensorial algebra of the Pauli matrices* [SU(2)]. In this paper, when ambiguities arise, the operators will be distinguished by a cap.

As is wellknown, in the usual *hydrodynamical picture of fluids*, every physical quantity depends not only on time, but also on the considered space point. In other words, every quantity is a *local* or *field* quantity:

$$G \equiv G(x), \qquad x = (t; \mathbf{x}) \tag{1}$$

In the Pauli Algebra the *local* value of G may be expressed as follows:

$$G(x) = \frac{\langle \psi \hat{G} \psi^\dagger \rangle_0}{\psi \psi^\dagger} \tag{2}$$

where $\langle \ \rangle_0$ indicates the *scalar part* of the *Clifford product* of the quantities appearing within the brackets.[4] Let us translate Eq. (2) into the ordinary tensorial language:

$$G(x) = \frac{\mathrm{Re}[\psi^\dagger \hat{G} \psi]}{\psi^\dagger \psi} \tag{3}$$

[4] The expression $\psi\psi^\dagger$ should not be confused with the similar expression met in standard tensor algebra, since here the dagger † indicates the reversion (Clifford conjugation). In the present RPA, moreover, the quantity $\psi\psi^\dagger$ turns out to be a scalar.(1)

It is easy to see, and remarkable, that this definition for $G(x)$ is equivalent to the *real part* of the so-called dual representation for *bilinear* operators, sometimes utilized in the literature.[8-10] In the ordinary approaches, such operators are commonly, ever if *implicitly* only, employed for obtaining the probability densities of various quantities entering the Schrödinger, Klein–Gordon, or Dirac wave equations. In this sense, one can say that definition (3) agrees with the theoretical apparatus of ordinary wave mechanics. In connection with the first two mentioned wave equations (and in the Dirac case, when confining ourselves to the translational-convective part of the well-known Gordon decomposition of the Dirac current,[11]) the energy density may be put into the following form:

$$\frac{i\hbar}{2}[\psi^\dagger(\partial_t\psi)-(\partial_t\psi^\dagger)\psi]\equiv\frac{1}{2}\psi^\dagger i\hbar\overleftrightarrow{\partial_t}\psi \tag{4}$$

as easily obtained from Eq. (3) for the Hamiltonian $\hat{G}\equiv H\equiv i\hbar\,\partial_t$. Analogously, for the current density one can write

$$-\frac{i\hbar}{2m}[\psi^\dagger(\nabla\psi)-(\nabla\psi^\dagger)\psi]\equiv\frac{1}{2m}\psi^\dagger(-i\hbar\overleftrightarrow{\nabla})\psi \tag{5}$$

as required by Eq. (3) if $G\equiv \mathbf{p}/m$ and $\hat{G}\equiv -i\hbar\nabla/m$. Therefore, the use of the bilinear operators $\overleftrightarrow{\partial_t}$ and $\overleftrightarrow{\nabla}$ does allow us to write the above densities in the form expected for quantum-mechanical densities, namely, in the form $\psi^\dagger\hat{X}\psi$. Let us notice that, even if $\hat{G}\neq\hat{G}^\dagger$ [nonhermiticity], the quantity $G(x)$ computed by means of Eq. (3) will be always *real*. The only difference with respect to the case of hermitian operators is that the mean value $\langle G\rangle$ and the eigenvalues G_i will not be real; but this does *not* necessarily mean that G is unobservable. From the very definition of eigenstate in quantum mechanics, in fact, an eigenstate of $\hat{G}$ from a "local point of view" is characterized by a function $G(x)$ *uniformly distributed* (spatially homogeneous and constant in time): $G(x)=G_i$ for any x. Then, one can conclude[1] that the necessary condition for the hermiticity of G, and the consequent existence of real eigenvalues G_i, is the possibility of creating and observing a *uniform distribution for quantity* G in correspondence with the chosen eigenstates. The inverse does not hold: it is possible to have locally uniform quantities not corresponding to hermitian operators. A noticeable example of this occurrence is given by the nonhermitian *velocity operator* proposed below. In spite of its nonhermiticity, we shall see for *plane waves* [$\mathbf{p}=$ const] that is nonhermitian part will give no distribution, so that the velocity field will be *locally uniform* and equal to $\mathbf{p}/m$.

