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Effects of spin on the cyclotron frequency for a Dirac electron ☆

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Abstract

The Barut–Zanghi (BZ) theory – that constitutes a natural 'classical limit' of the Dirac theory and can be regarded to be a satisfactory picture of a classical spinning electron – has been analytically studied in some previous papers of ours in the case of free particles. By contrast, in this letter we consider the case of external fields, and a previously found equation of motion is generalized for a *non*-free spin-$\frac{1}{2}$ particle. In the important case of a spinning charge in a uniform magnetic field, we find that its angular frequency (around the magnetic field direction) is slightly different from the classical 'cyclotron frequency' $\omega \equiv eH/m$ expected for spinless charges. As a matter of fact, the angular frequency does depend on the spin orientation. As a consequence, the electrons with magnetic moment $\boldsymbol{\mu}$ parallel to the magnetic field do rotate with a frequency greater than that of electrons endowed with a $\boldsymbol{\mu}$ antiparallel to $\boldsymbol{H}$.

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1. The Barut–Zanghi theory and the free-particle solutions

In the last twenty years, renewed interest arose for classical theories of electrons and spin (see Refs. [1–23] and references therein; cf. also Refs. [24–28]); in particular, for those approaches – as the Barut–Zanghi's (BZ) theory of the relativistic spinning electron [29–48], which we shall refer to in this paper – that involve the so-called *Zitterbewegung* (zbw) [49–75]. In the Barut–Zanghi theory the classical electron was actually characterized, besides by the usual pair of conjugate variables (x^μ, p^μ), also by a second pair of conjugate (classical) spinorial variables $(\psi, \bar{\psi})$, representing *internal* degrees of freedom, which were functions of the (proper) time τ measured in the center-of-mass (CM) frame; the CM frame being the one in which it is $\boldsymbol{p} = 0$ identically at any instant of time. Barut and Zanghi, then, introduced a classical Lagrangian that writes [A^μ is

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the electromagnetic potential, and the vacuum light speed c is set equal to 1]:

$$\mathscr{L} = \tfrac{1}{2} i\lambda\left(\dot{\bar{\psi}}\psi - \bar{\psi}\dot{\psi}\right) + p_\mu\left(\dot{x}^\mu - \bar{\psi}\gamma^\mu\psi\right) + eA_\mu\bar{\psi}\gamma^\mu\psi\,, \tag{1}$$

where λ has the dimension of an action, and ψ and $\bar{\psi} \equiv \psi^\dagger\gamma^0$ are ordinary $\mathbb{C}^4$-bispinors, the dot meaning derivation with respect to τ. The four Euler–Lagrange equations, with $-\lambda = \hbar = 1$, obtained by varying $\mathscr{L}$ with respect to $\psi, \bar{\psi}, x^\mu, p^\mu$, are the following:

$$\dot{\psi} = -i\not{\pi}\psi\,, \quad \dot{\bar{\psi}} = i\bar{\psi}\not{\pi}\,, \tag{2a}$$

$$\dot{\pi}^\mu = eF^{\mu\nu}\dot{x}_\nu\,, \quad \dot{x}^\mu = \bar{\psi}\gamma^\mu\psi\,, \tag{2b}$$

where $F^{\mu\nu}$ is the electromagnetic tensor, and π is the kinetic impulse

$$\pi^\mu \equiv p^\mu - eA^\mu\,, \quad \not{\pi} \equiv \gamma_\mu\pi^\mu. \tag{3}$$

Furthermore, $\pi_\mu\bar{\psi}\gamma^\mu\psi \equiv \pi_\mu\dot{x}^\mu$ results [39] to be always a conserved quantity [equal to m: see Eq. (9a) below]. Notice that, instead of adopting the variables ψ and $\bar{\psi}$, one can work in terms of the following set of independent dynamical variables

$$x^\mu, \pi^\mu\,, \quad v^\mu, S^{\mu\nu} \tag{4a}$$

where

$$S^{\mu\nu} \equiv \frac{i}{4}\,\bar{\psi}\,[\gamma^\mu, \gamma^\nu]\,\psi \tag{4b}$$

is the *spin tensor* met in the Dirac theory; then, one gets the following equations of motion:

