

ON NON-SELF-ADJOINT OPERATORS FOR OBSERVABLES IN QUANTUM MECHANICS AND QUANTUM FIELD THEORY*

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Received 7 June 2009

The aim of this paper is to show the possible significance, and usefulness, of various non-self-adjoint operators for suitable Observables in nonrelativistic and relativistic quantum mechanics, and in quantum electrodynamics. More specifically, this work deals with: (i) the maximal Hermitian (but not self-adjoint) *time* operator in nonrelativistic quantum mechanics and in quantum electrodynamics; (ii) the problem of the four-position and four-momentum operators, each one with its Hermitian and anti-Hermitian parts, for relativistic spin-zero particles. Afterwards, other physically important applications of non-self-adjoint (and even non-Hermitian) operators are discussed: in particular, (iii) we reanalyze in detail the interesting possibility of associating quasi-Hermitian Hamiltonians with (decaying) *unstable states* in nuclear physics. Finally, we briefly mention the cases of quantum dissipation, as well as of the nuclear optical potential.

Keywords: Time operator; space–time operator; time–“Hamiltonian”; non-self-adjoint operators; non-Hermitian operators; bilinear operators; time operator for discrete energy spectra; time–energy uncertainty relations; unstable states; quasi-Hermitian Hamiltonians; Klein–Gordon equation; quantum dissipation; nuclear optical model.

PACS numbers: 03.65.Ta, 03.65.-w, 03.65.Pm, 03.70.+k, 03.65.Xp, 11.10.St, 11.10.-z, 11.90.+t, 02.00.00, 03.00.00, 24.10.Ht, 03.65.Yz, 21.60.-u, 11.10.Ef, 03.65.Fd

*Work partially supported by INFN and MURST. A preliminary version of this paper was posted in the arXiv as arXiv:0903.3187[quant-ph].

1. Introduction

Time, as well as 3-position, sometimes is a parameter, but sometimes is an observable that in quantum theory would be expected to be associated with an operator. However, almost from the birth of quantum mechanics (cf. e.g. Refs. 1 and 2), it is known that time cannot be represented by a self-adjoint operator, except in the case of special systems (such as an electrically charged particle in an infinite uniform electric field).^a The list of papers devoted to the problem of time in quantum mechanics is extremely large (see, for instance, Refs. 3–37, and references therein). The same situation had to be faced also in quantum electrodynamics and, more in general, in relativistic quantum field theory (see, for instance, Refs. 3, 4, 26, 27).

As to quantum mechanics, the very first relevant papers are probably Refs. 3–15 and references therein. A second set of papers on time in quantum physics^{16–37} appeared in the nineties, stimulated partially by the need of a consistent definition for the tunneling time. It is noticeable, and let us stress it right now, that this second set of papers seems however to have ignored Naimark's theorem,³⁸ which had previously constituted (directly or indirectly) an important basis for the results in Refs. 3–15. Moreover, all the papers Refs. 16–23 attempted at solving the problem of time as a quantum observable by means of formal mathematical operations performed *outside* the usual Hilbert space of conventional quantum mechanics. Let us recall that Naimark's theorem states³⁸ that the nonorthogonal spectral decomposition of a Hermitian operator *can be approximated* by an orthogonal spectral function (which corresponds to a self-adjoint operator), in a weak convergence, *with any desired accuracy*.

The main goal of the first part of the present paper is to justify the use of time as a quantum observable, basing ourselves on the properties of the Hermitian (or, rather, maximal Hermitian) operators for the case of continuous energy spectra: cf. e.g. Refs. 24–27).

The question of time as a quantum-theoretical observable is conceptually connected with the much more general problem of the four-position operator and of the canonically conjugate four-momentum operator, both endowed with a Hermitian and an anti-Hermitian part, for relativistic spin-zero particles: this problem is analyzed in the second part of the present paper.

In the third part of this paper, it is shown how non-Hermitian operators can be meaningfully and extensively used, for instance, for describing *unstable states* (decaying resonances). Brief mentions are added of the cases of quantum dissipation, and of the nuclear optical potential.

^aThis is a consequence of the semiboundedness of the continuous energy spectra from below (usually from zero). Only for an electrically charged particle in an infinite uniform electric field, and other very rare special systems, the continuous energy spectrum is not bounded and extends over the whole axis from $-\infty$ to $+\infty$. It is curious that for systems with continuous energy spectra bounded from above and from below, the time operator is however self-adjoint and yields a discrete time spectrum.