2. A NEW NONRELATIVISTIC VELOCITY OPERATOR ENDOWED WITH ZITTERBEWEGUNG

In the framework of the RPA, the local velocity is obtained from the usual operator $\hat{\mathbf{p}}/m$, once it is "translated" into the new algebraic language. Thus we shall have, following the standard rules for that translation,

$$i\hbar \rightarrow \mathbf{i}2\hat{\mathbf{s}} \equiv i\hbar \times \tag{6a}$$

$$\hat{\mathbf{p}}/m \equiv -i\hbar\nabla/m \rightarrow -\nabla\, \mathbf{i}\hbar \times /m \tag{6b}$$

where $\hat{\mathbf{s}}$ represents the *spin vector operator*, $\times$ indicates the usual 2×2 Pauli matrices, and **i** the Pauli algebra *pseudoscalar unity* (which corresponds to the matrix $\sigma_x\sigma_y\sigma_z$, so that $\mathbf{i}^2 = -$). Therefore, we can write for the velocity field:

$$\mathbf{v}(x) = -\frac{1}{m}\frac{\langle \psi\nabla\, \mathbf{i}\hbar \times \psi^{\dagger}\rangle_0}{\psi\psi^{\dagger}} \tag{7}$$

The "tensorial version" of this expression is the velocity operator[5]

$$\hat{\mathbf{v}} = -\frac{i\hbar}{m} \times(\nabla \cdot \times) \tag{8}$$

Due to the mathematical identity

$$\times(\mathbf{a} \cdot \times) \equiv \mathbf{a} + i\mathbf{a} \times \times \tag{9}$$

a being a generic 3-vector, we shall finally get

$$\hat{\mathbf{v}} = -\frac{i\hbar}{m}\nabla + \frac{\hbar}{m}(\nabla \times \times) \equiv \frac{\hat{\mathbf{p}}}{m} + \frac{i}{m}(\hat{\mathbf{p}} \times \times) \tag{10}$$

The above operator turns out to be composed of a hermitian part, $\hat{\mathbf{p}}/m$, and by a nonhermitian part, $i(\hat{\mathbf{p}} \times \times)/m$. The hermitian part reduces to the ordinary (but "incomplete": see below) nonrelativistic operator for wave mechanics, usually written as $i[\hat{H}, \hat{\mathbf{x}}]/\hbar \equiv i[\hat{\mathbf{p}}^2/2m, \hat{\mathbf{x}}]/\hbar$. The nonhermitian part is strictly related to the so-called *Zitterbewegung* (zbw),[12, 13] which is the spin motion, or "internal" motion—since it is observed in the CMF—expected to exist for spinning particles. In Refs. 12 and 13 it has

[5] Let us recall that, with regard to the *vectorial* basis $\times^1, \times^2, \times^3$ of the Pauli multivector algebra, we have, by definition, $\nabla \equiv \sigma^i\nabla_i$, indicating by ∇_i the ith component of vector ∇.

been shown that such an internal motion can be attributed to a pointlike "constituent" $\mathcal{Q}$ (carrying the electric charge), whose motion at the classical limit is helical. It should be noted that an internal motion of the electric charge $\mathcal{Q}$ appears only for particles endowed with spin,[13] and in such a case is to be added to the drift–translational, or "external," motion of the CM, $\mathbf{p}/m$ (which is the only one taking place for scalar particles). Actually, in the Dirac theory $\hat{\mathbf{v}}$ and $\hat{\mathbf{p}}$ are not parallel:

$$\hat{\mathbf{v}} \neq \hat{\mathbf{p}}/m \tag{11}$$

Moreover, while $[\hat{\mathbf{p}}, \hat{H}] = 0$ so that $\mathbf{p}$ is a conserved quantity, $\mathbf{v}$ is *not* a constant of the motion: $[\hat{\mathbf{v}}, \hat{H}] \neq 0$ (quantity $\hat{\mathbf{v}} \equiv : \equiv \gamma^0 \in$ being the usual vector matrix of the Dirac theory). Let us notice that in case of zbw it is highly convenient[12, 13] to split the motion variables as follows (the dot meaning derivative with respect to time):

$$\hat{\mathbf{x}} \equiv \hat{!} + \hat{\mathbf{X}}, \qquad \dot{\hat{\mathbf{x}}} \equiv \hat{\mathbf{v}} = \hat{\mathbf{w}} + \hat{\mathbf{V}} \tag{12}$$

