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NON-NEWTONIAN MECHANICS*

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The classical motion of spinning particles can be described without recourse to particular models or special formalisms, and without employing Grassmann variables or Clifford algebras, but simply by generalizing the usual spinless theory. We only assume the invariance with respect to the Poincaré group; and only requiring the conservation of the linear and angular momenta, we derive the *zitterbewegung*, namely the decomposition of the four-velocity in the usual Newtonian constant term p^μ/m and in a non-Newtonian time-oscillating spacelike term. Consequently, free classical particles do not obey, in general, the Principle of Inertia. Superluminal motions are also allowed, without violating special relativity, provided that the energy–momentum moves along the worldline of the center-of-mass. Moreover, a nonlinear, nonconstant relation holds between the time durations measured in different reference frames. Newtonian mechanics is reobtained as a particular case of the present theory: namely for spinless systems with no zitterbewegung. Then we analyze the strict analogy between the classical zitterbewegung equation and the quantum Gordon-decomposition of the Dirac current. It is possible a variational formulation of the theory, through a Lagrangian containing also derivatives of the four-velocity: we get an equation of the motion, actually a generalization of the Newton law $a = F/m$, where non-Newtonian zitterbewegung-terms appear. Requiring the rotational symmetry and the reparametrization invariance we derive the classical spin vector and the conserved scalar Hamiltonian, respectively. We derive also the classical Dirac spin $(\boldsymbol{a} \times \boldsymbol{v})/4m$ and analyze the general solution of the Eulero–Lagrange equation oscillating with the Compton frequency $\omega = 2m$. The interesting case of *spinning* systems with zero intrinsic angular momentum is also studied.

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"If a spinning particle is not quite a point particle, nor a solid three-dimensional top, what can it be?"

Asim O. Barut

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1. Classical Non-Newtonian Systems

1.1. *Introduction*

The theory we are going to put forward concerns *classical systems* (CS's) in the most general meaning of the word, namely *nonquantum systems*. As special relativity allows, a CS can own "internal" degrees of freedom and a spin angular momentum. The set of CS's contains as a special subset, the one of *spinless systems*, that we shall call also *Newtonian systems* (NS's), e.g. macroscopic bodies studied in Newtonian classical mechanics. Since the works by Compton,[1] Uhlenbeck and Goudsmith,[2] and Frenkel,[3] many classical theories of spinning particles have been investigated for about eighty years.[4] Grassmann variables in classical actions for spinning systems have been employed by Berezin and Marinov,[5] Ikemori[6] and Casalbuoni.[7] In the last twenty years, a renewed interest has arisen towards classical theoretical approaches to microsystems, especially in applications to (super)strings and membranes, in view of a possible unification of the elementary forces of Nature. In this section we shall obtain important properties, constraints and equations which rule the kinematics of free classical particles endowed with spin, without any recourse to *ad hoc* theories or additional assumptions besides the requirements of the usual space–time symmetries, and without any recourse to the noncommuting numbers of the Grassmann algebra or to the multivectors of the Clifford "space–time" algebra.[8]

In the absence of external fields the space–time isotropy implies the conservation of the total angular momentum:

$$\dot{J}^{\mu\nu} = \dot{L}^{\mu\nu} + \dot{S}^{\mu\nu} = 0\,, \tag{1}$$

tensor $L^{\mu\nu} \equiv x^\mu p^\nu - x^\mu p^\nu$ being the orbital angular momentum, and tensor $S^{\mu\nu}$ the spin angular momentum. The derivation is taken with respect to the proper time τ, defined as *the time measured in the center-of-mass frame* (CMF) where, by definition, the three-momentum vanishes, $\boldsymbol{p} = 0$. The adopted metric is $(+;-,-,-)$. The symmetry under space–time translations involves the conservation of the four-momentum $p^\mu \equiv (p^0;\boldsymbol{p})$:

$$\dot{p}^\mu = 0\,. \tag{2}$$

We want to stress that the above conservation laws will be sufficient to derive all the equations and constraints of the motion. The consequent theory will be the most general one and will not be a result of a particular theoretical model adopted. Hereafter we shall choose units such that $c = 1$. Being $L^{\mu\nu} \equiv x^\mu p^\nu - x^\nu p^\mu$, from (1) and (2) we have

$$\dot{S}^{\mu\nu} = -\dot{L}^{\mu\nu} = p^\mu v^\nu - p^\nu v^\mu\,, \tag{3}$$

where, as usual, the four-velocity is defined as the proper-time derivative of the space–time coordinate:

$$v^\mu \equiv \dot{x}^\mu \equiv \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}; \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau}\right). \tag{4}$$

We are not forced, because of mathematical or physical reasons, to assume *a priori* that the CMF, where $\boldsymbol{p} = 0$, must coincide with the reference system where $\boldsymbol{v} \equiv \mathrm{d}\boldsymbol{x}/\mathrm{d}\tau = 0$, namely the *rest frame* (RF), where by definition the speed vanishes. Then, except particular initial or boundary conditions, in general we can write

$$\boldsymbol{v}_{\mathrm{CMF}} \equiv \left.\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau}\right|_{\mathrm{CMF}} \neq 0\,. \tag{5}$$

On the other hand the above statement agrees with the physical structure of quantum probability currents and quantum velocity operators for spinning systems, as we shall later see in Subsec. 1.4. The Lorentz invariant $v^2 = {v^0}^2 - \boldsymbol{v}^2$ can be evaluated in the CMF, where $v^0_{\mathrm{CMF}} = \mathrm{d}\tau/\mathrm{d}\tau = 1$ identically (hereafter whichever quantity referred to the CMF will be labeled by $\star$):

$$v^2 = 1 - \boldsymbol{v}_\star^2\,; \tag{6}$$

from which, taking in account Eq. (5),

$$v^2 \neq 1\,. \tag{7}$$

It follows that v^μ cannot be put in the usual form

$$v^\mu \neq \left(\frac{1}{\sqrt{1-w^2}}; \frac{\boldsymbol{w}}{\sqrt{1-w^2}}\right).$$

Furthermore, v^2 is not, *a priori*, required to be a time-constant quantity.

Let us write down the two basic invariant constraints, found also in classical theory of spinless systems:

$$p^2 = m^2\,; \tag{8}$$

$$p_\mu v^\mu = m\,. \tag{9}$$

The first constraint — which expresses the conservation of p^μ given by Eq. (2) — implies the second one. In fact the relativistic invariant $p_\mu v^\mu \equiv p_0 v^0 - \boldsymbol{p}\cdot\boldsymbol{v}$ is nothing but the energy in the CMF $p^0_\star$, because in the CMF $v^0_\star = 1$ and $\boldsymbol{p}_\star = 0$ by definition: $p_\mu v^\mu = p^0_\star$. Because of (8) $\boldsymbol{p}_\star = 0$ implies $p^0_\star = \pm m$ ($m \equiv \sqrt{p^2} > 0$); and, if we choose the positive sign for the CMF energy, $p_\mu v^\mu$ results to be equal just to m.

Before going on, we want to remark that the wide generality and novelty of the results we shall obtain is due in particular to our assumption that *the proper time is the time elapsed in the CMF and not in the RF*, and then that $v^2 \neq 1$. By contrast, in the literature (with some exception as the Barut–Zanghi model[9]) the proper time τ is defined as the RF time, $\tau \equiv t_{\mathrm{RF}}$. In the latter case we have as usual $v^2 = 1$ in any frame, since $v^2 \equiv (\mathrm{d}t/\mathrm{d}\tau)^2 - (\mathrm{d}\boldsymbol{x}/\mathrm{d}\tau)^2$ is actually equal to 1 in the RF, where by definition $\mathrm{d}\boldsymbol{x}/\mathrm{d}\tau = 0$. Notice also that the above deduction of (9) from (8) does not hold anymore with such a definition of τ, since now $v^0_\star \equiv \mathrm{d}t_\star/\mathrm{d}\tau \neq 1$, and then

$$p_\mu v^\mu \equiv m v^0_\star \equiv \mu \neq m\,. \tag{10}$$

Notwithstanding, in literature both constraints $p_\mu v^\mu = m$ and $v^2 = 1$, mutually excluding for particles endowed with spin, are simultaneously assumed.

1.2. *Zitterbewegung*

Let us come back to our proper time approach, $\tau \equiv t_{\rm CMF}$. By multiplying both sides of (3) times p_ν and exploiting conditions (8) and (9), we derive[a]

$$\boxed{v^\mu = \frac{p^\mu}{m} - \frac{\dot{S}^{\mu\nu} p_\nu}{m^2}}\,. \tag{11}$$

The above equation can be rewritten also in terms of orbital angular momentum rather than of spin tensor:

$$v^\mu = \frac{p^\mu}{m} + \frac{\dot{L}^{\mu\nu} p_\nu}{m^2}\,. \tag{12}$$

The peculiar occurrence that in general the velocity is not constant and not parallel to the momentum, is the so-called *zitterbewegung*.[12–14] We then shall call Eq. (11) *zitterbewegung equation for a free particle*. The global velocity contains a "translational," "Newtonian," time-constant component p^μ/m related to the motion of the CM; and a "rotational," "non-Newtonian," time-varying component related to the presence of the spin. As a consequence the RF, where $\boldsymbol{v} = 0$, and the CMF, where $\boldsymbol{p} = 0$, in general do not coincide. In particular, the presence of zitterbewegung implies a motion even in the CMF, and then in the *nonrelativistic limit*: in fact, for $\boldsymbol{p} \to 0$ we have $\dot{S}^{ik} \to 0$ (since the spin three-vector conserves in nonrelativistic mechanics) but $\dot{S}^{i0} \not\to 0$ (S^{i0} is not required to conserve), so that from Eq. (11) we have $v^i \to -\dot{S}^{i0}_\star/m \neq 0$.