2. Time Operator in Nonrelativistic Quantum Mechanics and in Quantum Electrodynamics

2.1. On time as an observable in nonrelativistic quantum mechanics for systems with continuous energy spectra

The last part of the above-mentioned list^{17–37} of papers, in particular Refs. 18–23, 29–37 appeared in the nineties, devoted to the problem of time in nonrelativistic quantum mechanics, essentially because of the need to define the tunneling time. As remarked, those papers did not refer to the Naimark theorem^{38,b} which had mathematically supported, on the contrary, the results in Refs. 3–15 and afterwards in Refs. 24–28. Indeed, already in the seventies (in Refs. 3–9 while more detailed presentations and reviews can be found in Refs. 10–13 and independently in Refs. 14 and 15), it was proved that for systems with continuous energy spectra time *is* a quantum-mechanical observable, canonically conjugate to energy. Namely, it had been shown the time operator

$$\hat{t} = \begin{cases} t, & \text{in the time } (t) \text{ representation,} & (1a) \\ -i\hbar \frac{\partial}{\partial E}, & \text{in the energy } (E) \text{ representation} & (1b) \end{cases}$$

to be not self-adjoint but Hermitian, and to act on square-integrable space-time wave packets in the representation (1a) and on their Fourier-transforms in (1b), once point $E = 0$ is eliminated (i.e. once one deals only with moving packets, excluding any *nonmoving* rear tails and the cases with zero fluxes).^c In Refs. 10–13 and Refs. 24–28, the operator \hat{t} (in the t -representation) had the property that any averages over time, in the one-dimensional (1D) scalar case, were to be obtained by use of the following *measure* (or weight):

$$W(t, x) dt = \frac{j(x, t) dt}{\int_{-\infty}^{+\infty} j(x, t) dt}, \quad (2)$$

where the the flux density $j(x, t)$ corresponds to the (temporal) probability for a particle to pass through point x during the unit time centered at t , when traveling in the positive x -direction. Such a measure is not postulated but is a direct consequence of the well-known probabilistic *spatial* interpretation of $\rho(x, t)$ and of the continuity relation $\partial\rho(x, t)/\partial t + \text{div } j(x, t) = 0$. Quantity $\rho(x, t)$ is, as usual, the probability of finding the considered moving particle inside a unit space interval, centered at point x , at time t .

^bThe Naimark's theorem states in particular the following:³⁸ the nonorthogonal spectral decomposition of a maximal Hermitian operator can be approximated by an orthogonal spectral function (which corresponds to a self-adjoint operator), in a weak convergence, with any desired accuracy. ^cSuch a condition is enough for operator (1a), (1b) to be a *Hermitian*, or more precisely a *maximal Hermitian*^{4–10} operator (see also Refs. 24–28); but it can be dispensed with, by recourse to bilinear forms (see, e.g. Refs. 8, 9, 39 and references therein), as we shall see below.

Quantities $\rho(x, t)$ and $j(x, t)$ are related to the wave function $\Psi(x, t)$ by the ordinary definitions $\rho(x, t) = |\Psi(x, t)|^2$ and $j(x, t) = \Re[\Psi^*(x, t)(\hbar/i\mu)\Psi(x, t)]$. When the flux density $j(x, t)$ changes its sign, quantity $W(x, t)dt$ is no longer positive-definite and, as in Refs. 10, 24–28, it acquires the physical meaning of a probability density *only* during those partial time-intervals in which the flux density $j(x, t)$ does keep its sign. Therefore, let us introduce the *two* measures^{24–27} by separating the positive and the negative flux-direction values (that is, the flux signs)

$$W_{\pm}(t, x)dt = \frac{j_{\pm}(x, t)dt}{\int_{-\infty}^{+\infty} j_{\pm}(x, t)dt}, \tag{3}$$

with $j_{\pm}(x, t) \equiv j(x, t)\theta(\pm j)$.

Then, the mean value $\langle t_{\pm}(x) \rangle$ of the time t at which the particle passes through position x , *when traveling in the positive or negative direction*, is, respectively,

$$\begin{aligned} \langle t_{\pm}(x) \rangle &= \frac{\int_{-\infty}^{+\infty} t j_{\pm}(x, t)dt}{\int_{-\infty}^{+\infty} j_{\pm}(x, t)dt} \\ &= \frac{\int_0^{+\infty} \frac{1}{2} [G^{*}(x, E)\hat{t}vG(x, E) + vG^{*}(x, E)\hat{t}G(x, E)]dE}{\int_0^{+\infty} v|G(x, E)|^2dE}, \end{aligned} \tag{4}$$

where $G(x, E)$ is the Fourier-transform of the moving 1D wave-packet

$$\Psi(x, t) = \int_0^{+\infty} G(x, E) \exp\left(-\frac{iEt}{\hbar}\right)dE = \int_0^{+\infty} g(E)\varphi(x, E) \exp\left(-\frac{iEt}{\hbar}\right)dE$$

when going on from the time to the energy representation. For free motion, one has $G(x, E) = g(E) \exp(ikx)$ and $\varphi(x, E) = \exp(ikx)$, while $E = \mu\hbar^2k^2/2 = \mu v^2/2$. In Refs. 24–27, there were defined the mean time *durations* for the particle 1D transmission from x_i to $x_f > x_i$, and reflection from the region $(x_i, +\infty)$ back to the interval $x_f \leq x_i$. Namely

$$\langle \tau_T(x_i, x_f) \rangle = \langle t_+(x_f) \rangle - \langle t_+(x_i) \rangle \tag{5}$$

and

$$\langle \tau_R(x_i, x_f) \rangle = \langle t_-(x_f) \rangle - \langle t_+(x_i) \rangle, \tag{6}$$

respectively. The 3D generalization for the mean durations of quantum collisions and nuclear reactions appeared in Refs. 10–13. Finally, suitable definitions of the averages $\langle t^n \rangle$ on time of t^n , with $n = 1, 2, \dots$ and of $\langle f(t) \rangle$, quantity $f(t)$ being any analytical function of time, can be found in Refs. 27 and 40, where expressions have been explicitly written down which result to be *unique*.