where $\hat{!}$ and $\hat{\mathbf{w}} \equiv \dot{\hat{!}}$ describe the motion of the CM, while $\hat{\mathbf{X}}$ and $\hat{\mathbf{V}} \equiv \dot{\hat{\mathbf{X}}}$ describe the zbw motion. From an electrodynamical point of view, the conserved electric current is associated with the trajectories of $\mathcal{Q}$ (i.e., with $\hat{\mathbf{x}}$), while the center of the particle Coulomb field—obtained[14] by a time average over the field produced by the quickly oscillating charge—coincides with the particle CM (i.e., with $\hat{\mathbf{w}}$) and therefore, for free particles, with the geometrical center of the helical trajectory. As a consequence, it is $\mathcal{Q}$ which performs the *total motion*, while the CM undergoes the *mean motion* only. The resulting electron can be regarded as "extended-like,"[13] because of the existence in the CMF of an internal spin motion.

As required by Eq. (11), one has to assume the existence of zbw also in the standard Dirac theory. In fact, the above decomposition for the total motion shows up in two well-known relativistic quantum-mechanical procedures: namely, in the above-mentioned *Gordon decomposition* of the Dirac current, and in the *decomposition of the Dirac velocity operator and Dirac position operator* proposed by Schrödinger in his pioneering works.[15] The Gordon decomposition of the Dirac current reads (hereafter we shall choose units such that $c = 1$)

$$\bar{\psi}\gamma^{\mu}\psi = \frac{1}{2m}[\bar{\psi}\hat{p}^{\mu}\psi - (\hat{p}^{\mu}\bar{\psi})\psi] - \frac{i}{m}p_{\nu}(\bar{\psi}S^{\mu\nu}\psi) \tag{13}$$

$\bar{\psi}$ being the "adjoint" spinor of ψ, the quantity $\hat{p}^{\mu} \equiv i\partial^{\mu}$ the 4-dimensional impulse operator, and $S^{\mu\nu} \equiv (i/4)(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu})$ the spin-tensor operator. The ordinary interpretation of Eq. (13) is in total analogy with the decomposition given in Eq. (10). The first term on the r.h.s. of Eq. (13) is associated with the translational motion of the CM (the *scalar* part of the current, corresponding to the traditional Klein–Gordon current). By contrast, the second term on the r.h.s. is related to the existence of spin, and describes the zbw motion.

In the above quoted papers, Schrödinger started from the Heisenberg equation for the time evolution of the acceleration operator in Dirac theory

$$\hat{\mathbf{a}} \equiv \frac{d\hat{\mathbf{v}}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{\mathbf{v}}] = \frac{i}{\hbar} 2(\hat{H}\hat{\mathbf{v}} - \hat{\mathbf{p}}) \tag{14}$$

where $\hat{H}$ is equal as usual to $\hat{\mathbf{v}} \cdot \hat{\mathbf{p}} + \beta m$ (where $\hat{\mathbf{v}} \equiv :$). By integrating once this operator equation over time, he obtained

$$\hat{\mathbf{v}} = \hat{H}^{-1}\, \hat{\mathbf{p}} + \hat{\eta}(0)\, e^{-2\, iHt/\hbar} \qquad (\hat{\eta} \equiv \hat{\mathbf{v}} - \hat{H}^{-1}\, \hat{\mathbf{p}}) \tag{15}$$

After a little algebra, we may get a more interesting form for the velocity decomposition:

$$\hat{\mathbf{v}} = \hat{H}^{-1}\, \hat{\mathbf{p}} - \tfrac{1}{2}\, i\hbar \hat{H}^{-1}\, \hat{\mathbf{a}} \tag{16}$$

By integrating a second time, Schrödinger ended up also with the spatial part of the decomposition:

$$\hat{\mathbf{x}} \equiv \hat{!} + \hat{\mathbf{X}} \tag{17}$$

where we have

$$\hat{!} = \hat{\mathbf{r}} + \hat{H}^{-1}\, \hat{\mathbf{p}} t \tag{18}$$

linked to the motion of the CM, and

$$\hat{\mathbf{X}} = \tfrac{1}{2}\, i\hbar \hat{\eta} \hat{H}^{-1} \tag{19}$$

linked to the zbw motion.

We can therefore consider decomposition (10) of our velocity operator as the *nonrelativistic analogue of decomposition* (16) *of the relativistic velocity operator*. It is not at all surprising that (besides spin and the related intrinsic magnetic moment) also another "spin effect," zbw, does not vanish in the nonrelativistic limit, i.e., for small velocities of the CM $[\mathbf{p} \to 0]$. Therefore also the Schrödinger electron, being endowed with a zbw motion, does actually show its spinning nature, and is not a "scalar" particle (as often assumed)! As a matter of fact, when constructing atoms, we have necessarily to introduce "by hand" the Pauli exclusion principle; and in the Schrödinger equation the Planck constant $\hbar$ implicitly denounces the presence of spin. In Refs. 16 we have proved that the nonhermitian (zbw) part of our velocity field gives origin to the *quantum potential* of the Madelung fluid,[17] as well as to the related *zero-point energy* of the Schrödinger theory.