From Eq. (11) it follows that in a generic frame the trajectory will be a helix around the constant direction of $\boldsymbol{p}$. Notice that, because identically $\dot{S}^{\mu\nu} p_\nu p_\mu = 0$ (the contraction of an antisymmetric tensor with a symmetric tensor always vanishes), *the spin term in the global velocity results always spacelike and orthogonal to the timelike Newtonian component p^μ/m.* This property recalls the known dispersion relation $|\boldsymbol{v}|\,|\boldsymbol{V}| = c^2$ between the timelike (external) group-speed and the spacelike (internal) phase-speed found, for instance, in de Broglie's "pilot-wave" ("double-solution") theory. From (11) we see that the *zitterbewegung originates from the nonconservation of the orbital angular momentum and of the spin angular momentum* (even if their sum conserves)

$$\dot{S} \neq 0\,, \qquad \dot{L} \neq 0\,. \tag{13}$$

Let us underline that in some papers $S^{\mu\nu} p_\nu = 0$ is arbitrarily assumed, so that $\dot{S}^{\mu\nu} p_\nu = 0$: i.e. for (11), no zitterbewegung, in spite of the presence of spin. Here

[a] Another zitterbewegung equation similar, but *not equivalent* to (11), is the well-known *Corben–Papapetrou equation*[10,11] $v^\mu = p^\mu/\mu - \dot{S}^{\mu\nu} v_\nu/\mu$, where $\mu \equiv p_\mu v^\mu \neq m$. We have also to take into account that in Corben's theory any derivative is taken with respect to the RF time, and not to the CMF time, as in the present approach. Notice also that, by contrast with (11), in the Corben–Papapetrou equation the zitterbewegung term is *not* in general orthogonal to the Newtonian term p^μ/m. Obviously, for spinless NS's which do not show zitterbewegung, $\dot{L}^{\mu\nu} = \dot{S}^{\mu\nu} = 0$, both equations of the motion reduce to the usual Newtonian relation $v^\mu = p^\mu/m$, with $v^2 = 1$.

we will not in any way limit the generality of the theory and shall not make further assumptions: And in Sec. 4 we shall see that the motion of classical Dirac particles undergoes zitterbewegung with $S^{\mu\nu}p_\nu = a^\mu/4 \neq 0$, see Eq. (80). From the derivation of both sides of Eq. (11) we get

$$a^\mu = -\frac{\ddot{S}^{\mu\nu}p_\nu}{m^2}\,, \tag{14}$$

or also

$$a^\mu = \frac{\ddot{L}^{\mu\nu}p_\nu}{m^2}\,. \tag{15}$$

Therefore, while for NS's $a^\mu = 0$ in the absence of external forces, for CS's in general $a^\mu \neq 0$ so that *the Galileo–Newton Principle of Inertia does not hold anymore.*

From (3) we obtain

$$\dot{S}^{0i} = p^0 v^i - p^i v^0\,, \quad (i = 1, 2, 3)$$

and then

$$v^i = \frac{p^i v^0}{p^0} + \frac{\dot{S}^{0i}}{p^0}\,.$$

Dividing both sides for v^0, and taking into account (4) and the identity

$$\frac{v^i}{v^0} \equiv \left(\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^{-1} \equiv \frac{\mathrm{d}x^i}{\mathrm{d}t}\,,$$

we get

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{p^i}{p^0} + u^i\,, \tag{16}$$

where $u^i \equiv \dot{S}^{0i}/v^0 p^0$. Notice that, whilst the speed $|\boldsymbol{p}|/p^0$ of the CM is always smaller than the speed of light in vacuum c, the zitterbewegung speed $|\boldsymbol{u}|$ is not constrained at all (see below). Therefore, *without violating special relativity*, *we can allow superluminal motions* of spinning charges, provided that signals and momenta are carried by the CM (it follows also that the reference systems, as expected, are endowed with subluminal relative speeds).

1.3. *General properties of the inertial motion*

1.3.1. *Constraints on v^2 and motions in the CMF*

Let us write the zitterbewegung equation (11) in a compact form

$$v^\mu = w^\mu + V^\mu\,, \tag{17}$$

where $w^\mu \equiv (1/\sqrt{1-w^2}; \boldsymbol{w}/\sqrt{1-w^2}) \equiv p^\mu/m$ and $V^\mu \equiv -\dot{S}^{\mu\nu}p_\nu/m^2$.[b] Since, as was already seen,

$$w_\mu V^\mu = -\dot{S}^{\mu\nu}p_\nu p_\mu/m^3 = 0 \tag{18}$$

(four-orthogonality between Newtonian and non-Newtonian terms in v^μ) and $w^2 = p^2/m^2 = 1 > 0$, we have

$$V^2 \leq 0\,. \tag{19}$$

[On the other hand we have identically $v^0_\star = w^0_\star = 1$ which for (17) implies

$$V^0_\star = 0\,, \tag{20}$$

which in its turn involves just $V^2 = 0 - \boldsymbol{V}^2_\star \leq 0$]. Because of (18), (19), and of the decomposition $v^2 = w^2 + V^2 + 2w_\mu V^\mu = 1 + V^2$, the following constraint holds:

$$-\infty < v^2 \leq 1\,. \tag{21}$$

If, at a given time, $0 < v^2 = 1 - \boldsymbol{v}^2_\star < 1$ (*timelike* case), the corresponding motion is subluminal in the CMF, in that $\boldsymbol{v}^2_\star < 1$. If instead we have $v^2 < 0$ (*spacelike* case) the motion is superluminal, "tachyonic," since $\boldsymbol{v}^2_\star > 1$. In the special *lightlike* case, $v^2 = 0$, we have $\boldsymbol{v}^2_\star = 1$, and the charge moves in the CMF at the speed of light c.[c]

By a little algebra we can also obtain the following relations:

$$v^2 = 1 + \frac{\dot{S}^{\mu\nu}\dot{S}_{\mu\nu}}{2m^2}\,, \tag{22}$$

$$v^2 = 1 + \frac{\dot{S}^{\mu\nu}p_\mu v_\nu}{m^2}\,. \tag{23}$$

1.3.2. *"Longitudinal" and "intrinsic" zitterbewegung*

Equation (18) implies $\boldsymbol{p}\cdot\boldsymbol{V} - p_0V^0 = 0$; then we have orthogonality between $\boldsymbol{p}$ and $\boldsymbol{V}$, $\boldsymbol{p}\cdot\boldsymbol{V} = 0$, only in all those frames where

$$V^0 = 0\,.$$

The reference frames where the time-component of a given spacelike four-vector A^μ vanishes are named "standard frames for A^μ".[15] Therefore we have $\boldsymbol{p}\cdot\boldsymbol{V} = 0$ only in the standard frames for V^μ, among which the CMF is a particular case [since $V^0_\star = 0$, Eq. (20)]. In these reference systems it holds a particular case of the general

[b]Even if endowed with different transformation properties, the space parts of four-vectors (as, e.g. $\boldsymbol{v}$, $\boldsymbol{p}$, $\boldsymbol{x}$) and the three-vectors (as, e.g. $\boldsymbol{w}$ or the spin vector $\boldsymbol{s}$) will be for convenience labeled by means of boldface symbols.

[c]Let us suppose that for all the massive elementary (not composed) particles, i.e. for electrons and quarks, it *always* is $v^2 = 0$. We might therefore state that c must not be meant as the *maximum* speed, but actually as the *unique* speed of any pointlike charge. In such a way eachever subluminal speed observed — energy and momentum travel at a slower-than-light speed w — is to be realized as the component parallel to the momentum of the total velocity (obviously smaller than the modulus c of the total velocity).

constraint $p_\mu a^\mu = 0$ [obtained by time-derivating side by side Eq. (9)], namely: $\dot{V}^0 = a^0 = \boldsymbol{p} \cdot \boldsymbol{a} = 0$. The considered frames can be obtained by applying to the CMF a Lorentz boost $\boldsymbol{w}$ orthogonal to the zitterbewegung plane, $\boldsymbol{w} \cdot \boldsymbol{V}_\star = 0$. In such a way, as it is easy to see, the spacelike four-vector V^μ, equal in the CMF to $(0; \boldsymbol{V}_\star)$, will transform in itself (namely, it is an eigenvector of the matrix operating the considered Lorentz transformation), so that V^0 still vanishes.[d] In Sec. 4, in studying the classical Dirac theory, we shall see that the standard frames, in which we observe a pure transverse ($\boldsymbol{p} \cdot \boldsymbol{V} = 0$) zitterbewegung, are the frames in which it appears as a polarized particle, i.e. with the spin aligned along the momentum. By contrast, *in a generic frame we have also a component of the zitterbewegung parallel to the momentum, with* (cf. Sec. 4) *oscillations of the charge along the rectilinear trajectory of the* CM ("longitudinal" or "extrinsic" zitterbewegung), besides the oscillations transverse to the momentum ("transverse" or "intrinsic" zitterbewegung).

1.3.3. *Nonconstant times-ratio*

The quantity $v^0 = w^0 + V^0 = \mathrm{d}t/\mathrm{d}\tau$ may be defined as "times-ratio," in that it measures the ratio between the time durations referred to a generic reference system ($\mathrm{d}t$) and to the CMF ($\mathrm{d}\tau$). It generalizes the concept of Lorentz factor in the present theory. But, whilst for NS's $v^0 = w^0 = \gamma \equiv 1/\sqrt{1 - w^2}$ is always a constant quantity due to (2), for generic CS's $v^0 \neq 1/\sqrt{1 - w^2}$ is time-varying (and in particular time-oscillating, cf. Sec. 4) since V^0 is not forced to be a constant quantity. Thus, *the times-ratio is not time-constant* anymore, as it instead occurs in special relativity for spinless NS's. In a sense, we might speak of a *nonconstant Lorentz factor*. Moreover, the times-ratio is not necessarily larger than 1: we may also have a time-contraction, besides the usual time-dilation, see Sec. 4. (By contrast, in the standard frames for V^0 we have $V^0 = 0$, and then the times-ratio turns out to be the usual constant quantity γ[e]). In general it is easy to see that *a nonlinear relation occurs between the time durations measured in different reference systems* (see Sec. 4).

All four components of v^μ not being constant, we may also say that the trajectory is a helix not only in the ordinary space R^3 but also in the Minkowski space–time M^4, spiralizing around the constant four-vector p^μ.