The two canonically conjugate operators, the time operator (1) and the energy operator

$$\hat{E} = \begin{cases} E, & \text{in the energy } (E) \text{ representation,} \\ i\hbar \frac{\partial}{\partial t}, & \text{in the time } (t) \text{ representation} \end{cases} \tag{7a}$$

$$\tag{7b}$$

do clearly satisfy the commutation relation^{8,9,27,40}

$$[\hat{E}, \hat{t}] = i\hbar. \tag{8}$$

The Stone and von Neumann theorem⁴¹ has been always interpreted as establishing a commutation relation like (8) for the pair of the canonically conjugate operators (1) and (7), in both representations, for self-adjoint operators only. However, it can be generalized for (maximal) Hermitian operators once one introduces \hat{t} by means of the *single-valued* Fourier–Laplace transformation from the t -axis ($-\infty < t < \infty$) to the E -semiaxis ($0 < E < \infty$) and utilizes the properties^{42,43} of the “(maximal) Hermitian” operators: this has been shown, e.g. in the last one of Refs. 3, 4 as well as in Refs. 27 and 40.

Indeed, from Eq. (8) the uncertainty relation

$$\Delta E \Delta t \geq \frac{\hbar}{2} \tag{9}$$

(where the standard deviations are $\Delta a = \sqrt{Da}$, quantity Da being the variance $Da = \langle a^2 \rangle - \langle a \rangle^2$, and $a = E, t$, while $\langle \dots \rangle$ denotes the average over t with the measures $W(x, t)dt$ or $W_{\pm}(x, t)dt$ in the t -representation) can be derived also for operators which are only Hermitian by a straightforward generalization of the procedures which are common in the case of *self-adjoint* (canonically conjugate) quantities, like coordinate \hat{x} and momentum \hat{p}_x . Moreover, relation (8) satisfies^{27,40} the Dirac “correspondence” principle, since the classical Poisson brackets $\{q_0, p_0\}$, with $q_0 = t$ and $p_0 = -E$, are equal to 1. In Refs. 6–10, 27 and 40, it was also shown that *the differences* between the mean times at which a wave-packet passes through a *pair* of points obey the Ehrenfest correspondence principle.

As a consequence, one can state that, for systems with continuous energy spectra, the mathematical properties of (maximal) Hermitian operators, like \hat{t} in Eq. (1), are sufficient for considering them as quantum observables. Namely, the *uniqueness*⁴² of the spectral decomposition (although not orthogonal) for operators \hat{t} , and \hat{t}^n ($n > 1$), guarantees the “equivalence” of the mean values of any analytical function of time when evaluated in the t and in the E -representations. In other words, such an expansion is equivalent to a completeness relation, for the (approximate) eigenfunctions of \hat{t}^n ($n > 1$) which *with any accuracy* can be regarded as orthogonal, and correspond to the actual eigenvalues for the continuous spectrum. These approximate eigenfunctions belong to the space of the square-integrable functions of the energy E (see, for instance Refs. 8–13, 27 and references therein).

From this point of view, there is no *practical* difference between self-adjoint and maximal Hermitian operators for systems with continuous energy spectra. Let us repeat that the mathematical properties of \hat{t}^n ($n > 1$) are enough for considering time as a quantum mechanical observable (like energy, momentum, space coordinates, etc.) *without having to introduce any new physical postulates*.

It is remarkable that von Neumann himself,⁴⁴ before confining himself for simplicity to self-adjoint operators, stressed that operators like our time \hat{t} may represent physical observables, even if they are not self-adjoint. Namely, he explicitly

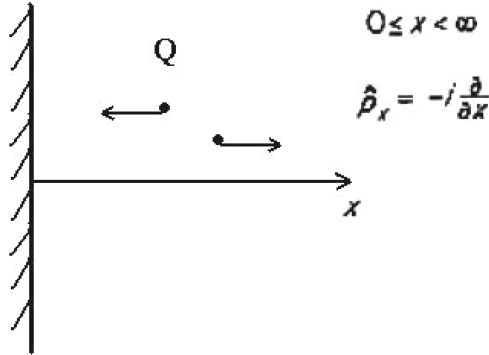


Fig. 1. For a particle Q free to move in a semi-space, bounded by a rigid wall located at $x = 0$, the operator $-i\hbar\partial/\partial x$ has the clear physical meaning of the particle momentum x -component even if it is *not* self-adjoint (cf. von Neumann⁴⁴ and Refs. 8 and 9) (see the text).

considered the example of the operator $-i\hbar\partial/\partial x$ associated with a particle living in the right semi-space bounded by a rigid wall located at $x = 0$; that operator is not self-adjoint (acting on wave packets defined on the positive x -axis only), nevertheless it obviously corresponds to the x -component of the observable *momentum* for that particle: see Fig. 1.