$$\dot{\pi}^\mu = eF^{\mu\nu}v_\nu\,, \tag{5a}$$

$$\dot{x}^\mu = v^\mu\,, \tag{5b}$$

$$\dot{v}^\mu = 4\,S^{\mu\nu}\pi_\nu\,, \tag{5c}$$

$$\dot{S}^{\mu\nu} = v^\nu\pi^\mu - v^\mu\pi^\nu\,. \tag{5d}$$

The first equation is the well-known Lorentz equation of the motion for a (spinless) charged particle inside an electromagnetic (em) field [even if, the present case of spinning electrons, we have $\pi^\mu \neq mv^\mu$ (see below)!]; the second one is nothing but the definition for the 4-velocity; the third equation is formally very similar to the first one, with v^μ and π^μ interchanged and the spin tensor replacing the em tensor. The last equation expresses the conservation of the total angular momentum $J^{\mu\nu}$, the sum of the orbital angular momentum $L^{\mu\nu}$ and of $S^{\mu\nu}$:

$$\dot{J}^{\mu\nu} = \dot{L}^{\mu\nu} + \dot{S}^{\mu\nu} = 0\,, \tag{6}$$

where $\dot{L}^{\mu\nu} = v^\mu\pi^\nu - v^\nu\pi^\mu$ from the very definition of L. Notice that only the first couple of equations is present in the case of spinless particles, while the second couple of equations is directly related to the existence of spin. Starting from these Euler-Lagrange equations, in Refs. [40–42] we worked out the equation of the motion in the space-time coordinates x^μ for the particular case of *free* electrons ($A^\mu = 0$):

$$v^\mu = \frac{p^\mu}{m} - \frac{\ddot{v}^\mu}{4m^2}\,, \tag{7}$$

when assuming $p^\mu =$ const. and $p^2 \equiv p_\mu p^\mu = m^2$. The general solution of this equation was shown to be [29–33]:

$$v_i^\mu = \frac{p^\mu}{m} + \left[v^\mu(0) - \frac{p^\mu}{m}\right]\cos(\omega\tau) + \frac{\dot{v}^\mu(0)}{2m}\sin(\omega\tau) \tag{8a}$$

with[1]

$$\omega = 2m\,. \tag{8b}$$

The particular solution corresponding to $v^\mu(0) = p^\mu/m$, $\dot{v}^\mu(0) = 0$, is associated with a rectilinear uniform motion with constant velocity p^μ/m, like in the case of a macroscopic free body or of a non-spinning free particle. The general solution, by contrast, oscillates with frequency ω: this denounces explicitly the presence of spin and zbw. The zbw is nothing but the spin motion, or 'internal motion' [since it can be observed in the CM frame, where by definition $\boldsymbol{p} = 0$], which is expected to exist for spinning particles only. Let us recall that it arises because the motion of the electrical charge does not coincide with the motion of the CM, so that spinning particles actually appear as extended-like objects [39]. In the Dirac theory, indeed, the operators velocity

[1] Obviously the quantity $\omega \equiv (\mathrm{d}\theta/\mathrm{d}\tau)$ can be considered as a proper angular velocity only in the CM frame or in non-relativistic frames; otherwise one has to multiply the value given in Eqs. (8b) by the Lorentz factor $m/\mathscr{E}$ (getting the so-called relativistic decrease of the frequency).

($\boldsymbol{\alpha}$) and impulse ($-i\boldsymbol{\nabla}$) are *not* parallel, in general. Therefore a zbw motion is to be added to the translational, or 'external', motion of the CM, whose velocity is p^{μ}/m.

Let us explicitly observe that the general solution (4c) represents a *helical* motion in the ordinary 3-space (a result met also in other models and approaches[2] which imply a zbw). Moreover, free polarized particles (with the spin projection s_z along the z-axis equal to $\pm\frac{1}{2}$) are endowed [39–46] with internal *uniform circular* motions around the z-axis. In such a way, the *classical* values for s_z corresponding to classical uniform motions in the CM frame belong to the *discrete* spectrum $\pm\frac{1}{2}$. The orbit radius in the CM frame was found to be equal to $|\boldsymbol{V}|/2m$ (quantity $\boldsymbol{V}$ being the orbital 3-velocity), which, in the special case of a *light-like* zbw ($|\boldsymbol{V}|=1$), turns out to be equal to half the Compton wavelength.

In the next section we want to generalize Eq. (7) for the case of an electron in an external em field, and to write down its analytical solutions in the special case of a uniform magnetic field.