1.4. *Correspondences between the classical velocity and the quantum probability current*

The most impressive correspondence between the present classical theory and the standard wave-mechanics may be found in the celebrated *Gordon decomposition* of

[d] By contrast, any Lorentz boost changes the time-component of a generic *timelike* four-vector.
[e] This result can be alternatively derived by considering Eq. (9) $p_\mu v^\mu = m$. In fact, in the standard frames we have $\boldsymbol{p} \cdot \boldsymbol{V} = 0$, which, for (17), implies $\boldsymbol{p} \cdot \boldsymbol{v} = \boldsymbol{p} \cdot \boldsymbol{w}/\sqrt{1 - w^2}$. It follows that $p_\mu v^\mu \equiv p_0 v^0 - \boldsymbol{p} \cdot \boldsymbol{v} = p_0 v^0 - \boldsymbol{p} \cdot \boldsymbol{w}/\sqrt{1 - w^2} = m$, from which (exploiting also $p^\mu \equiv m w^\mu$) we get the constant times-ratio $v^0 = 1/\sqrt{1 - w^2}$.

the conserved current of the Dirac equation,[16] which writes (hereafter we assume $\hbar = 1$):

$$j^\mu = \bar{\psi}\gamma^\mu\psi = \frac{1}{2m}[\bar{\psi}(\hat{p}^\mu\psi) - (\hat{p}^\mu\bar{\psi})\psi] + \frac{1}{m}\partial_\nu(\bar{\psi}\hat{S}^{\mu\nu}\psi)\,, \tag{24}$$

where $\bar{\psi} \equiv \psi^\dagger\gamma^0$ is the usual Hermitian-adjoint bispinor, $\hat{p}^\mu \equiv i\partial^\mu$ is the four-dimensional momentum operator, and $\hat{S}^{\mu\nu} \equiv i(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)/4$ represents the spin-tensor operator. In fact, the standard interpretation of the above decomposition quite agrees with our zitterbewegung equation (11). The first term in the r.h.s. is associated with the translational motion of the CM (the scalar part of the current, corresponding to the Klein–Gordon current). As a matter of fact, for a momentum eigenstate, i.e. for a plane-wave, this term turns out to be proportional to p^μ/m. By contrast, the non-Newtonian term in the r.h.s. is related to the existence of the spin, and describes the zitterbewegung rotational motion. The correspondences and analogies between classical and quantum laws do not concern only the probability current, but concern also the operators of the basic kinematic quantities. In Dirac theory, indeed, both the four-velocity operator γ^μ and the three-velocity operator $\boldsymbol{\alpha} \equiv \gamma^0\boldsymbol{\gamma}$ do not commute with the Dirac Hamiltonian $\hat{H} = \boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}} + m\gamma^0$. Therefore, such as it happens for CS's, also in quantum mechanics those quantities, differently from the momentum, are *not* time-constant. Let us recall that the zitterbewegung actually occurs also for nonrelativistic particles, in the framework of the Pauli and Schrödinger theories.[8,13,17] In fact, following Landau,[18] we can write a nonrelativistic Gordon-like decomposition of the conserved Pauli current

$$\boldsymbol{j} = \frac{i}{2m}[(\boldsymbol{\nabla}\psi^\dagger)\psi - \psi^\dagger\boldsymbol{\nabla}\psi] + \frac{1}{m}\boldsymbol{\nabla}\times(\psi^\dagger\boldsymbol{\sigma}\psi)\,, \tag{25}$$

where ψ is a Pauli two-component spinor and $\boldsymbol{\sigma}$ is the usual Pauli vector (2×2) matrix. Also the above current appears as a sum of a Newtonian part which, at the classical limit ($\hbar \to 0$), is parallel to the classical momentum (equal to $\hbar$ times the gradient of the action); and of a non-Newtonian part due to the spin, which vanishes only at the classical limit, i.e. for spinless bodies, *but not in the nonrelativistic limit*, i.e. for a small momentum $\boldsymbol{p}$.

Analogous Gordon-like decompositions of the conserved four-currents can be found also for spin-1 bosons and for spin-$\frac{3}{2}$ fermions in the Proca and Rarita–Schwinger theories, respectively.

2. Lagrangian Theory

2.1. *Generalized Newton equation*

A Lagrangian for a free spinless NS, which must be invariant under the Lorentz group as well as under the space and time inversions, can depend only on the four-velocity squared and is often written in the following form[19] (hereafter a generic four-vector a^μ is for simplicity indicated only by a, and a scalar product $a_\mu b^\mu$ by ab)

$$\mathcal{L} = \frac{1}{2}mv^2\,. \tag{26}$$

Searching the minimum of the action $\mathcal{S} = \int \mathcal{L}\, \mathrm{d}\tau$ leads to the Eulero–Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial \dot{\mathcal{L}}}{\partial \dot{x}}\,,$$

which implies the usual rectilinear uniform motion of free Newtonian bodies

$$a = 0\,. \tag{27}$$

The corresponding momentum is

$$p \equiv \frac{\partial \mathcal{L}}{\partial \dot{x}} = mv\,, \tag{28}$$

with — as expected for NS's — no zitterbewegung. By contrast, the general inertial motion of a CS is endowed with zitterbewegung and nonconstant velocity. Actually, *a priori*, the derivatives of the velocity do not vanish. By requiring the symmetry under the Poincaré group, the Lagrangian can depend, besides on the four-velocity squared, also on its (any order) derivatives squared. Therefore a Lagrangian for a CS may be taken as follows:

$$\mathcal{L} \equiv \frac{1}{2}\, mv^2 + \frac{1}{2} k_1 \dot{v}^2 + \frac{1}{2} k_2 \ddot{v}^2 + \cdots \equiv \sum_{i=0}^{\infty} \frac{1}{2} k_i {v^{(\mathrm{i})}}^2\,, \tag{29}$$

where the k_i are constant scalar coefficients, $k_0 = m$, and

$$v^{(\mathrm{i})} \equiv \frac{\mathrm{d}^{\mathrm{i}} v}{\mathrm{d}\tau^{\mathrm{i}}}\,.$$

As we are going to see, the above Lagrangian will imply the expected four-velocity, the sum of a translational part and of a spin part. In the presence of an external force the generalization can be made as usual, with the introduction of a scalar potential $U(x)$

$$\mathcal{L} \equiv \sum_{i=0}^{\infty} \frac{1}{2} k_i {v^{(\mathrm{i})}}^2 - U\,. \tag{30}$$

The coefficients k_i — which may be chosen equal to zero for i larger than a given integer, see below — might be functions of the self-interaction of the particle and of its mass and charge: Let us recall, for comparison, the well-known infinite-terms equation of the self-radiating classical electron or the "cronon" theory of the electron (reviewed at the end of the present section). In other theoretical frameworks, the coefficients k_i can be related to the underlying string structure (or membrane or n-brane structure) of a spinning particle. Polyakov and others[20,21] have proposed a classical string action in which, besides the ordinary Nambu–Goto term, appear additional terms dependent on the so-called "rigidity" or on the so-called "extrinsic curvature": then on the four-acceleration squared. Classical equations of the motion for a rigid n-dimensional world sheet, either in flat or curved background space–times, have been derived from Lagrangians containing also terms dependent on higher derivatives of the four-velocity ("torsion"-terms, etc.).

The Eulero–Lagrange equation for a generic Lagrangian $\mathcal{L}(x, \dot{x}, \ddot{x}, \ldots)$ derived from the Principle of Least Action is

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial \dot{\mathcal{L}}}{\partial \dot{x}} - \frac{\partial \ddot{\mathcal{L}}}{\partial \ddot{x}} + \frac{\partial \dddot{\mathcal{L}}}{\partial \dddot{x}} - \cdots . \tag{31}$$

From Eqs. (30) and (31) we can write the *generalized Newton equation of the motion*:

$$\boxed{-\frac{\partial U}{\partial x} = ma - k_1 \ddot{a} + k_2 \ddddot{a} - \cdots \equiv \sum_{i=0}^{\infty} (-1)^{\mathrm{i}} k_i a^{(2\mathrm{i})}} . \tag{32}$$

The four-momentum p is that quantity which is conserved under four-translations for free systems ($\dot{p} = 0$ if $U = 0$) and whose time derivative is the four-force F:

$$\dot{p} = \frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial U}{\partial x} = F . \tag{33}$$

The requirement of space–time homogeneity

$$\frac{\partial \mathcal{L}}{\partial x} = 0$$

because Eq. (31) implies

$$0 = \frac{\partial \dot{\mathcal{L}}}{\partial \dot{x}} - \frac{\partial \ddot{\mathcal{L}}}{\partial \ddot{x}} + \frac{\partial \dddot{\mathcal{L}}}{\partial \dddot{x}} - \cdots = \frac{\mathrm{d}}{\mathrm{d}\tau} \left[\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \dot{\mathcal{L}}}{\partial \ddot{x}} + \frac{\partial \ddot{\mathcal{L}}}{\partial \dddot{x}} - \cdots \right] .$$

Then the quantity in square brackets is the conserved momentum of a CS:

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \dot{\mathcal{L}}}{\partial \ddot{x}} + \frac{\partial \ddot{\mathcal{L}}}{\partial \dddot{x}} - \cdots \tag{34}$$

which, for Lagrangians (30) can be written as

$$p = mv - k_1 \ddot{v} + k_2 \ddddot{v} - \cdots \equiv \sum_{i=0}^{\infty} (-1)^{\mathrm{i}} k_i v^{(2\mathrm{i})} . \tag{35}$$

The zitterbewegung part of the velocity $v_{\mathrm{zbw}} = v - p/m$ reads:

$$v_{\mathrm{zbw}} = \frac{1}{m}(k_1 \ddot{v} - k_2 \ddddot{v} + \cdots) \equiv -\frac{1}{m} \sum_{i=1}^{\infty} (-1)^{\mathrm{i}} k_i v^{(2\mathrm{i})} . \tag{36}$$

The orbital angular momentum of a CS is the sum of the usual Newtonian term and of a non-Newtonian zitterbewegung term

$$\boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{p} = \boldsymbol{x} \times m\boldsymbol{v} + \sum_{i=1}^{\infty} (-1)^{\mathrm{i}} \boldsymbol{x} \times k_i \boldsymbol{v}^{(2\mathrm{i})} . \tag{37}$$

For free particles ($U = 0$) the generalized Newton equation (32) reduces to

$$0 = ma - k_1 \ddot{a} + k_2 \ddddot{a} - \cdots . \tag{38}$$

As far as we know, only Caldirola's classical theory of the electron — based on the existence of an elementary time duration: the "cronon"[22] — is an infinite-order

Lagrangian of the same kind of (30). In the cronon theory, which applies to charged leptons, the constant coefficients of the derivatives of the velocity are linked to the electrical charge e:

$$k_i \equiv (-1)^i \frac{mT^{2i}}{(2i+1)!}\,, \tag{39}$$

where T is the already mentioned cronon

$$T \equiv \frac{4}{3}\frac{e^2}{mc^3}\,.$$

According to this choice, and assuming $U \equiv eA_\mu v^\mu$, the Eulero–Lagrange equation results to be the following *finite-differences equation* (herafter we come back to the previous asssumption $c = 1$)

$$m\frac{v^\mu(\tau+T) - v^\mu(\tau-T) + v^\mu(\tau)v^\nu(\tau)[v_\nu(\tau+T) - v_\nu(\tau-T)]}{2T} = eF^{\mu\nu}(\tau)v_\nu(\tau)\,,$$

which appears as a (non-Newtonian) time-symmetrical discretization of the Lorentz nonradiating equation for the motion of a (Newtonian) spinless charge

$$m\dot{v}^\mu = eF^{\mu\nu}v_\nu\,.$$

The cronon theory is rather interesting, among other things, for it seems to overcome well-known problems due to the electric self-interaction as the so-called "runaway solutions" of the Lorentz–Dirac equation of the electron. Moreover, Caldirola's theory seems to explain the origin of the "classical (Schwinger's) part," $e\hbar/2mc \cdot \alpha/2\pi = e^3/4\pi mc^2$ of the anomalous magnetic momentum of the electron as well as the mass spectrum of charged leptons. Because of the classical, non-quantum character of his theory, Caldirola excluded *a priori* the existence of spin contributions or zitterbewegung terms in its theory. We instead are going to show, in the next subsection, the arising of a very intrinsic angular momentum for any given $\mathcal{L}$.