At this point, let us emphasize that our previously assumed boundary condition $E \neq 0$ can be dispensed with by having recourse^{3,4,8,9} to the *bi-linear* Hermitian operator

$$\hat{t} = \frac{-i\hbar}{2} \frac{\overleftarrow{\partial}}{\partial E}, \tag{10}$$

where the meaning of the sign \leftrightarrow is clear from the definition

$$(f, \hat{t}g) = \left(f, -\frac{i\hbar}{2} \frac{\partial}{\partial E} g \right) + \left(-\frac{i\hbar}{2} \frac{\partial}{\partial E} f, g \right).$$

By adopting this expression for the time operator, the algebraic sum of the two terms on the r.h.s. of the last relation results to be automatically zero at point $E = 0$. This question will be exploited below, in Sec. 3 (when dealing with the more general case of the four-position operator). Incidentally, such an “elimination”^{3,4,8,9} of point $E = 0$ is not only simpler, but also more physical than other kinds of elimination obtained much later in papers like Refs. 33 and 34.

In connection with the last quotation, let us briefly comment on the so-called *positive-operator-value-measure* (POVM) approach, often used or discussed in the second set of papers on time in quantum physics mentioned in our Introduction. Actually, an analogous procedure had been proposed since the sixties,⁴⁵ in some approaches to the quantum theory of measurements. Afterwards and much later the POVM approach has been applied, in a simplified and shortened form, to the time-operator problem in the case of one-dimensional free motion: for instance, in Refs. 16, 18, 21, 29–37 and especially in Refs. 33 and 34. Those papers stated that

a generalized decomposition of unity (or “POV measure”) could be obtained from self-adjoint extensions of the time operator inside an extended Hilbert space (for instance, adding the negative values of the energy, too), by exploiting the Naimark dilation-theorem:⁴⁶ but such a program has been realized till now only in the simple cases of free particle one-dimensional motion.

By contrast, our own approach is based on a different Naimark’s theorem,³⁸ which, as already mentioned above, allows a much more direct, simple and general –and at the same time not less rigorous – introduction of a quantum operator for time. More precisely, our approach is based on the so-called *Carleman theorem*,⁴⁷ utilized in Ref. 38, about approximating a Hermitian operator by suitable successions of “bounded” self-adjoint operators: that is, of self-adjoint operators whose spectral functions do weakly converge to the nonorthogonal spectral function of the considered hermitian operator. And our approach is applicable to a large family of particle three-dimensional (3D) collisions with all possible Hamiltonians. Actually, our approach was proposed in the early Refs. 3–10 and in the first one of Refs. 24, 25, and applied therein for the time analysis of quantum collisions, nuclear reactions and tunneling processes.

2.2. On the momentum representation of the time operator

In the continuous spectrum case, instead of the E -representation, with $0 < E < +\infty$, in Eqs. (1)–(4) one can also use the k -representation^{14,15} with the advantage that $-\infty < k < +\infty$:

$$\Psi(x, t) = \int_{-\infty}^{+\infty} g(k)\varphi(x, k) \exp\left(-\frac{iEt}{\hbar}\right) dk \tag{11}$$

with $E = \hbar^2 k^2 / 2\mu$ and $k \neq 0$.

For the extension of the momentum representation to the case of $\langle t^n \rangle$ with $n > 1$ we confine ourselves here to refer the reader to Refs. 27 and 40.

2.3. An alternative weight for time averages (in the cases of particle dwelling inside a certain spatial region)

We recall that the weight (2) [as well as its modifications (3)] has the meaning of a probability for the considered particle to pass through point x during the time interval $(t, t + dt)$. Let us follow the procedure presented in Refs. 24–28 and references therein and analyze the consequences of the equality

$$\int_{-\infty}^{+\infty} j(x, t) dt = \int_{-\infty}^{+\infty} |\Psi(x, t)|^2 dx \tag{12}$$

obtained from the 1D continuity equation. One can easily realize that a second, alternative weight can be adopted:

$$dP(x, t) \equiv Z(x, t) dx = \frac{|\Psi(x, t)|^2 dx}{\int_{-\infty}^{+\infty} |\Psi(x, t)|^2 dx} \tag{13}$$

which possesses the meaning of probability for the particle to be located (or to sojourn, i.e. to *dwell*) inside the infinitesimal space region $(x, x + dx)$ at the instant t , independently of its motion properties. Then, the quantity

$$P(x_1, x_2, t) = \frac{\int_{x_1}^{x_2} |\Psi(x, t)|^2 dx}{\int_{-\infty}^{+\infty} |\Psi(x, t)|^2 dx} \quad (14)$$

will have the meaning of probability for the particle to dwell inside the spatial interval (x_1, x_2) at the instant t .

As it is known (see, for instance, Refs. 24–27 and references therein), the *mean dwell time* can be written in the *two* equivalent forms:

$$\langle \tau(x_i, x_f) \rangle = \frac{\int_{-\infty}^{+\infty} dt \int_{x_i}^{x_f} |\Psi(x, t)|^2 dx}{\int_{-\infty}^{+\infty} j_{\text{in}}(x_i, t) dt}, \quad (15)$$

$$\langle \tau(x_i, x_f) \rangle = \frac{\int_{-\infty}^{+\infty} t j(x_f, t) dt - \int_{-\infty}^{+\infty} t j(x_i, t) dt}{\int_{-\infty}^{+\infty} j_{\text{in}}(x_i, t) dt}, \quad (16)$$

where it has been taken account, in particular, of relation (12), which follows — as already said — from the continuity equation.