2. The motion of a classical Dirac electron in a uniform magnetic field

Before going on, we have to assume some important constraints for physical consistency with the standard relativistic quantum mechanics; namely:

$$\pi_{\mu} v^{\mu} = m\,, \tag{9a}$$

$$\pi^2 = m^2 + eF^{\mu\nu}S_{\mu\nu}\,. \tag{9b}$$

For free electrons, the condition $p_{\mu}v^{\mu}=m$ represents the ''classical limit'' of the standard Dirac equation $\hat{p}_{\mu}\gamma^{\mu}\psi = m\,\psi$ (with $\hat{p}\equiv i\partial$), as was shown in previous works [29–46]. Analogously, in the presence of external em fields, Eq. (9a) may be regarded as the 'classical limit' of the Dirac equation in an external em field, namely $\hat{\pi}_{\mu}\gamma^{\mu}\psi = m\,\psi$ (with $\hat{\pi}_{\mu}\equiv i\partial_{\mu} - eA_{\mu}$). Notice that for spinless particles this constraint reduces to an identity; in fact, in the absence of spin, the kinetic impulse and the velocity $v_{\mu} = \pi_{\mu}/m$ get parallel, so that Eq. (9a) follows by multiplying both members by v^{μ}.

The second condition is nothing but the ''classical limit'' of the so-called ''second-order Dirac equation'', obtained by left-multiplying [76] the usual Dirac equation by $\hat{\pi}_{\mu}\gamma^{\mu}+m$. In fact, from

$$\left(\hat{\pi}_{\mu}\gamma^{\mu}+m\right)\left(\hat{\pi}_{\mu}\gamma^{\mu}-m\right)\psi = 0\,,$$

it follows

$$\hat{\pi}^2\,\psi = \left(m^2 + eF^{\mu\nu}\hat{S}_{\mu\nu}\right)\psi\,, \tag{10}$$

where $\hat{S}_{\mu\nu}\equiv\frac{i}{4}[\gamma_{\mu},\gamma_{\nu}]$ indicates the spin tensor operator. For free ($F^{\mu\nu}=0$) spinning particles this constraint reduces to $\pi^2\equiv\pi_{\mu}\pi^{\mu}=m^2$, which *for scalar particles* holds both in the presence and in the absence of external fields (of course, the spin-field term $eF^{\mu\nu}S_{\mu\nu}$ is not present for spinless particles). For the case of a purely magnetic field ($\boldsymbol{E}=0$), that we are going to analyse, the constraint (9b) assumes the form:

$$\mathscr{E} = \sqrt{m^2 + \boldsymbol{\pi}^2 - 2e\boldsymbol{s}\cdot\boldsymbol{H}}\,, \tag{11}$$

since for the energy $\mathscr{E}$ it holds $\mathscr{E}\equiv p^0\equiv\pi^0+e\varphi=\pi^0$.

In the non-relativistic limit, $\pi^2\ll m^2$, from Eq. (11) we easily get just the expected Hamiltonian for a spin-$\frac{1}{2}$ particle in a magnetic field:

$$\mathscr{E}\sim m + \frac{\boldsymbol{\pi}^2}{2m} - \frac{e\boldsymbol{s}}{m}\cdot\boldsymbol{H}\,, \tag{12}$$

with the correct gyromagnetic ratio $g=2$.

By means of a procedure analogous to the one followed for the free case, we can now deduce the equation of the motion in the presence of an external em field. By deriving Eq. (5c) one gets:

$$\ddot{v}^{\mu} = 4\dot{S}^{\mu\nu}\pi_{\nu} + 4S^{\mu\nu}\dot{\pi}_{\nu}\,. \tag{13}$$

[2] A physical role of the zbw has been found, and studied, even in the non-relativistic framework [39,57–75] and recently extended to supersymmetry and superstrings [34–38].

Inserting Eqs. (5d) and (5a) into Eq. (13), and exploiting constraints (9a) and (9b), we finally obtain[3] the equation:

$$\ddot{v}^{\mu} - 4m\pi^{\mu} + 4m^2 v^{\mu} + 4eV^{\mu}F_{\lambda\rho}S^{\lambda\rho} - 4eS^{\mu\nu}F_{\nu\rho}v^{\rho} = 0. \quad (14)$$