3. THE VELOCITY FIELD OF THE MADELUNG FLUID

A spinning nonrelativistic particle can be represented by means of a Pauli 2-component spinor:

$$\psi \equiv \sqrt{\rho}\, \Phi \tag{20}$$

where, if we require $|\psi|^2 = \rho$, the quantity Φ must obey the normalization constraint

$$\Phi^\dagger \Phi = 1 \tag{21}$$

Inserting the factorization (20) into definition (3), we have

$$\mathbf{s} \equiv \frac{\mathrm{Re}\{\psi^\dagger(\hbar/2)\, \times \psi\}}{\psi^\dagger \psi} \equiv (\hbar/2)\, \Phi^\dagger \hat{\times} \Phi \tag{22}$$

if G is the spin vector; and

$$\mathbf{p} \equiv \frac{\mathrm{Re}\{\psi^\dagger(-i\hbar\, \nabla)\, \psi\}}{\psi^\dagger \psi} = \frac{i\hbar}{2m}\, [(\nabla \Phi^\dagger)\, \Phi - \Phi^\dagger\, \nabla \Phi] \tag{23}$$

if G is the impulse. By definition, quantities (22), (23) represent the *local spin vector* and the *local impulse*, respectively. In the most general case (*Pauli* generalization of Schrödinger theory), when an external potential $\mathbf{A} \neq 0$ is present, we have to replace ("minimal prescription") in the translational term of expression (10) the canonic impulse $\hat{\mathbf{p}}$ by the kinetic impulse $\hat{\mathbf{p}} - e\mathbf{A}$, where e is the particle electric charge. Let us substitute in definition (3) this "generalized" velocity operator for G, and the factorization (20) for ψ. We finally obtain the following decomposition for the velocity field of the nonrelativistic probabilistic fluid:

$$\mathbf{v} = \frac{p - e\mathbf{A}}{m} + \frac{\nabla \times (\rho \mathbf{s})}{m\rho} \tag{24}$$

This expression may be considered as the nonrelativistic analogue of the Gordon decomposition (13) of the Dirac current. Even if the above operator $\hat{\mathbf{v}}$ is not hermitian, the local velocity $\mathbf{v}(x)$, as expected, turns out to be always a real quantity.

We want to stress that decomposition (24), just now derived by means of definition (3), may be also obtained within standard wave mechanics: and this constitutes a further test of the validity of our operator. It is sufficient, in fact, to take the familiar expression of the Pauli current (i.e., the nonrelativistic limit of the Gordon decomposition[18])

$$\mathbf{j} \equiv \rho \mathbf{v} = \frac{i\hbar}{2m} [(\nabla \psi^{\dagger}) \psi - \psi^{\dagger} \nabla \psi] - \frac{e\mathbf{A}}{m} \psi^{\dagger} \psi + \frac{\hbar}{2m} \nabla \times (\psi^{\dagger} \times \psi) \tag{25}$$

and to insert into it factorization (20) in place of ψ, for obtaining the velocity distribution in Eq. (24).

Let us single out in the total velocity field the zbw component:

$$\mathbf{V} \equiv \frac{\nabla \times (\rho \mathbf{s})}{m\rho} \tag{26}$$

Further, it is decomposable into two distinct parts:

(A) $\mathbf{V}_1 \equiv (\nabla \rho \times \mathbf{s})/m\rho$: due to the presence of the gradient of ρ, this term refers to local motions, in which the constant density surfaces $[\nabla \rho = 0]$ do rotate around the spin axis; and it vanishes identically for the plane waves $[\mathbf{p} = \text{const}]$, for which $\nabla \rho = 0$;

(B) $\mathbf{V}_2 \equiv (\text{rot}\, \mathbf{s})/m$: such a term does not depend on the density ρ and is different from 0 only in the presence of external nonuniform magnetic fields (when the wavefunctions are *not* spin eigenstates).

Let us remark that the usual requirement of continuity at every point for ψ and for its gradient $\nabla \psi$ (in order to have conservation of the current) automatically assures the analytic continuity of the zbw field, and the analytic regularity of the trajectories.