Thus, in correspondence with the two measures (2) and (13), when integrating over time one gets *two* different kinds of time distributions (mean values, variances, ...), which refer to the particle traversal time in the case of measure (2), and to the particle dwelling in the case of measure (13). Some examples for 1D tunneling are contained in Refs. 24–27.

2.4. Time as a quantum-theoretical observable in the case of photons

As is known (see, for instance, Refs. 26, 48, 49), in first quantization the single-photon wave function can be probabilistically described in the 1D case by the wave-packet^d

$$\mathbf{A}(\mathbf{r}, t) = \int_{k_0} \frac{d^3 k}{k_0} \chi(\mathbf{k}) \varphi(\mathbf{k}, \mathbf{r}) \exp(-ik_0 t), \quad (17)$$

where as usual $\mathbf{A}(\mathbf{r}, t)$ is the electromagnetic vector potential, while $\mathbf{r} = \{x, y, z\}$, $\mathbf{k} = \{k_x, k_y, k_z\}$, $k_0 \equiv \omega/c = \varepsilon/\hbar c$ and $k \equiv |\mathbf{k}| = k_0$. The axis x has been chosen as the propagation direction. Let us notice that $\chi(\mathbf{k}) = \sum_{i=y,z} \chi_i(\mathbf{k}) e_i(\mathbf{k})$ with $e_i e_j = \delta_{ij}$ and $x_i, x_j = y, z$, while $\chi_i(\mathbf{k})$ is the probability amplitude for the photon to have momentum \mathbf{k} and polarization e_j along x_j . Moreover, it is $\varphi(\mathbf{k}, \mathbf{r}) = \exp(ik_x x)$ in the case of plane waves, while $\varphi(\mathbf{k}, \mathbf{r})$ is a linear combination of evanescent (decreasing) and antievanescent (increasing) waves in the case of “photon barriers” (i.e. band-gap filters, or even undersized segments of waveguides for microwaves, or

^dThe gauge condition $\text{div } \mathbf{A} = 0$ is assumed.

frustrated total-internal-reflection regions for light, and so on). Although it is not easy to localize a photon in the direction of its polarization,^{48,49} nevertheless for 1D propagations it is possible to use the space–time probabilistic interpretation of Eq. (17) and define the quantity

$$\rho_{em}(x, t)dx = \frac{S_0 dx}{\int S_0 dx}, \quad S_0 = \iint s_0 dy dz \tag{18}$$

($s_0 = [\mathbf{E}^* \cdot \mathbf{E} + \mathbf{H}^* \cdot \mathbf{H}]/4\pi$ being the energy density, with the electromagnetic field $\mathbf{H} = \text{rot } \mathbf{A}$, and $\mathbf{E} = -1/c\partial\mathbf{A}/\partial t$), which represents the probability density of a photon to be found (localized) in the spatial interval $(x, x + dx)$ along the x -axis at the instant t ; and the quantity

$$j_{em}(x, t)dt = \frac{S_x dt}{\int S_x(x, t)dt}, \quad S_x(x, t) = \iint s_x dy dz \tag{19}$$

($s_x = c \Re[\mathbf{E}^* \wedge \mathbf{H}]_x/8\pi$ being the energy flux density), which represents the flux probability density of a photon to pass through point x in the time interval $(t, t + dt)$: in full analogy with the probabilistic quantities for nonrelativistic particles. The justification and convenience of such definitions is self-evident, when the wave-packet group velocity coincides with the velocity of the energy transport; in particular: (i) the wave-packet (17) is quite similar to wave-packets for nonrelativistic particles, and (ii) in analogy with conventional non-relativistic quantum mechanics, one can define the “mean time instant” for a photon (i.e. an electromagnetic wave-packet) to pass through point x as follows:

$$\langle t(x) \rangle = \int_{-\infty}^{+\infty} t J_{em,x} dt = \frac{\int_{-\infty}^{+\infty} t S_x(x, t) dt}{\int_{-\infty}^{+\infty} S_x(x, t) dt}.$$

As a consequence [in the same way as in the case of Eqs. (1), (2)], the form (1) for the time operator in the energy representation is valid also for photons, with the same boundary conditions adopted in the case of particles, that is, with $\chi_i(0) = \chi_i(\infty)$ and with $E = \hbar ck_0$.

The energy density s_0 and energy flux density s_x satisfy the relevant continuity equation

$$\frac{\partial s_0}{\partial t} + \frac{\partial s_x}{\partial x} = 0 \tag{20}$$

which is Lorentz-invariant for 1D spatial propagation^{26,27} processes.