By comparison with Eq. (7), which holds for free particles, in the r.h.s. of the general equation of motion (14) we see the appearance of two additional spin-field terms. The analytical solutions of Eq. (14) can be easily found in the simple, but important, case of an external *uniform magnetic field* $\boldsymbol{H}$. Let us take the magnetic field oriented along the z-axis at all times:

$$\boldsymbol{H} = (0,0,H). \quad (15)$$

The quantum mechanical theory (the Dirac equation) entails the conservation of the z-component of the spin vector: $s_z = \pm\frac{1}{2}$. According to the correspondence principle between quantum mean values and classical values, during the precession of the classical spin vector it will be $S^{12} = s_z =$ constant. Furthermore, the only nonzero components of the em tensor are $F^{21} = -F^{12} = H$. We restrict ourselves to the xy-plane where (by analogy with the behaviour of spinless charges involved by the Maxwell equations) we expect to have a uniform circular motion of the spinning charge due to the Lorentz force. From Eq. (14) we get:

$$\ddot{v}_x - 4m\pi_x + 4\left(m^2 - 3es_z H\right) v_x = 0, \quad (16a)$$

$$\ddot{v}_y - 4m\pi_y + 4\left(m^2 - 3es_z H\right) v_y = 0. \quad (16b)$$

By deriving with respect to time and exploiting Eq. (5a), we finally get:

$$\dddot{v}_x + 4\left(m^2 - 3es_z H\right) \dot{v}_x - 4meHv_y = 0, \quad (17a)$$

$$\dddot{v}_y + 4\left(m^2 - 3es_z H\right) \dot{v}_y + 4meHv_x = 0. \quad (17b)$$

This system of equations yields uniform circular motions whose angular velocities $\omega_i\,(i = 1,2,3)$ are roots of the 'characteristic' 3-order algebraic equation:

$$\omega^3 - 4\left(m^2 - 3es_z H\right)\omega + 4meH = 0. \quad (18)$$

Even with the highest magnetic fields today experimentally achievable in laboratory, the following condition always holds between the 'intrinsic' frequency $2m$ (that is, the zbw angular frequency for free particles) and the 'external' *cyclotron frequency* ω_H:

$$\omega_H \equiv \frac{eH}{m} \ll 2m. \quad (19)$$

As a consequence, we may write down the characteristic frequencies as follows:

$$\omega_1 \approx \omega_H\left[1 + \frac{3\omega_H}{m}s_z\right]; \quad \omega_2 \approx 2m\left[1 - \frac{3\omega_H}{2m}s_z\right];$$

$$\omega_3 \approx -2m\left[1 - \frac{3\omega_H}{2m}s_z\right]. \quad (20)$$

The global motion of the charge is obtained by the linear combination of the three uniform circular motions with the three frequencies ω_i. The 'internal' frequencies ω_2, ω_3 are related to the zbw motion and are, because of (19), substantially identical to the ones ($\pm 2m$) found for free particles [notice that the free-particle solution (8a) does not vary if we simultaneously invert the signs of the frequency (8b) and of the initial condition $\dot{v}^{\mu}(0)$]. For zero external fields, i.e., $H = 0$, $\omega_H = 0$, the frequency ω_1 vanishes and the general solution of the equations of motion turns out – of course – to be equal to the one found for free particles, Eq. (8a).

The important point is that *the 'external' angular frequency ω_1 results slightly different from the cyclotron frequency ω_H* which is typical of ordinary (spinless) charges in a magnetic field. The frequency shift is a function of the spin vector orientation, so that *the spin-up and the spin-down polarized electrons rotate with different angular velocities*:

$$\frac{\Delta\omega}{\omega_H} \equiv \frac{\omega_{1/2} - \omega_{-1/2}}{\omega_H} \approx 3\,\frac{\omega_H}{m}. \quad (21)$$

Thus, we found that spinning charges with their magnetic moment $\boldsymbol{\mu} \equiv -e\boldsymbol{s}/m$ antiparallel to the

[3] In the original paper by Barut and Zanghi [29] a *different* equation of the motion in the presence of an em field was deduced. Actually, those authors, besides having adopted some peculiar constraints, were working not in the ordinary spacetime, but in a particular 5-*dimensional manifold.*

magnetic field rotate with a frequency *greater* than the one of spinning charges having $\boldsymbol{\mu}$ parallel to $\boldsymbol{H}$. The present, classical approach to the problem of a charge in a uniform magnetic field turns out to be very suitable for describing *(unbound) electrons performing large orbits in vacuum*, which behave as classical bodies.

In fact, in cyclotrons, magnetic rings or bottles, and non-linear accelerators, the angular frequency is usually assumed to be equal to the classical value ω_H for both the possible polarizations $s_z = \pm\frac{1}{2}$. The small frequency shift predicted, on the contrary, by Eq. (21) could be observed by means of an ad hoc experiment, in which a device measuring s_z (or, anyway, interacting in a *different* way with the different polarizations) is placed at a point along the orbit. For Eq. (21), a particle beam, initially containing both the spin components, will progressively spread along the orbital motion: so that the spin-up particles (which rotate faster) will slowly separate from the spin-down particles.