For each finite n, we define a nth order Lagrangian as follows:

$$\mathcal{L}^{(n)} \equiv \sum_{i=0}^{n} \frac{1}{2} k_i {v^{(\mathrm{i})}}^2\,, \tag{40}$$

and look for the consequent motions of the system. The generalized Newton equation (38) is now a linear constant-coefficients differential equation of nth order:

$$0 = ma - k_1\ddot{a} + k_2\ddddot{a} - \cdots + (-1)^{\mathrm{n}}k_n a^{(2\mathrm{n})}\,.$$

The associated "characteristic equation" is

$$0 = m - k_1 z^2 + k_2 z^4 - \cdots + (-1)^{\mathrm{n}} k_n z^{2n}\,.$$

If we forbid exponentially spreading or collapsing motions, but, on the contrary, we ask finite, periodic zitterbewegung motions (around the uniform translation of the CM) the signs of the coefficients k_i must be alternate. In fact, to have only oscillating motions, each solution ${z_i}^2$ $(i = 1, \ldots, n)$ of the characteristic equation must satisfy

$$z_i^2 < 0\,,$$

since $z_i^2 = -\omega_i^2$, where ω_i is the ith frequency of the motion. Therefore, because of the Descartes rule, we have to ask

$$\mathrm{sign}(k_i) = (-1)^{\mathrm{i}}\,. \tag{41}$$

Notice that the above condition is satisfied in the cronon theory, Eq. (39), as well as in our future applications: see, e.g. Secs. 3 and 4.

2.2. *Classical spin*

Any Poincaré-invariant Lagrangian $\mathcal{L}$[f] is also invariant under the four-rotations group. Then, for the Nöther theorem, the angular momentum tensor $J^{\mu\nu}$ is conserved. For the deduction of $J^{\mu\nu}$, we firstly work supposing only that our $\mathcal{L}$ be a Poincaré-invariant function of x and of its time derivatives

$$\mathcal{L} \equiv f(\tau; x, v, \dot{v}, \ddot{v}, \ldots)\,, \tag{42}$$

and without recourse to the more specific form given by (30). We then ask that the Lagrangian after an infinitesimal four-rotation of the reference frame does not vary:

$$0 = \delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial x}\delta x + \frac{\partial\mathcal{L}}{\partial v}\delta v + \frac{\partial\mathcal{L}}{\partial \dot{v}}\delta \dot{v} + \frac{\partial\mathcal{L}}{\partial \ddot{v}}\delta \ddot{v} + \frac{\partial\mathcal{L}}{\partial \dddot{v}}\delta \dddot{v} + \cdots\,. \tag{43}$$

If $\delta\Omega^{\mu\nu}$ is the antisymmetric four-tensor giving the infinitesimal rotation angles around an axis orthogonal to the plane $x^\mu x^\nu$ we shall have

$$\begin{aligned} 0 = \delta\mathcal{L} = {} & \frac{\partial\mathcal{L}}{\partial x^\mu}\delta\Omega^{\mu\nu}x_\nu + \frac{\partial\mathcal{L}}{\partial v^\mu}\delta\Omega^{\mu\nu}v_\nu + \frac{\partial\mathcal{L}}{\partial \dot{v}^\mu}\delta\Omega^{\mu\nu}\dot{v}_\nu \\ & + \frac{\partial\mathcal{L}}{\partial \ddot{v}^\mu}\delta\Omega^{\mu\nu}\ddot{v}_\nu + \frac{\partial\mathcal{L}}{\partial \dddot{v}^\mu}\delta\Omega^{\mu\nu}\dddot{v}_\nu + \cdots\,. \end{aligned} \tag{44}$$

If we consider the first-order Lagrangian $\mathcal{L}(\tau; x, v)$, i.e. the usual Newtonian Lagrangian, we can write, being $p_\mu = \partial\mathcal{L}/\partial v^\mu$ and $\partial\mathcal{L}/\partial x^\mu = \dot{p}_\mu$,

$$\begin{aligned} 0 = \delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial x^\mu}\delta\Omega^{\mu\nu}x_\nu + \frac{\partial\mathcal{L}}{\partial v^\mu}\delta\Omega^{\mu\nu}v_\nu = \dot{p}_\mu\delta\Omega^{\mu\nu}x_\nu + p_\mu\delta\Omega^{\mu\nu}v_\nu \\ &= \frac{1}{2}\delta\Omega^{\mu\nu}[(\dot{p}_\mu x_\nu - \dot{p}_\nu x_\mu) + (p_\mu v_\nu - p_\nu v_\mu)] = \frac{1}{2}\delta\Omega^{\mu\nu}\frac{\mathrm{d}}{\mathrm{d}\tau}(p_\mu x_\nu - p_\nu x_\mu)\,, \end{aligned}$$

where we have exploited the antisymmetry of $\Omega^{\mu\nu}$ and the derivation rule

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(fg) = \dot{f}g + f\dot{g}\,. \tag{45}$$

[f]Let us underline that each result obtained in this subsection holds not only for free systems, but equally in the presence of an external *scalar* potential U.

For the arbitrariness of $\delta\Omega^{\mu\nu}$ we eventually obtain the conservation of the total angular momentum

$$\frac{\mathrm{d}J^{\mu\nu}}{\mathrm{d}\tau} = 0\,,$$

with

$$J^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu\,. \tag{46}$$

We can work analogously for Lagrangians of the second order $\mathcal{L}(\tau; x, v, \dot{v})$:

$$0 = \delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial x^\mu}\delta\Omega^{\mu\nu}x_\nu + \frac{\partial\mathcal{L}}{\partial v^\mu}\delta\Omega^{\mu\nu}v_\nu + \frac{\partial\mathcal{L}}{\partial\dot{v}^\mu}\delta\Omega^{\mu\nu}\dot{v}_\nu\,.$$

Being Eq. (34), $p_\mu = \partial\mathcal{L}/\partial v^\mu - \mathrm{d}(\partial\mathcal{L}/\partial a^\mu)/\mathrm{d}\tau$ and $\partial\mathcal{L}/\partial x^\mu = \dot{p}_\mu$ we can also write

$$\begin{aligned}
0 &= \dot{p}_\mu\delta\Omega^{\mu\nu}x_\nu + \left(p_\mu + \frac{\partial\dot{\mathcal{L}}}{\partial a^\mu}\right)\delta\Omega^{\mu\nu}v_\nu + \frac{\partial\mathcal{L}}{\partial a^\mu}\delta\Omega^{\mu\nu}a_\nu \\
&= \frac{1}{2}\delta\Omega^{\mu\nu}\Bigg[(\dot{p}_\mu x_\nu - \dot{p}_\nu x_\mu) + (p_\mu v_\nu - p_\nu v_\mu) \\
&\quad + \left(\frac{\partial\dot{\mathcal{L}}}{\partial a^\mu}v_\nu - \frac{\partial\dot{\mathcal{L}}}{\partial a^\nu}v_\mu\right) + \left(\frac{\partial\mathcal{L}}{\partial a^\mu}a_\nu - \frac{\partial\mathcal{L}}{\partial a^\nu}a_\mu\right)\Bigg] \\
&= \frac{1}{2}\Omega^{\mu\nu}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[(p_\mu x_\nu - p_\nu x_\mu) + \left(\frac{\partial\mathcal{L}}{\partial a^\mu}v_\nu - \frac{\partial\mathcal{L}}{\partial a^\nu}v_\mu\right)\right].
\end{aligned}$$

We again obtain the conservation of the total angular momentum

$$\frac{\mathrm{d}J^{\mu\nu}}{\mathrm{d}\tau} = 0\,,$$

with

$$J^{\mu\nu} = [x^\mu p^\nu - x^\nu p^\mu] + \left[v^\mu\frac{\partial\mathcal{L}}{\partial a_\nu} - v^\nu\frac{\partial\mathcal{L}}{\partial a_\mu}\right]. \tag{47}$$

Let us consider also the Lagrangians of third and fourth order: $\mathcal{L}(\tau; x, v, \dot{v}, \ddot{v})$, $\mathcal{L}(\tau; x, v, \dot{v}, \ddot{v}, \dddot{v})$. By some algebra, exploiting Eq. (33) and the expression of the canonical momentum (34), as well as the identities

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(\dot{f}g - f\dot{g}) = \ddot{f}g - f\ddot{g}\,, \quad \frac{\mathrm{d}}{\mathrm{d}\tau}(\ddot{f}g - \dot{f}\dot{g} + f\ddot{g}) = \dddot{f}g + f\dddot{g}\,, \tag{48}$$

from (44) we get the following conserved total angular momentum

$$\begin{aligned}
J^{\mu\nu} &= [x^\mu p^\nu - x^\nu p^\mu] + \left[v^\mu\frac{\partial\mathcal{L}}{\partial a_\nu} - v^\nu\frac{\partial\mathcal{L}}{\partial a_\mu}\right] \\
&\quad + \left[\left(a^\mu\frac{\partial\mathcal{L}}{\partial\dot{a}_\nu} - a^\nu\frac{\partial\mathcal{L}}{\partial\dot{a}_\mu}\right) - \left(v^\mu\frac{\partial\dot{\mathcal{L}}}{\partial\dot{a}_\nu} - v^\nu\frac{\partial\dot{\mathcal{L}}}{\partial\dot{a}_\mu}\right)\right]
\end{aligned} \tag{49}$$

for $\mathcal{L}(\tau; x, v, \dot{v}, \ddot{v})$; and

$$J^{\mu\nu} = [x^\mu p^\nu - x^\nu p^\mu] + \left[v^\mu \frac{\partial\mathcal{L}}{\partial a_\nu} - v^\nu \frac{\partial\mathcal{L}}{\partial a_\mu}\right] + \left[\left(a^\mu \frac{\partial\mathcal{L}}{\partial \dot{a}_\nu} - a^\nu \frac{\partial\mathcal{L}}{\partial \dot{a}_\mu}\right) - \left(v^\mu \frac{\partial\dot{\mathcal{L}}}{\partial \dot{a}_\nu} - v^\nu \frac{\partial\dot{\mathcal{L}}}{\partial \dot{a}_\mu}\right)\right] + \left[\left(v^\mu \frac{\partial\ddot{\mathcal{L}}}{\partial \ddot{a}_\nu} - v^\nu \frac{\partial\dot{\mathcal{L}}}{\partial \dot{a}_\mu}\right) - \left(a^\mu \frac{\partial\dot{\mathcal{L}}}{\partial \ddot{a}_\nu} - a^\nu \frac{\partial\dot{\mathcal{L}}}{\partial \ddot{a}_\mu}\right) + \left(\dot{a}^\mu \frac{\partial\mathcal{L}}{\partial \ddot{a}_\nu} - \dot{a}^\nu \frac{\partial\mathcal{L}}{\partial \ddot{a}_\mu}\right)\right] \tag{50}$$

for $\mathcal{L}(\tau; x, v, \dot{v}, \ddot{v}, \dddot{v})$. Thus for $\mathcal{L}(\tau; x, v)$ we have no spin, $J_{\mu\nu} = L_{\mu\nu}$, $S_{\mu\nu} = 0$, as expected for NS's; the first spin term appears for $\mathcal{L}(\tau; x, v, \dot{v})$ where we have for (47)

$$\boxed{S^{\mu\nu} = v^\mu \frac{\partial\mathcal{L}}{\partial a_\nu} - v^\nu \frac{\partial\mathcal{L}}{\partial a_\mu}}. \tag{51}$$