2.5. Introducing the analogue of the “Hamiltonian” for the case of the Time operator: A new Hamiltonian approach

In nonrelativistic quantum theory, the Energy operator acquires (cf. e.g. Refs. 11–13 and 27) the two forms: (i) $i\hbar \frac{\partial}{\partial t}$ in the t -representation, and (ii) $\hat{H}(\hat{p}_x, \hat{x}, \dots)$ in the Hamiltonian formalism. The “duality” of these two forms can be easily inferred from the Schrödinger equation itself, $\hat{H}\Psi = i\hbar \frac{\partial \Psi}{\partial t}$. One can introduce in quantum

mechanics a similar duality for the case of *time*: besides the general form (1) for the time operator in the energy representation, which is valid for any physical systems in the region of continuous energy spectra, one can *express the time operator also in a “Hamiltonian form,”* i.e. in terms of the coordinate and momentum operators, by having recourse to their commutation relations. Thus, by the replacements

$$\hat{E} \rightarrow \hat{H}(\hat{p}_x, \hat{x}, \dots), \quad \hat{t} \rightarrow \hat{T}(\hat{p}_x, \hat{x}, \dots), \tag{21}$$

and on using the commutation relation [similar to Eq. (3)]

$$[\hat{H}, \hat{T}] = i\hbar, \tag{22}$$

one can obtain,⁵⁰ given a specific ordinary Hamiltonian, the corresponding explicit expression for $\hat{T}(\hat{p}_x, \hat{x}, \dots)$.

Indeed, this procedure can be adopted for any physical system with a known Hamiltonian $\hat{H}(\hat{p}_x, \hat{x}, \dots)$, and we are going to see a concrete example. By going on from the coordinate to the momentum representation, one realizes that the *formal* expressions of *both* the Hamiltonian-type operators $\hat{H}(\hat{p}_x, \hat{x}, \dots)$ and $\hat{T}(\hat{p}_x, \hat{x}, \dots)$ *do not change*, except for an obvious change of sign for operator $\hat{T}(\hat{p}_x, \hat{x}, \dots)$.

As an explicit example, let us address the simple case of a free particle whose Hamiltonian is

$$\hat{H} = \begin{cases} \frac{\hat{p}_x^2}{2\mu}, \hat{p}_x = -i\hbar \frac{\partial}{\partial x} & \text{in the coordinate representation,} \\ \frac{p_x^2}{2\mu} & \text{in the momentum representation.} \end{cases} \tag{23a}$$

$$\tag{23b}$$

Correspondingly, the Hamilton-type *time operator* in its symmetrized form, will read

$$\hat{T} = \begin{cases} \frac{\mu}{2} \left(\hat{p}_x^{-1} x + x \hat{p}_x^{-1} + \frac{i\hbar}{\hat{p}_x^2} \right) & \text{in the coordinate representation;} \\ -\frac{\mu}{2} \left(p_x^{-1} \hat{x} + \hat{x} p_x^{-1} + \frac{i\hbar}{p_x^2} \right) & \text{in the momentum representation,} \end{cases} \tag{24a}$$

$$\tag{24b}$$

where

$$\hat{p}_x^{-1} = \frac{i}{\hbar} \int dx \dots, \quad \hat{x} = i\hbar \frac{\partial}{\partial p_x}.$$

Incidentally, operator (24b) is equivalent to $-i\hbar \frac{\partial}{\partial E}$, since $E = p_x^2/2\mu$; and therefore it is also a (maximal) *Hermitian* operator. Indeed, by applying the operator $\hat{T}(\hat{p}_x, \hat{x}, \dots)$, for instance, to a plane-wave of the type $\exp(ikx)$, we obtain the same result in both the coordinate and the momentum representations:

$$\hat{T} \exp(ikx) = \frac{x}{v} \exp(ikx) \tag{25}$$

quantity x/v being the free-motion time (of a particle with velocity v) for traveling the distance x .

On the basis of what precedes, it is possible to show that the wave function $\Psi(x, t)$ of a quantum system satisfies the two (dual) equations

$$\hat{H}\Psi = i\hbar \frac{\partial \Psi}{\partial t} \quad \text{and} \quad \hat{T}\Psi = t\Psi. \tag{26}$$

In the energy representation and in the stationary case we obtain again *two* (dual) equations

$$\hat{H}\varphi_t = \varepsilon\varphi_t \quad \text{and} \quad \hat{T}\varphi_t = -i\hbar \frac{\partial \varphi_t}{\partial \varepsilon}, \tag{27}$$

with quantity φ_t being the Fourier-transform of Ψ :

$$\varphi_t = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} \Psi(x, t) e^{i\varepsilon t/\hbar} dt. \tag{28}$$

It might be interesting to apply the two pairs of the last dual equations also for investigating tunneling processes through the quantum gravitational barrier, which appears during inflation, or at the beginning of the big-bang expansion whenever a quasilinear Schrödinger-type equation does approximately show up.

2.6. Time as an observable (and the time-energy uncertainty relation), for quantum-mechanical systems with discrete energy spectra

For describing the time evolution of nonrelativistic quantum systems endowed with a purely *discrete* (or a continuous *and discrete*) spectrum, let us now introduce wave-packets of the form^{11-13,27,40}

$$\psi(x, t) = \sum_{n=0} g_n \varphi_n(x) \exp \left[-\frac{i(\varepsilon_n - \varepsilon_0)t}{\hbar} \right], \tag{29}$$

where $\varphi_n(x)$ are orthogonal and normalized bound states which satisfy the equation $\hat{H}\varphi_n(x) = \varepsilon_n\varphi_n(x)$, quantity \hat{H} being the Hamiltonian of the system; while the coefficients g_n are normalized: $\sum_{n=0} |g_n|^2 = 1$. We omitted the nonsignificant phase factor $\exp(-i\varepsilon_0 t/\hbar)$ of the fundamental state.