For example, for a cyclotron orbit with a diameter $d = 1$ m only and a magnetic field $H = 3.4 \times 10^{-4}$ T, we obtain, for electrons, the orbital speed $v = \frac{1}{10}c$ (which may be still considered as non-relativistic, so that the relativistic frequency decrease is negligible), and the angular frequency $\omega \approx 6 \times 10^7$ Hz. As a consequence, the frequency shift $\Delta\omega$ will be about 1.34×10^{-6} Hz, which implies a phase difference of 2π (corresponding to one full orbit) in about 7.5×10^6 s. After a time $\Delta t = 40$ minutes, the spin-up electrons are expected to precede the spin-down ones by a distance $\Delta l = 1$ mm. As a consequence, the spin-up particles will interact with a suitable detector at a time $\Delta\tau = 4.27 \times 10^{-10}$ s earlier than the spin-down particles. Of course, the effect will be much bigger for higher intensity magnetic fields, since $\Delta l \propto H^2$.

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References

[1] H.C. Corben, Phys. Rev. 121 (1961) 1833.
[2] H.C. Corben, Classical and quantum theories of spinning particles, Holden-Day, San Francisco, 1968.
[3] H.C. Corben, Phys. Rev. D 30 (1984) 2683.
[4] H.C. Corben, Am.J. Phys. 45 (1977) 658.
[5] H.C. Corben, Am.J. Phys. 61 (1993) 551.
[6] H.C. Corben, Int. J. Theor. Phys. 34 (1995) 19.
[7] A.J. Kálnay et al., Phys. Rev. 158 (1967) 1484.
[8] A.J. Kálnay et al., Phys. Rev. D 1 (1970) 1092.
[9] A.J. Kálnay et al., Phys. Rev. D 3 (1971) 2357.
[10] A.J. Kálnay et al., Phys. Rev. D 3 (1971) 2977.
[11] H. Jehle, Phys. Rev. D 3 (1971) 306.
[12] F. Riewe, Lett. Nuovo Cim. 1 (1971) 807.
[13] F.A. Berezin, M.S. Marinov, J.E.T.P. Lett. 21 (1975) 320.
[14] R. Casalbuoni, Nuovo Cimento A 33 (1976) 389.
[15] G.A. Perkins, Found. Phys. 6 (1976) 237.
[16] D. Gutkowski, M. Moles, J.P. Vigier, Nuovo Cim. B 39 (1977) 193.
[17] A.O. Barut, Z. Naturforsch. A 33 (1978) 993.
[18] J.A. Lock, Am.J. Phys. 47 (1979) 797.
[19] M. Pauri: in Lecture Notes in Physics, vol. 135, Springer, Berlin, 1980, p. 615.
[20] G. Cavalleri, Nuovo Cim. B 55 (1980) 392.
[21] G. Cavalleri, Phys. Rev. D 23 (1981) 363.
[22] G. Cavalleri, Phys. Rev. C 6 (1983) 239.
[23] G. Cavalleri, Lett. Nuovo Cim. 43 (1985) 285.
[24] F.A. Ikemori, Phys. Lett. B 199 (1987) 239.
[25] Ph. Gueret, Lectures at the Bari university, Bari, 1989.
[26] M.H. McGregor, The enigmatic electron, Kluwer, Dordrecht, 1992.
[27] W.A. Rodrigues, J. Vaz, E. Recami, Found. Phys. 23 (1993) 459.
[28] G. Cavalleri, G. Salesi, in: Proc. of Physical Relativity Theory, British Soc. Phil. Sc., London, 9-12 September, 1994.
[29] A.O. Barut, N. Zanghi, Phys. Rev. Lett. 52 (1984) 2009.
[30] A.O. Barut, A.J. Bracken: Phys. Rev. D 23 (1981) 2454.
[31] A.O. Barut, Phys. Rev. D 24 (1981) 3333.
[32] A.O. Barut, I.H. Duru, Phys. Rev. Lett. 53 (1984) 2355.