A more specific form of the spin vector $\boldsymbol{s} \equiv (S^{23}, S^{31}, S^{12})$ can be found for the Lagrangians $\mathcal{L}^{(n)}$ given by (40). We get

$$\boxed{\boldsymbol{s} = k_1(\boldsymbol{v} \times \boldsymbol{a})} \tag{52}$$

for $\mathcal{L}^{(1)}$;

$$\boxed{\boldsymbol{s} = k_1(\boldsymbol{v} \times \boldsymbol{a}) + k_2(\boldsymbol{a} \times \dot{\boldsymbol{a}} - \boldsymbol{v} \times \ddot{\boldsymbol{a}})} \tag{53}$$

for $\mathcal{L}^{(2)}$; and

$$\boxed{\boldsymbol{s} = k_1(\boldsymbol{v} \times \boldsymbol{a}) + k_2(\boldsymbol{a} \times \dot{\boldsymbol{a}} - \boldsymbol{v} \times \ddot{\boldsymbol{a}}) + k_3(\dot{\boldsymbol{a}} \times \ddot{\boldsymbol{a}} - \boldsymbol{a} \times \ddot{\boldsymbol{a}} + \boldsymbol{v} \times \dddot{\boldsymbol{a}})} \tag{54}$$

for $\mathcal{L}^{(3)}$. And so on for larger n.

2.3. *Hamiltonian*

Lagrangian (42) describes free systems: it cannot explicitly depend on the proper-time τ (reparametrization invariance); actually, the Lagrangians $\mathcal{L}^{(n)}$ are not explicit functions of the time parameter τ. Also in the presence of external forces we generally suppose that the potential U does not depend on the proper time. Therefore, because of the Nöther theorem, we can always get out a conserved scalar Hamiltonian. Let us write the total time derivative $[(\partial\mathcal{L}/\partial a)b \equiv (\partial\mathcal{L}/\partial a^\mu)b^\mu$ as before$]$

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\tau} = \frac{\partial\mathcal{L}}{\partial\tau} + \frac{\partial\mathcal{L}}{\partial x}\dot{x} + \frac{\partial\mathcal{L}}{\partial v}\dot{v} + \frac{\partial\mathcal{L}}{\partial a}\dot{a} + \frac{\partial\mathcal{L}}{\partial \dot{a}}\ddot{a} + \cdots . \tag{55}$$

Imposing the reparametrization invariance $\partial\mathcal{L}/\partial\tau = 0$, we get

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\tau} = \frac{\partial\mathcal{L}}{\partial x}\dot{x} + \frac{\partial\mathcal{L}}{\partial v}\dot{v} + \frac{\partial\mathcal{L}}{\partial a}\dot{a} + \frac{\partial\mathcal{L}}{\partial \dot{a}}\ddot{a} + \cdots . \tag{56}$$

Let us now consider, for brevity, only the Lagrangians of the first 3 orders. By using the Eulero–Lagrange equation (31), as well as identities (45) and (48), we can rewrite the above equation in the following forms:

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\mathcal{L}}{\partial v}v\right) \tag{57}$$

for $\mathcal{L}(\tau; x, v)$;

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\mathcal{L}}{\partial v}v\right) + \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\mathcal{L}}{\partial a}a - \frac{\partial\dot{\mathcal{L}}}{\partial a}v\right) \tag{58}$$

for $\mathcal{L}(\tau; x, v, \dot{v})$;

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\mathcal{L}}{\partial v}v\right) + \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\mathcal{L}}{\partial a}a - \frac{\partial\dot{\mathcal{L}}}{\partial a}v\right) + \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\ddot{\mathcal{L}}}{\partial\dot{a}}v + \frac{\partial\mathcal{L}}{\partial\dot{a}}\dot{a} - \frac{\partial\dot{\mathcal{L}}}{\partial\dot{a}}a\right) \tag{59}$$

for $\mathcal{L}(\tau; x, v, \dot{v}, \ddot{v})$. Hereby we have the following conserved Hamiltonians:

$$\mathcal{H} = \frac{\partial\mathcal{L}}{\partial v}v - \mathcal{L} \tag{60}$$

for $\mathcal{L}(\tau; x, v)$;

$$\mathcal{H} = \frac{\partial\mathcal{L}}{\partial v}v + \left(\frac{\partial\mathcal{L}}{\partial a}a - \frac{\partial\dot{\mathcal{L}}}{\partial a}v\right) - \mathcal{L} \tag{61}$$

for $\mathcal{L}(\tau; x, v, \dot{v})$;

$$\mathcal{H} = \frac{\partial\mathcal{L}}{\partial v}v + \left(\frac{\partial\mathcal{L}}{\partial a}a - \frac{\partial\dot{\mathcal{L}}}{\partial a}v\right) + \left(\frac{\partial\ddot{\mathcal{L}}}{\partial\dot{a}}v + \frac{\partial\mathcal{L}}{\partial\dot{a}}\dot{a} - \frac{\partial\dot{\mathcal{L}}}{\partial\dot{a}}a\right) - \mathcal{L} \tag{62}$$

for $\mathcal{L}(\tau; x, v, \dot{v}, \ddot{v})$. We see that for spinless NS's we have Eq. (60), that is the usual Hamiltonian $pv - \mathcal{L}$. The Hamiltonians involved by Lagrangians (40) write

$$\boxed{\mathcal{H} = \frac{1}{2}mv^2} \tag{63}$$

for $\mathcal{L}^{(0)}$;

$$\boxed{\mathcal{H} = \frac{1}{2}mv^2 + \left(\frac{1}{2}k_1a^2 - k_1\dot{a}v\right)} \tag{64}$$

for $\mathcal{L}^{(1)}$;

$$\boxed{\mathcal{H} = \frac{1}{2}mv^2 + \left(\frac{1}{2}k_1a^2 - k_1\dot{a}v\right) + \left(\frac{1}{2}k_2\dot{a}^2 + k_2\dddot{a}v - k_2\ddot{a}a\right)} \tag{65}$$

for $\mathcal{L}^{(2)}$.

Let us now pass to write, for the first-order Lagrangians $\mathcal{L}(\tau; x, v, \dot{v})$, "Hamilton equations" fully equivalent to the Eulero–Lagrange equation.

Besides the first-order momentum $p \equiv \partial\mathcal{L}/\partial\dot{x} - \mathrm{d}(\partial\mathcal{L}/\partial\ddot{x})/\mathrm{d}\tau$ given by (34), let us define a "second-order momentum":

$$\pi \equiv \frac{\partial\mathcal{L}}{\partial\dot{v}}\,. \tag{66}$$

Consequently the above Hamiltonian, Eq. (61), may be rewritten as follows:

$$\mathcal{H}(\tau;\ x,p;\ v,\pi) = p\dot{x} + \pi\dot{v} - \mathcal{L}\,. \tag{67}$$

Using the differential $\mathrm{d}\mathcal{H}$ of the Hamiltonian

$$\mathrm{d}\mathcal{H} = \dot{x}\mathrm{d}p + p\mathrm{d}v + \dot{v}\mathrm{d}\pi + \pi\mathrm{d}\dot{v} - \mathrm{d}\mathcal{L}\,,$$

the above definitions of the momenta, and Eq. (33), we obtain

$$\frac{\partial\mathcal{H}}{\partial x} = -\frac{\partial\mathcal{L}}{\partial x} = -\dot{p}\,,$$

$$\frac{\partial\mathcal{H}}{\partial v} = p - \frac{\partial\mathcal{L}}{\partial v} = -\dot{\overline{\frac{\partial\mathcal{L}}{\partial\dot{v}}}} = -\dot{\pi}\,.$$

Thus we finally can write the following *double couple of Hamilton equations*:

$$\begin{cases} \dfrac{\partial\mathcal{H}}{\partial p} = \dot{x}\,, \\[2mm] \dfrac{\partial\mathcal{H}}{\partial x} = -\dot{p}\,, \end{cases} \qquad \begin{cases} \dfrac{\partial\mathcal{H}}{\partial \pi} = \dot{v}\,, \\[2mm] \dfrac{\partial\mathcal{H}}{\partial v} = -\dot{\pi}\,. \end{cases}$$

Thus, besides the standard couple of Hamilton equations, we have a new non-Newtonian couple of Hamilton equations applying to the second-order pair of canonical variables (v, π). An identical result has been found in Ref. 21, but employing a different Hamiltonian.

3. Classical Dirac Particles

Since we are employing the proper time, let us shortly recall the so-called "proper time formulation" of the Dirac theory.[23] In this formalism we can rewrite the Dirac equation $\hat{p}_\mu\gamma^\mu\psi = m\psi$ in the form of a Schrödinger eigenvalues-equation, introducing the scalar "proper Hamiltonian" $\hat{H} \equiv \hat{p}_\mu\gamma^\mu$:

$$\hat{H}\psi = i\partial_\tau\psi = m\psi\,, \tag{68}$$

quantity τ being, as before, the CMF time and m representing the energy eigenvalue in the CMF. The Heisenberg equation for the proper-time derivative of a generic operator $\hat{G}$ writes

$$\dot{\hat{G}} = i[\hat{H}, \hat{G}]\,. \tag{69}$$

Applying such an equation to the space–time coordinate x^ν we get

$$\dot{\hat{x}}^\nu \equiv \hat{v}^\nu = i[\hat{H}, x^\nu] = i[\hat{p}_\mu\gamma^\mu, x^\nu] = \gamma^\nu\,. \tag{70}$$

Hereby we might say that *the quantum equivalent of classical constraint* (9), $p_\mu v^\mu = m$, is actually the Dirac equation itself $\hat{p}_\mu \gamma^\mu \psi = m\psi$. Also the classical conservation equations (2) and (3) are recovered in the operatorial form:

$$\dot{\hat{p}}^\nu = i[\hat{H}, \hat{p}^\nu] = i[\hat{p}_\mu \gamma^\mu, \hat{p}^\nu] = 0\,; \tag{71}$$

$$\dot{\hat{S}}^{\mu\nu} = i[\hat{H}, \hat{S}^{\mu\nu}] = i\left[\hat{p}_\rho \gamma^\rho, \frac{i}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\right] = \hat{p}^\mu\gamma^\nu - \hat{p}^\nu\gamma^\mu\,. \tag{72}$$

For the four-acceleration operator we have

$$\hat{a}^\nu = i[\hat{H}, \hat{v}^\nu] = i[\hat{p}_\mu\gamma^\mu, \gamma^\nu] = 4\hat{S}^{\nu\rho}\hat{p}_\rho\,. \tag{73}$$

Equations (70)–(73) define the Dirac operatorial algebra (cf. 8, 9).