Let us first consider the systems whose energy levels are separated by intervals admitting a maximum common divisor D (e.g., harmonic oscillator, particle in a rigid box, and spherical spinning top), so that the wave-packet (29) is a periodic function of time possessing as period the Poincaré cycle time $T = 2\pi\hbar/D$. For such systems it is possible^{11-13,27,40} to construct a *self-adjoint* time operator with the form (in the time representation) of a saw-tooth function of t , choosing $t = 0$ as the initial time instant:

$$\hat{t} = t - T \sum_{n=0}^{\infty} \Theta(t - [2n + 1]T/2) + T \sum_{n=0}^{\infty} \Theta(-t - [2n + 1]T/2). \tag{30}$$

This periodic function for the time operator is a linear (increasing) function of time t within each Poincaré cycle (see Fig. 2).

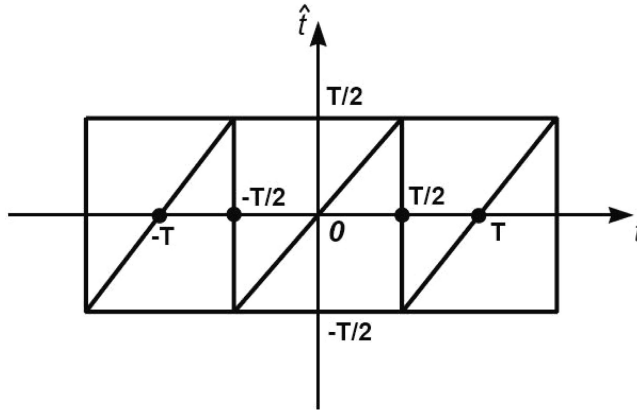


Fig. 2. The periodic saw-tooth function for the time operator in the case of quantum mechanical systems with *discrete* energy spectra, namely for the case of Eq. (30).

The commutation relation of the energy and time operators, now both self-adjoint, acquires in the case of discrete energies and of a periodic time operator the form

$$[\hat{E}, \hat{t}] = i\hbar \left\{ 1 - T \sum_{n=0}^{\infty} \delta(t - [2n + 1]T) \right\}, \tag{31}$$

wherefrom the uncertainty relation follows in the new form

$$(\Delta E)^2 (\Delta t)^2 = \hbar^2 \left[1 - \frac{T |\psi(T/2 + \gamma)|^2}{\int_{-T/2}^{T/2} |\psi(t)|^2 dt} \right], \tag{32}$$

where, following A. S. Davydov, it has been introduced a parameter γ with $-T/2 < \gamma < T/2$ in order to assure that the r.h.s. integral is single-valued.^{27,40}

When $\Delta E \rightarrow 0$ (that is, when $|g_n| \rightarrow \delta_{nn'}$), the r.h.s. of Eq. (32) tends to zero too since $|\psi(t)|^2$ tends to a constant value. In such a case, the distribution of the time instants at which the wave-packet passes through point x becomes flat within each Poincaré cycle. When, by contrast, $\Delta E \gg D$ and $|\psi(T + \gamma)|^2 \ll \left(\int_{-T/2}^{T/2} |\psi(t)|^2 dt \right) / T$, the periodicity condition may become inessential since $\Delta t \ll t$. In other words, our uncertainty relation (32) transforms into the ordinary uncertainty relation for systems with continuous spectra.

In more general cases, for excited states of nuclei, atoms and molecules, the *energy-level intervals, for discrete and quasidiscrete (resonance) spectra, are not multiples of a maximum common divisor*, and hence the Poincaré cycle is not well defined for such systems. Nevertheless, even for those systems one can introduce an approximate description (sometimes with any desired degree of accuracy) in terms of Poincaré quasicycles and a quasiperiodical evolution; so that for sufficiently long time intervals the behavior of the wave-packets can be associated with a *periodical*

motion (*oscillation*), sometimes — e.g. for very narrow resonances — with any desired accuracy. For them, when choosing an approximate Poincaré-cycle time, one can include in one cycle as many quasicycles as it is necessary for the demanded accuracy. Then, within the chosen accuracy a *quasi-self-adjoint time operator* can be introduced.

3. On Four-Position Operators in Quantum Field Theory, in Terms of *Bilinear* Operators

In this section we approach the *relativistic* case, taking into consideration — therefore — the space–time (4D) “position” operator, starting however with an analysis of the 3D (spatial) position operator in the simple relativistic case of the Klein–Gordon equation. Actually, this analysis will already lead us to tackle with non-Hermitian operators. Moreover, while performing it, we shall meet the opportunity of introducing *bilinear* operators, which will be used even more in the next case of the full 4-position operator.