[33] A.O. Barut, M. Pavšič, Class. Quantum Grav. 4 (1987) L131.
[34] A.O. Barut, M. Pavšič, Phys. Lett. B 216 (1989) 297.
[35] A.O. Barut, M. Pavšič, The spinning minimal surfaces without the Grassmann variables, preprint IC/88/2, ICTP; Trieste, 1988.
[36] M. Pavšič, Phys. Lett. B 221 (1989) 264.
[37] M. Pavšič, Phys. Lett. B 205 (1988) 231.
[38] M. Pavšič, Class. Quant. Grav. 7 (1990) L187.
[39] E. Recami, G. Salesi, Phys. Rev. A 57 (1998) 98, and refs. therein.
[40] E. Recami, G. Salesi, Found. Phys. 28 (1998) 763.
[41] E. Recami, G. Salesi, Adv. Appl. Cliff. Alg 6 (1996) 27.
[42] E. Recami, G. Salesi, in: P. Pronin, G. Sardanashvily (Eds.), Gravity, Particles and Space-Time, World Scientific, Singapore, 1996, pp. 345–368.
[43] J. Vaz, W.A. Rodrigues, Phys. Lett. B 319 (1993) 203.
[44] W.A. Rodrigues, J. Vaz, E. Recami, G. Salesi, Phys. Lett. B 318 (1993) 623.
[45] M. Pavšič, E. Recami, W.A. Rodrigues, G.D. Maccarrone, F. Raciti, G. Salesi, Hadronic J 18 (1995) 97.
[46] M. Pavšič, E. Recami, W.A. Rodrigues, G.D. Maccarrone, F. Raciti, G. Salesi, Phys. Lett. B 318 (1993) 481.
[47] G. Salesi, E. Recami, Phys. Lett. A 190 (1994) 137.
[48] G. Salesi, E. Recami, Phys. Lett. A 195 (1994) E389.
[49] E. Schrödinger, Sitzunger. Preuss. Akad. Wiss. Phys.-Math. Kl. 24 (1930) 418.
[50] E. Schrödinger, Sitzunger. Preuss. Akad. Wiss. Phys.-Math. Kl. 25 (1931) 1.
[51] A.H. Compton, Phys. Rev. 14 (1919) 20, 247, and refs. therein.
[52] G.E. Uhlenbeck, S.A. Goudsmit, Nature 117 (1926) 264.
[53] J. Frenkel, Z. Phys. 37 (1926) 243.
[54] T. Takabayasi, J.-P. Vigier, Progr. Theor. Phys. 18 (1957) 573.
[55] P.A.M. Dirac, The Principles of Quantum Mechanics, Claredon; Oxford, 1958, 4th ed., p. 262.
[56] J. Maddox, Nature 325 (1987) 306.
[57] G. Salesi, Mod. Phys. Lett. A 11 (1996) 1815.
[58] G. Salesi, E. Recami, Found. Phys. Lett. 10 (1997) 533.
[59] G. Salesi, E. Recami, in: J. Keller, Z. Oziewicz (Eds.), The Theory of the Electron, UNAM Press, Mexico City, 1997, pp. 269–278.
[60] D. Hestenes, Am.J. Phys. 39 (1971) 1028.
[61] D. Hestenes, Am.J. Phys. 39 (1971) 1013.
[62] D. Hestenes, Am.J. Phys. 47 (1979) 399.
[63] D. Hestenes, J. Math. Phys. 14 (1973) 893.
[64] D. Hestenes, J. Math. Phys. 16 (1975) 573.
[65] D. Hestenes, J. Math. Phys. 16 (1975) 556.
[66] D. Hestenes, J. Math. Phys. 8 (1979) 798.
[67] D. Hestenes, Found. Phys. 15 (1985) 63.
[68] D. Hestenes, Found. Phys. 20 (1990) 1213.
[69] D. Hestenes, Found. Phys. 23 (1993) 365.
[70] D. Hestenes, Space-time algebra, Gordon & Breach, New York, 1966; New Foundations for Classical Mechanics, Kluwer; Dordrecht, 1986.
[71] D. Hestenes, G. Sobczyk: Clifford Algebra to Geometric Calculus, Reidel; Dordrecht, 1984.
[72] D. Hestenes, A. Weingartshofer (Eds.), The electron, Kluwer, Dordrecht, 1991.
[73] M. Rivas, J. Math. Phys. 35 (1994) 3380, and references therein.
[74] M. Rivas, unpublished.
[75] W.M. Pezzaglia Jr., Polydimensional relativity, a classical generalization of the automorphism invariance principle, submitted for pub.
[76] L.D. Landau, E.M. Lifshitz: Fisica Teorica, vol. IV: Teoria quantistica relativistica, Ed.i Riuniti–Edizioni MIR; Roma, 1978, p. 148.