Let us now prove that the first order Lagrangian

$$\mathcal{L}^{(1)} = \frac{1}{2}mv^2 + \frac{1}{2}k_1 a^2 \tag{74}$$

can describe a free Dirac particle provided that we assume

$$k_1 = -\frac{1}{4m}\,, \tag{75}$$

so that

$$\boxed{\mathcal{L}_{\text{Dirac}} = \frac{1}{2}mv^2 - \frac{a^2}{8m}}\,. \tag{76}$$

In fact Eqs. (71) and (72), expressing the conservation of four-momentum and total angular momentum, are carried by any $\mathcal{L}^{(n)}$. The third equation

$$a^\mu = 4S^{\mu\nu}p_\nu \tag{77}$$

holds instead only in the special case of the Dirac theory, and is substantially equivalent to the Eulero–Lagrange equation for $\mathcal{L}^{(1)}$ — i.e. to the generalized Newton equation (38) — with $k_1 = -1/4m$; $k_i = 0, i \geq 2$:

$$ma^\mu + \frac{1}{4m}\ddot{a}^\mu = 0\,. \tag{78}$$

In fact, by derivating side by side Eq. (77) we get

$$\dot{a}^\mu = 4\dot{S}^{\mu\nu}p_\nu\,, \tag{79}$$

and then

$$\dot{S}^{\mu\nu}p_\nu = \frac{\dot{a}^\mu}{4}\,. \tag{80}$$

But if we insert this result in the general zitterbewegung equation (11) we obtain

$$mv^\mu + \frac{\ddot{v}^\mu}{4m} = p^\mu\,, \tag{81}$$

which is nothing but Eq. (78) after its integration with respect to τ. Therefore the spin vector of a classical Dirac particle is given by Eq. (52) with $k_1 = -1/4m$:

$$\boldsymbol{s} = \frac{1}{4m}(\boldsymbol{a} \times \boldsymbol{v})\,; \tag{82}$$

while the spin tensor (51) now writes

$$S^{\mu\nu} = \frac{1}{4m}(a^\mu v^\nu - a^\nu v^\mu)\,. \tag{83}$$

The equation of the motion (81) is a four-vectorial, constant-coefficients, second-order differential equation. Its general solution writes

$$\boxed{v^\mu = \frac{p^\mu}{m} + E^\mu \cos(2m\tau) + H^\mu \sin(2m\tau)}\,, \tag{84}$$

where, being Eq. (81) of the second order, we can fix two initial conditions by choosing the constant four-vectors E^μ and H^μ. The general solution exhibits the special zitterbewegung foreseen by Schrödinger[14] since we have an oscillating motion around the CM with the characteristic "Compton frequency" $2m$. This result has been found also in other, alternative approaches and in very particular models.[8–11,13,21] The equivalence between the above oscillatory solution and the solutions obtained in the quoted papers is often merely formal. In Ref. 21, the time parameter τ is chosen, like in the present work, equal to the CMF time. Nevertheless, because of the presence of square roots of $\dot{x}^2$ in the adopted action $[H^\mu \equiv (\sqrt{\dot{x}^2})^{-1}\mathrm{d}/\mathrm{d}\tau(\dot{x}^\mu/\sqrt{\dot{x}^2})]$

$$\mathcal{L} \equiv \sqrt{\dot{x}^2}\left(m - \frac{\mu}{\sqrt{\dot{x}^2}}H^2\right), \tag{85}$$

either lightlike or spacelike motions ($\dot{x}^2 \leq 0$) are forbidden if we will finite and real four-velocities. By contrast, our theory allows these motions: for example, we shall see later that the lightlike motion, $\dot{x}^2 = 0$, is the only involving both spin 1/2 and uniform motion in the CMF. Moreover, we easily see, after explicitation, that Lagrangian (85) is *not* time-reversal invariant, differently from our T-symmetric Lagrangian (76). Consequently, Lagrangian (85) allows also unphysical nonstationary motions in the CMF, exponentially growing or damping.

In the CMF, according to Eq. (82), the spin vector is

$$\boldsymbol{s}_\star = \frac{1}{2}(\boldsymbol{H}_\star \times \boldsymbol{E}_\star)\,, \tag{86}$$

and then it is orthogonal to the orbital plane defined by $\boldsymbol{E}_\star$ and $\boldsymbol{H}_\star$. As expected, in the CMF the spin vector is time-constant, whilst in a generic frame it is constant only its projection $\boldsymbol{s}_p$ along the momentum (helicity), in its turn equal to the projection of $\boldsymbol{s}_\star$ along the momentum

$$\boldsymbol{s}_p = (\boldsymbol{s}_\star)_p = \frac{1}{2}(\boldsymbol{H}_\star^\perp \times \boldsymbol{E}_\star^\perp) = \text{constant}\,,$$

where $\boldsymbol{H}_\star^\perp$ and $\boldsymbol{E}_\star^\perp$ are, respectively, the components of $\boldsymbol{H}_\star$ and $\boldsymbol{E}_\star$ orthogonal to $\boldsymbol{p}$.

Because of the "Correspondence Principle" we can relate the (classical) spin vector $\boldsymbol{s}_\star$ in the CMF to the mean *nonrelativistic* (quantum) spin vector, averaged in a given (spin-$\frac{1}{2}$) state ψ, $\bar{\boldsymbol{s}}_{\rm qu} = \int \psi^\dagger \boldsymbol{\sigma} \psi {\rm d}V/2$. We ask these vectors to have equal magnitudo. It is well known[24] that the modulus of the mean spin vector, anyever be the considered quantum state ψ, is always equal to 1/2:

$$|\bar{\boldsymbol{s}}_{\rm qu}| = \frac{1}{2}\left| \int \psi^\dagger \boldsymbol{\sigma} \psi \, {\rm d}V \right| = \frac{1}{2} \,. \tag{87}$$

Taking account of Eq. (86) let us require

$$|\boldsymbol{s}_\star| = \frac{1}{2}|\boldsymbol{H}_\star \times \boldsymbol{E}_\star| = \frac{1}{2} \,. \tag{88}$$

Hereby, by exploiting the known algebraic identity $(\boldsymbol{a} \times \boldsymbol{b})^2 = \boldsymbol{a}^2 \boldsymbol{b}^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2$, we have the following condition for Dirac spin-$\frac{1}{2}$ particles:

$$\boldsymbol{E}_\star^2 \boldsymbol{H}_\star^2 - (\boldsymbol{E}_\star \cdot \boldsymbol{H}_\star)^2 = 1 \,. \tag{89}$$

[The same result can be obtained requiring that, in an arbitrary frame, the (classical) value of $|\boldsymbol{s}|$ averaged over a zitterbewegung period be equal to the magnitudo of the (quantized) helicity ($\frac{1}{2}$)]. Notice that the above constraint, implying $\boldsymbol{E}_\star$ and $\boldsymbol{H}_\star$ to be not parallel, does not allow pure linear oscillations for Dirac spin-$\frac{1}{2}$ particles.

Besides the basic condition (89), from (84) and from (9) [or from (18)] we derive the following useful constraint:

$$p_\mu E^\mu = p_\mu H^\mu = 0 \,, \tag{90}$$

that implies either $E^2, H^2 < 0$ (spacelike vectors), or $E^\mu = H^\mu = (0; 0, 0, 0)$ (null vectors). As we shall see below, the latter case implies a vanishing spin [cf. (86)] and then refers only to spinless NS's, and not to Dirac systems for which therefore the spacelike case always holds. In the CMF, according to (20) and to (90), we always have

$$E_\star^0 = H_\star^0 = 0 \,, \tag{91}$$

that is, the CMF is a standard frame for the spacelike four-vectors E^μ and H^μ. From (90) we have also

$$E^0 = \boldsymbol{w} \cdot \boldsymbol{E} \,, \tag{92}$$

and

$$H^0 = \boldsymbol{w} \cdot \boldsymbol{H} \,. \tag{93}$$

Let us write down the explicit expression of the three-velocity referred to a generic frame

$$\frac{{\rm d}\boldsymbol{x}}{{\rm d}t} = \frac{{\rm d}\boldsymbol{x}}{{\rm d}\tau}\frac{{\rm d}\tau}{{\rm d}t} \equiv \frac{\boldsymbol{v}}{v^0} = \frac{\boldsymbol{p} + m\boldsymbol{E}\cos(2m\tau) + m\boldsymbol{H}\sin(2m\tau)}{p^0 + m\boldsymbol{w} \cdot \boldsymbol{E}\cos(2m\tau) + m\boldsymbol{w} \cdot \boldsymbol{H}\sin(2m\tau)} \,, \tag{94}$$

where Eqs. (84), (92) and (93) have been applied. In the absence of spin we have obviously the usual expression $\mathrm{d}\boldsymbol{x}/\mathrm{d}t = \boldsymbol{p}/p^0 = \boldsymbol{w}$. The times-ratio

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} \equiv v^0 = \frac{p^0}{m} + \boldsymbol{w}\cdot\boldsymbol{E}\cos(2m\tau) + \boldsymbol{w}\cdot\boldsymbol{H}\sin(2m\tau)\,, \tag{95}$$

is time-oscillating around its mean value, namely around the Lorentz factor $p^0/m = \gamma$. In general, during a zitterbewegung cycle we may observe both time-dilation ($\mathrm{d}t/\mathrm{d}\tau > 1$) and time-contraction ($\mathrm{d}t/\mathrm{d}\tau < 1$), and even time-inversion ($\mathrm{d}t/\mathrm{d}\tau < 0$). The overall effect, measured at the end of each oscillation is, of course, the usual time-dilation: the ratio between two zitterbewegung periods (referred to the laboratory and to the CMF, respectively) is always equal to the Lorentz factor. An analogous nonconstant relation holds between the times elapsed in two generic reference systems:

$$\frac{\mathrm{d}t}{\mathrm{d}t'} \equiv \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t'} \equiv \frac{v_0}{{v_0}'} = \frac{p_0 + m\boldsymbol{w}\cdot\boldsymbol{E}\cos(2m\tau) + m\boldsymbol{w}\cdot\boldsymbol{H}\sin(2m\tau)}{p_0' + m\boldsymbol{w}'\cdot\boldsymbol{E}\cos(2m\tau) + m\boldsymbol{w}'\cdot\boldsymbol{H}\sin(2m\tau)}\,. \tag{96}$$

By integrating equation (84) we derive the generic equation of the trajectory of a free Dirac CS

$$x^\mu = x^\mu(0) + \frac{p^\mu}{m}\tau + \frac{1}{2m}E^\mu\sin(2m\tau) - \frac{1}{2m}H^\mu\cos(2m\tau)\,. \tag{97}$$

Thus a Dirac pointlike charge moves along a cylindrical helix, spiralizing around the direction of the constant momentum. As said in Subsec. 1.3.3, this happens not only in the ordinary three-space, but also in space–time since the proper time τ and the laboratory time t are not linearly linked as for NS's and the times-ratio $v^0 \equiv \mathrm{d}t/\mathrm{d}\tau$ oscillates.