The *Schrödinger theory* is of course a particular case of the present Pauli framework, corresponding to a *uniform local spin vector*. The wavefunction is then a *spin eigenstate* and may be factorized as the product of a "scalar" part $\sqrt{\rho}\, e^{i\varphi}$ and of a "spin" part χ (a 2-components spinor):

$$\psi \equiv \sqrt{\rho}\, e^{i(\varphi/\hbar)} \chi \tag{27}$$

the quantity χ being *constant* in space and time. Let us underline that, even if $\mathbf{s} = \chi^{\dagger}(\hbar/2) \times \chi =$ constant, *in the Schrödinger case the zbw does not vanish*

—except for the unrealistic case of pure plane waves—while the velocity **V** reduces to

$$\mathbf{V} = \mathbf{V}_1 \equiv \frac{\nabla\rho \times \mathbf{s}}{m\rho} \tag{28}$$

All this does actually contribute to remove some difficulties remaining in the semiclassical representation of the particle motion, and in the interpretation of the particle energies, for some stationary solutions of the Schrödinger equation. As is known, the eigenfunctions corresponding to nondegenerate energy eigenvalues are always *real* (φ is uniform and equal to a constant which for the "global gauge invariance" may be assumed equal to zero). (Even for some systems with degenerate energy levels, as for instance the spherical harmonic oscillator, one gets real eigenfunctions). Let us refer to the $l = 0$, $n \geqslant 1$ stationary states of a particle inside a well or a box, and of the hydrogen atom, as well as to the $m = 0$, $n \geqslant 0$ stationary states of a particle in a uniform magnetic field. For all such systems *the local velocity obtained from standard quantum mechanics*, $\mathbf{p}/m \equiv \nabla\varphi/m$, *is zero* everywhere, at any time. As was first remarked by Einstein and Perrin and by de Broglie,[19] such a result *seems to be really in contrast* from a classical point of view *with the nonvanishing of the energy eigenvalue for those stationary states.* But we now know, from our previous analysis, that $\mathbf{p}/m$ is the *mean* velocity, describing only the motion of the CM, while the zbw-component **V**—which depends on the ρ-gradient, and not on the phase-gradient of ψ—does not identically vanish, thus implying an internal motion around the spin axis. Let us show some examples of applications of Eq. (28).

For the ground state of the hydrogen atom, endowed with azimuthal quantum number $l = 0$, we have[20]

$$\mathbf{V} = \frac{1}{\pi r_B^3} e^{-2r/r_B} \tag{29}$$

where quantity $r_B \equiv \hbar^2/me^2$ is the Bohr radius. Therefore Eq. (28) yields

$$\mathbf{V} = -\frac{2}{mr_B}\frac{\mathbf{r}}{r} \times \mathbf{s} \tag{30}$$

The speed is maximum in the polar plain ($\theta = \pi/2$, the z-axis being parallel to the constant spin vector **s**) and vanishes along the z-axis ($\theta = 0, \pi$):

$$V = \frac{\hbar \sin\theta}{mr_B} \equiv \alpha \sin\theta \tag{31}$$

where $\alpha \equiv ke^2/\hbar c$ is the fine-structure constant. We therefore have for the zbw speed:

$$V \leqslant \alpha \simeq 7 \cdot 10^{-3}\ c \tag{32}$$

Analogously, for the hydrogen-like atoms we obtain (Z being the atomic number) $V = Z\alpha \sin\theta$.

Let us now consider the bound states of a (spinning) particle in a potential well endowed with spherical symmetry: $U(r) = 0$ for $r \leqslant a$; $U(r) = U_0 > 0$ for $r > a$. For every $n \geqslant 1$, $l = 0$ stationary state, corresponding to the energy eigenvalue $E_n (< U_0)$, we can write inside the well[(21)]

$$\rho = A^2 \frac{\sin^2 k_n r}{r^2}, \qquad \text{for} \quad r < a \tag{33}$$

where A is a suitable normalization factor, and $k_n \equiv \sqrt{2mE_n}/\hbar$ is implicitly determined from the following constraint (which actually constitutes an eigenvalue equation):

$$k_n \cot k_n a = -\frac{1}{\hbar}\sqrt{2m(U_0 - E_n)} \tag{34}$$

Inserting Eq. (33) into Eq. (28), we obtain the zbw field inside the well:

$$\mathbf{V} = \left(\frac{2k_n}{m}\cot k_n r - \frac{2}{mr}\right)\frac{\mathbf{r}}{r} \times \mathbf{s} \tag{35}$$

and then

$$V = \left|\frac{\hbar k_n}{m}\cot k_n r - \frac{\hbar}{mr}\right| \sin\theta \tag{36}$$