Notice that the trajectory is a *right* helix if the plane of the elliptical orbit containing $\boldsymbol{E}$ and $\boldsymbol{H}$ is orthogonal to $\boldsymbol{p}$: when $(\boldsymbol{H}\times\boldsymbol{E})\times\boldsymbol{p} = 0$ and $\boldsymbol{p}\cdot\boldsymbol{V} = 0$. In this case we shall have no longitudinal zitterbewegung along the straight path of the CM. For what seen in Subsec. 1.3.2, this happens in the standard frames for the four-vector V^μ, where $V^0 (= V^0_\star) = 0$ and the times-ratio $v^0 = w^0 + V^0$ is time-constant. Since $\boldsymbol{p}$ is orthogonal to the zitterbewegung plane containing $\boldsymbol{E}(= \boldsymbol{E}_\star)$ and $\boldsymbol{H}(= \boldsymbol{H}_\star)$ which for (86) is orthogonal to the spin, then $\boldsymbol{p}$ results parallel to $\boldsymbol{s}(= \boldsymbol{s}_\star)$: i.e. *in the standard frames for V^μ CS's appear to be polarized.* Summarizing: a polarized Dirac classical charge travels along a right helix, without forward and backward oscillations along the direction of the momentum, and the ratio between the CMF time and the laboratory time is the usual constant Lorentz factor.

Let us now look for all the solutions of (84) endowed with *constant* v^2. By superimposing such a claim we obtain the following conditions:

$$\begin{cases} E_\mu H^\mu = 0\,, \\ E^2 = H^2\,. \end{cases} \tag{98}$$

Of course, the constraint v^2 = constant does not involve in a generic frame a uniform circular motion, but only ${v_0}^2 - \boldsymbol{v}^2$ = constant. In the CMF, where $v_0 = 1$, we have instead a uniform circular motion, as it can be derived also by inserting $p^\mu = (m; 0, 0, 0)$ and (98) in (84) or (97). The orbital speed and the orbital radius are

$$|\boldsymbol{v}_\star| = \sqrt{-E^2} = \sqrt{\boldsymbol{E}_\star^2}\,, \tag{99}$$

$$R_\star = \frac{|\boldsymbol{v}_\star|}{\omega} = \frac{|\boldsymbol{v}_\star|}{2m} = \frac{\sqrt{-E^2}}{2m} = \frac{\sqrt{\boldsymbol{E}_\star^2}}{2m}\,, \tag{100}$$

where the vectors E may be indifferently replaced by the vectors H, since from (98) we have $\boldsymbol{E}_\star^2 = \boldsymbol{H}_\star^2$. Always confining ourselves to the v^2-constant solutions, we easily derive from (89) and (98) that $|\boldsymbol{s}| = 1/2$ only for the *lightlike* motion $v^2 = 0$, $|\boldsymbol{v}_\star| = 1$, with $R_\star$ equal to the Compton wavelength $(2m)^{-1}$. By inserting constraints (98) in Eq. (84), we get the constant four-velocity squared which results to be always less than 1 as expected from (21):

$$v^2 = 1 + E^2 = 1 + H^2 < 1\,. \tag{101}$$

Derivating now side by side Eq. (84) we get the four-acceleration:

$$a^\mu = -2mE^\mu \sin(2m\tau) + 2mH^\mu \cos(2m\tau)\,,$$

from which

$$a^2 = 4m^2E^2 = 4m^2(v^2 - 1) < 0\,. \tag{102}$$

Notice that the four-acceleration is always a spacelike four-vector, any be v^2 (even not constant), as expected from the general condition $p_\mu a^\mu = 0$.

The conserved Pauli–Lubanski "spin four-vector" (whose square, a Casimir-invariant of the Poincaré group, is equal to $-m^2\boldsymbol{s}_\star^2$) is defined as

$$W^\mu \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}J_{\nu\rho}p_\sigma = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}S_{\nu\rho}p_\sigma = (\boldsymbol{s}\cdot\boldsymbol{p}; p^0\boldsymbol{s} - \boldsymbol{p}\times\boldsymbol{k})\,, \tag{103}$$

where the three-vector $\boldsymbol{k}$ is the Lorentz-boosts generator $\boldsymbol{k} \equiv (S^{01}, S^{02}, S^{03})$. Here, for a classical Dirac particle, we have after some algebra

$$W^0 = \frac{1}{2}\boldsymbol{p}\cdot(\boldsymbol{H}\times\boldsymbol{E})\,, \tag{104}$$

$$\boldsymbol{W} = \frac{1}{2}[p^0(\boldsymbol{H}\times\boldsymbol{E}) + H^0(\boldsymbol{E}\times\boldsymbol{p}) + E^0(\boldsymbol{p}\times\boldsymbol{H})]\,. \tag{105}$$

Obviously in the CMF, $\boldsymbol{W}$ reduces to $m\boldsymbol{s}_\star$ [cf. Eq. (86)].

Another interesting quantity is the (nonconstant) three-vector $\boldsymbol{k}$, averaged over a zitterbewegung cycle. In the CMF this vector times e/m is a sort of "intrinsic electric-dipole momentum," by contrast with the intrinsic magnetic-dipole momentum, i.e. as usual, $\boldsymbol{s}$ times e/m. In fact in the CMF $e\boldsymbol{k}_\star/m$ is equal to the intrinsic electric-dipole momentum $\boldsymbol{d}_\star \equiv e\boldsymbol{r}_\star$ (oscillating *with zero average*). Actually, for Eq. (83) we have

$$\boldsymbol{k}_\star = \frac{a^0{}_\star\boldsymbol{v}_\star - v^0{}_\star\boldsymbol{a}_\star}{4m} = -\frac{1}{4m}\boldsymbol{a}_\star\,;$$

finally, exploiting the harmonic relation between $\boldsymbol{r}_\star$ and $\boldsymbol{a}_\star$, $\boldsymbol{a}_\star + 4m^2\boldsymbol{r}_\star = 0$ [cf. Eq. (97) in the CMF with $\boldsymbol{x}(0) = 0$],

$$\boldsymbol{d}_\star \equiv e\boldsymbol{r}_\star = -\frac{e}{4m^2}\boldsymbol{a}_\star = \frac{e}{m}\boldsymbol{k}_\star\,. \tag{106}$$

The proportionality factor e/m might perhaps be defined as "gyroelectric factor." Let us pass to a generic reference system other than the CMF. A straight calculation employing (84) gives, after averaging over a period $T = 2\pi/\omega = \pi/m$, a *nonzero* mean value for $\boldsymbol{k}$

$$\bar{\boldsymbol{k}} = \frac{1}{2}(H^0\boldsymbol{E} - E^0\boldsymbol{H})\,. \tag{107}$$

From (107) we see that the above quantity is zero only if E^0 and H^0 vanish:[g] i.e. for what above-seen, in the CMF and in all the frames where the system appears polarized. In general we can say that for a free classical Dirac particle $\bar{\boldsymbol{k}}$ is nonzero depending on the orientation of the spin.

At last let us reformulate the classical theory of Dirac particles in the Hamiltonian formalism introduced in Subsec. 2.3.

The classical Dirac Hamiltonian writes, according to (67) and (76)

$$\mathcal{H}(\tau; x, p; v, \pi) = p\dot{x} + \pi\dot{v} - \mathcal{L} = pv - 2m\pi^2 - \frac{1}{2}mv^2\,, \tag{108}$$

where the momenta p and π are, according to (34) and (66),

$$p \equiv \frac{\partial\mathcal{L}}{\partial\dot{x}} - \frac{\dot{\overline{\partial\mathcal{L}}}}{\partial\dot{v}} = mv + \frac{\ddot{v}}{4m}\,, \tag{109}$$

$$\pi \equiv \frac{\partial\mathcal{L}}{\partial\dot{v}} = -\frac{\dot{v}}{4m}\,. \tag{110}$$

Finally, the Hamilton equations

$$\begin{cases} \dfrac{\partial\mathcal{H}}{\partial p} = \dot{x}\,, \\ \dfrac{\partial\mathcal{H}}{\partial x} = -\dot{p}\,, \end{cases} \qquad \begin{cases} \dfrac{\partial\mathcal{H}}{\partial\pi} = \dot{v}\,, \\ \dfrac{\partial\mathcal{H}}{\partial v} = -\dot{\pi}\,, \end{cases}$$

now become

$$\begin{cases} v = \dot{x}\,, \\ 0 = -\dot{p}\,, \end{cases} \qquad \begin{cases} -4m\pi = \dot{v}\,, \\ p - mv = -\dot{\pi}\,. \end{cases}$$

By inserting the last but one equation in the last one we get out, as expected, the Dirac equation of the motion, Eq. (81)

$$v = \frac{p}{m} + \frac{\ddot{v}}{4m^2}\,.$$

[g] We have already seen that $\boldsymbol{E}$ and $\boldsymbol{H}$ cannot be parallel for Dirac particles.