At the center of the well, for $r \to 0$, we have a vanishing velocity, $V \to 0$. The maximum speed is found on the surface of the well ($r = a$) where, because of the condition (34), one has

$$V = \left(\sqrt{\frac{2(U_0 - E_n)}{m}} + \frac{\hbar}{ma}\right) \sin\theta \tag{37}$$

Outside the well the probability density reads

$$\rho = B^2 \frac{e^{-2\kappa_n}}{r^2}, \qquad \text{for} \quad r > a \tag{38}$$

with $\kappa_n \equiv \sqrt{2m(U_0 - E_n)}/\hbar$. Thus we obtain

$$\mathbf{V} = -\left(\frac{2\kappa_n}{m} + \frac{2}{mr}\right)\frac{\mathbf{r}}{r} \times \mathbf{s} \tag{39}$$

$$V = \left(\frac{\hbar\kappa}{m} + \frac{\hbar}{mr}\right)\sin\theta \tag{40}$$

Now the local velocity is composed of a constant term and of a linearly decreasing term, which vanishes for $r \to \infty$ (at large distances from the well).

Consider now an electron in a uniform magnetic field $\mathbf{H} = (0; 0; H)$. In the stationary ground state $\psi_{nmp_x} \equiv \psi_{000}$—with the radial number $n = 0$, the magnetic number $m = 0$ and the conserved azimuthal impulse $p_x = 0$—the density reads[22]

$$\rho = \frac{\hbar}{2\pi eH} e^{-eHd^2} \tag{41}$$

where d indicates the distance from the z-axis. In the present case only the component of the spin vector parallel to $\mathbf{H}$ is conserved, so that $\mathbf{s}$ necessarily points in the z-direction. From Eq. (28) we see that the zbw motion is endowed with cylindrical symmetry [$\mathbf{d} \equiv (x; y; 0)$]:

$$\mathbf{V} = -\frac{2eH}{m\hbar}\mathbf{d} \times \mathbf{s} \tag{42}$$

$$V = \frac{eH}{m} d \tag{43}$$

At any distance from the z-axis, the charge local flow around the magnetic field direction is endowed with an angular velocity equal to the well-known "cyclotron frequency" eH/m. Thus, in our (probabilistic) fluid every infinitesimal element of the electron charge behaves as a classical macroscopic electrical charge performing a uniform circular motion because of the Lorentz force.

Let us finally analyze the distribution for the *total* velocity, Eq. (24), always for the Schrödinger case, $\mathbf{A} = 0$, and stress some of its interesting properties.

Since the curl of a gradient is identically zero, we shall have $\text{rot}\,\mathbf{p} \equiv \text{rot}\,\nabla\varphi = 0$; and since $\mathbf{s}$ is furthermore a homogeneous quantity, we shall also have $\nabla \mathbf{s} = 0$. As a consequence, by employing the known property of the double vectorial product

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\, \mathbf{c}, \tag{44}$$

we can show that the local rotational properties of the Madelung fluid are actually given by the following expression:

$$\mathrm{rot}\, \mathbf{v} = \frac{1}{m}\left[\left(\frac{\nabla \rho}{\rho}\right)^2 - \frac{\Delta \rho}{\rho}\right] \mathbf{s} \tag{45}$$

Moreover, the zbw current field $\rho \mathbf{V}$ turns out to be *solenoidal* (and this happens also in the most general case of the Pauli fluid with a nonhomogeneous $\mathbf{s}$):

$$\mathrm{div}(\rho \mathbf{V}) \equiv \mathrm{div}[\mathrm{rot}(\rho \mathbf{s})]/m = 0 \tag{46}$$

The flux streamlines will be *closed lines*, as happens for the magnetic field: therefore, we expect *the zbw motion to be limited*, *finite*, *and periodical*. We can see eventually that the *local angular velocity* ω *is parallel to the spin vector* $\mathbf{s}$:

$$| = \frac{1}{2}\, \mathrm{rot}\, \mathbf{v} = \frac{1}{2m}\left[\left(\frac{\nabla \rho}{\rho}\right)^2 - \frac{\Delta \rho}{\rho}\right] \mathbf{s} \tag{47}$$

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