4. Spinning Systems with Zero Intrinsic Angular Momentum

Let us show that, for each $\mathcal{L}^{(n)}$ with n from 1 to ∞, can exist CS's *endowed, in an arbitrary frame, with nonzero spin three-vector and zitterbewegung, but with zero intrinsic* (i.e. in the CMF) *angular momentum*

$$\boldsymbol{s} \neq 0\,, \qquad \boldsymbol{s}_\star = 0\,. \tag{111}$$

The CS's satisfying Eq. (111) seem to be endowed — besides the usual orbital angular momentum $\boldsymbol{l} \equiv \boldsymbol{x} \times \boldsymbol{p}$ — with a kind of "extrinsic" spin, which arises only in the presence of the "external" motion the CM, and disappears in the CMF, just like the orbital angular momentum does. It is sufficient to consider, for a chosen $\mathcal{L}^{(n)}$, those solutions of the Eulero–Lagrange equation which entail a *rectilinear oscillatory motion* in the CMF.[h] To make an example, for $\mathcal{L}^{(1)}$ — which describes, as aforeseen, *also* Dirac spin-$\frac{1}{2}$ particles — it is enough to assume, whichever is the chosen value of k_1, $\boldsymbol{E}_\star \neq 0$; $\boldsymbol{H}_\star = 0$, or $\boldsymbol{E}_\star = 0$; $\boldsymbol{H}_\star \neq 0$, or $\boldsymbol{E}_\star$ parallel to $\boldsymbol{H}_\star$, for obtaining a linear harmonic motion in the CMF. As it easy to check from Eqs. (52)–(54) and from the analogous formulae for $n > 3$, in correspondence to CMF rectilinear motions the intrinsic angular momentum $\boldsymbol{s}_\star$ actually vanishes at any time since $\boldsymbol{v}_\star$ and its time derivatives are collinear vectors: $\boldsymbol{v}_\star // \boldsymbol{a}_\star // \dot{\boldsymbol{a}}_\star // \cdots$. By contrast, in a frame other than the CMF the space part of the spin tensor, S^{ik}, is in general nonzero. Let us apply to the CMF an arbitrary boost $\boldsymbol{w}$. Labeling with $\|(\perp)$ the components parallel (orthogonal) to the boost, and taking into account that $v^0_\star = 1$ and $a^0_\star = 0$, we can write for the Lorentz-transformed components of the four-vectors v^μ and a^μ

$$\begin{aligned} v^\| &= \gamma^{-1} v^\|_\star + v^0_\star w = \gamma^{-1} v^\|_\star + w\,, \qquad & v^\perp &= v^\perp_\star\,,\\ a^\| &= \gamma^{-1} a^\|_\star + a^0_\star w = \gamma^{-1} a^\|_\star\,, & a^\perp &= a^\perp_\star\,, \end{aligned}$$

and so on for the higher-order derivatives of the velocity. As a consequence, in the new frame the Lorentz-transformed velocity $\boldsymbol{v}$ and its Lorentz-transformed derivatives *are not anymore collinear*, so that $S^{ik} \neq 0$ and $\boldsymbol{s} \equiv (S^{23}, S^{31}, S^{12}) \neq 0$. It is easy to check that $\boldsymbol{v}$, $\boldsymbol{a}$, etc. belong to the plane α containing $\boldsymbol{v}_\star$, $\boldsymbol{a}_\star$, etc. and $\boldsymbol{w}$. Being for Eqs. (52)–(54) orthogonal to the plane α, the spin vector $\boldsymbol{s}$ is then normal to the momentum $\boldsymbol{p}(\equiv m\boldsymbol{w}/\sqrt{1-w^2})$. It follows that the helicity always vanishes:

$$\lambda \equiv \frac{\boldsymbol{s} \cdot \boldsymbol{p}}{|\boldsymbol{s}|\,|\boldsymbol{p}|} = 0\,. \tag{112}$$

This could be alternatively gotten from the well-known relativistic property that in any reference frame the (time-constant) projection of the spin onto the direction of the momentum is equal to the projection of $\boldsymbol{s}_\star$ along the same direction. Since $\boldsymbol{s}_\star = 0$ for (111), Eq. (112) follows.

[h] The linear oscillatory motion is harmonic only for $\mathcal{L}^{(1)}$; it is in general anharmonic for $n \geq 2$; notice that for $n \geq 2$ we might have a vanishing $\boldsymbol{s}_\star$ without necessarily imposing a rectilinear trajectory: cf. Eqs. (53) and (54).

The Pauli–Lubanski four-vector is here always a null vector:

$$W^\mu = (0; 0, 0, 0)\,. \tag{113}$$

Then we have, from (103) and (113) $\boldsymbol{s}\cdot\boldsymbol{p} = 0$, $\boldsymbol{s} = \boldsymbol{w}\times\boldsymbol{k}$, so that, as expected, the helicity is 0 in any frame and the spin three-vector vanishes in the CMF where $\boldsymbol{w} = 0$.

Let us choose a specific Lagrangian for a detailed picture of this phenomenon, i.e. $\mathcal{L}^{(1)}$. By integrating the generalized Newton equation for $\mathcal{L}^{(1)}$, that is Eq. (38) where $k_i = 0$ for $i \geq 2$, we may get, by imposing suitable boundary conditions, an oscillating *linear* motion. This motion implies, as aforesaid, the vanishing of $\boldsymbol{s}_\star$. We have

$$v^\mu = \frac{p^\mu}{m} + F^\mu \cos(\omega\tau)\,, \tag{114}$$

where $\omega \equiv \sqrt{-m/k_1}$ $(k_1 < 0)$. Thus the space trajectory turns out to be a tilted sinusoid-like path belonging to the aforesaid plane α, with the nodal axis parallel to $\boldsymbol{p}$. Actually the trajectory is quite different from the one of a spinless NS which is a straight line.

The spin vector is given by $k_1(\boldsymbol{v}\times\boldsymbol{a})$ for Eq. (52). Then we have, after some algebra (notice that, since the Lorentz boost $\boldsymbol{w}$ does not affect the components orthogonal to $\boldsymbol{p}$, we have $\boldsymbol{F}\times\boldsymbol{p} = \boldsymbol{F}^\perp\times\boldsymbol{p} = \boldsymbol{F}^\perp_\star\times\boldsymbol{p}$)

$$\boxed{\boldsymbol{s} = \frac{1}{\sqrt{\omega}}(\boldsymbol{F}^\perp_\star\times\boldsymbol{p})\sin(\omega\tau)}\,. \tag{115}$$

The spin does not preceed anymore as the spin of a Dirac particle does, but *linearly vibrates along a direction orthogonal to the momentum.* Furthermore, the average over a zitterbewegung period of the above vector turns out to be zero. By contrast, the time average of the spin squared does not vanish and results proportional to the square of the momentum:

$$\overline{\boldsymbol{s}^2} = \frac{1}{2\omega}|\boldsymbol{F}^\perp_\star\times\boldsymbol{p}|^2 = \frac{1}{2\omega}{\boldsymbol{F}^\perp_\star}^2\boldsymbol{p}^2\,. \tag{116}$$

A *quantum* analog of the present CS will be a particle endowed with zero helicity and, at the same time, endowed with spin in an arbitrary frame (different from the CMF). Such an object can be found in a recent (quantum) theory by Ahluwalia and Kirchbach.[25] As is known, the usual spin-1 Proca equation — due to the transverse Lorenz constraint $\partial_\mu\psi^\mu = 0$ which violates the completeness relation — does *not* describe the complete physical content of the $(1/2, 1/2)$ representation space of the Lorentz proper group. In the mentioned paper those authors write a (vector) wave equation which includes the Proca theory as a particular case and describes the whole representation space. They also show that the $(1/2, 1/2)$ representations can be divided into a triplet and a singlet of opposite relative intrinsic parities, but *do not carry a definite spin angular momentum.* In general both spin-1 and spin-0 particles are covariantly inseparable inhabitants of massive vector fields.[i] In particular, the state labeled[25] as w_4 does describe a particle endowed with spin 0 in the CMF, but *not* in an arbitrary frame where we have a superposition of the helicity-0 $s = 0$ and $s = 1$ eigenstates. As a consequence that solution just refers to a helicity-0 spinning particle.

5. Summary of the Results

The theory outlined in this paper appears very general because it is not based on particular models or special approaches. The classical spin is studied simply by generalizing and extending (through the usual tensorial algebra) the Newtonian theory, without any recourse to Grassmann variables or special Clifford algebras. The only necessary (for free classical systems) assumptions are space–time isotropy and homogeneity. Thus, starting from the conservation of the linear and angular momenta, $\dot{p}^\mu = \dot{J}^{\mu\nu} = 0$, in the first section we have obtained a zitterbewegung equation, in which appear besides the timelike Newtonian term p/m also a spacelike zitterbewegung-term. Among the consequences of the zitterbewegung we have:

(a) even in the absence of forces, differently from the momentum, the velocity is not required to be a constant quantity: the Principle of Inertia is not a general law for the classical free motion; or equivalently, *the RF is not an inertial reference frame*;
(b) the square of the four-velocity v^2 obeys nonordinary constraints;
(c) "global" superluminal motions are not forbidden, provided that the energy–momentum, and any related signal or information, move along the worldline of the CM, which travels always with a subluminal speed;
(d) in general, the zitterbewegung motion of a CS has a component along the momentum;
(e) the ratio between the time durations measured in a generic frame and in the CMF is not constant and differs from the Lorentz factor (as instead occurs for NS's). In general we can say that a nonlinear relation occurs between the time durations measured in different reference frames.

Newtonian mechanics and usual relativistic kinematics are of course recovered as a *particular case* of the present theory: namely the spinless case with no zitterbewegung. We have also analyzed the strict analogy holding between the classical zitterbewegung equation and the quantum Gordon decomposition of the Dirac current.

In the second section we have performed the Lagrangian formulation of our theory through a direct generalization (satisfying the relativistic covariance and

[i]From various considerations Ahluwalia and Kirchbach conclude that the $(1/2, 1/2)$ space appears to be very suitable for a consistent picture not only of $W^\pm$ and Z^0 vector gauge bosons but also of Higgs bosons.

the symmetry under space–time inversions) of the Newtonian Lagrangian to Lagrangians containing time derivatives of the velocity. The results are obtained without any particular choose of the coefficients appearing in the theory and without any *ad hoc* assumption. We have derived a constant-coefficients differential equation of the motion (generalization of the Newton law $a = F/m$) in which non-Newtonian zitterbewegung terms appear. Alternate signs are requested for the coefficients of the terms appearing in the Lagrangian if we want only stationary solutions and finite oscillatory motions. Through the Nöther theorem, by satisfying the rotational symmetry, the classical spin can be defined employing only classical kinematical quantities, without recourse to quantum quantities as the Planck constant $\hbar$, or to Grassmann noncommuting numbers. Imposing the reparametrization invariance the conserved Hamiltonian is also obtained. For the important case of the first-order Lagrangian it can be written a second couple of Hamilton equations for the canonically conjugate variables v and π, in addition to the usual couple of Hamilton equations referring to the canonical variables x and p.

In the third section we have shown that the first-order Lagrangian with $k_1 = -1/4m$ fully describes classical Dirac particles and derived the classical Dirac spin in the form $\boldsymbol{a} \times \boldsymbol{v}/4m$. The general solution of the Eulero–Lagrange equation oscillates with the Compton frequency $\omega = 2m$, and the space–time worldline is a four-dimensional helix. The particular solutions corresponding to polarized Dirac particles and to constant v^2 have been in detail studied. Finally, we have derived the explicit form of the Pauli–Lubanski four-vector for classical Dirac particles.

In the last section we have studied spinning CS's with zero intrinsic angular momentum and helicity, especially in the case of first-order Lagrangians. An interesting quantum analog has been recently found in a $(1/2, 1/2)$ vector representation of the Lorentz group, shown to be a superposition of spin-0 and spin-1 states, with $s = 0$ only in the CMF.

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