Let us recall that in Subsec. 2.1 we mentioned that the boundary condition $E \neq 0$, therein imposed to guarantee (maximal) hermiticity of the time operator, can be dispensed with just by having recourse to bilinear forms. Namely, by considering the bilinear Hermitian operator^{8,9,39} $\hat{t} = (-i\hbar \overleftrightarrow{\partial} / \partial E) / 2$, where the sign \leftrightarrow is defined through the accompanying equality $(f, \hat{t}g) = (f, -\frac{i\hbar}{2} \frac{\partial}{\partial E} g) + (-\frac{i\hbar}{2} \frac{\partial}{\partial E} f, g)$.

3.1. The Klein–Gordon case: Three-position operators

The standard position operators, being Hermitian and moreover self-adjoint, are known to possess real eigenvalues: i.e. they yield a *point-like* localization. J. M. Jauch showed, however, that a point-like localization would be in contrast with “unimodularity.” In the relativistic case, moreover, phenomena such as the pair production forbid a localization with precision better than one Compton wavelength. The eigenvalues of a realistic position operator \hat{z} are therefore expected to represent space *regions* rather than points. This can be obtained only by having recourse to non-Hermitian (and therefore non-self-adjoint) position operators \hat{z} (*a priori* one can have recourse either to nonnormal operators with commuting components, or to normal operators with noncommuting components). Following, e.g. the ideas in Refs. 51–55, we are going to show that the mean values of the *Hermitian (self-adjoint) part* of \hat{z} will yield a mean (point-like) position,^{56,57} while the mean values of the *anti-Hermitian (anti-self-adjoint) part* of \hat{z} will yield the sizes of the localization region.^{3,4}

Let us consider, e.g. the case of relativistic spin-zero particles, in natural units and with metric $(+ - - -)$. The position operator $i\nabla_{\mathbf{p}}$, is known to be actually non-Hermitian and may be in itself a good candidate for an extended-type position operator. To show this, we split^{51–55} it into its Hermitian and anti-Hermitian (or skew-Hermitian) parts.

Consider then a vector space V of complex differentiable functions on a 3D phase-space³⁹ equipped with an inner product defined by

$$(\Psi, \Phi) = \int \frac{d^3\mathbf{p}}{p_0} \Psi^*(\mathbf{p})\Phi(\mathbf{p}) \tag{33}$$

quantity p_0 being $\sqrt{\mathbf{p}^2 + m_0^2}$. Let the functions in V satisfy moreover the condition

$$\lim_{R \rightarrow \infty} \int_{S_R} \frac{dS}{p_0} \Psi^*(\mathbf{p})\Phi(\mathbf{p}) = 0, \tag{34}$$

where the integral is taken over the surface of a sphere of radius R . If $U : V \rightarrow V$ is a differential operator of degree one, condition (34) allows a definition of the transpose U^T by

$$(U^T \Psi, \Phi) = (\Psi, U\Phi) \quad \text{for all } \Psi, \Phi \in V, \tag{35}$$

where U is changed into U^T or vice versa by means of integration by parts.

This allows, further, to introduce a *dual representation*³⁹ (U_1, U_2) of a *single operator* $U_1^T + U_2$ by

$$(U_1 \Psi, \Phi) + (\Psi, U_2 \Phi) = (\Psi, (U_1^T + U_2)\Phi). \tag{36}$$

With such a dual representation, it is easy to split any operator into its Hermitian and anti-Hermitian parts

$$(\Psi, U\Phi) = \frac{1}{2}((\Psi, U\Phi) + (U^* \Psi, \Phi)) + \frac{1}{2}((\Psi, U\Phi) - (U^* \Psi, \Phi)). \tag{37}$$

Here the pair

$$\frac{1}{2}(U^*, U) \equiv \vec{U}_h, \tag{38}$$

corresponding to $(1/2)(U + U^{*T})$, represents the Hermitian part, while

$$\frac{1}{2}(-U^*, U) \equiv \vec{U}_a \tag{39}$$

represents the anti-Hermitian part.

Let us apply what precedes to the case of the Klein–Gordon position-operator $\hat{z} = i\nabla_{\mathbf{p}}$. When

$$U = i \frac{\partial}{\partial p_j} \tag{40}$$

we have^{5,7-9}

$$\frac{1}{2}(U^*, U) = \frac{1}{2} \left(-i \frac{\partial}{\partial p_j}, i \frac{\partial}{\partial p_j} \right) \equiv \frac{i}{2} \frac{\vec{\partial}}{\partial p_j}, \tag{41a}$$

$$\frac{1}{2}(-U^*, U) = \frac{1}{2} \left(i \frac{\partial}{\partial p_j}, i \frac{\partial}{\partial p_j} \right) \equiv \frac{i}{2} \frac{\vec{\partial}_{(+)}}{\partial p_j}. \tag{41b}$$

And the corresponding *single* operators turn out to be

$$\frac{1}{2}(U + U^{*T}) = i \frac{\partial}{\partial p_j} - \frac{i}{2} \frac{p_j}{\mathbf{p}^2 + m_0^2}, \tag{42a}$$

$$\frac{1}{2}(U - U^{*T}) = \frac{i}{2} \frac{p_j}{\mathbf{p}^2 + m_0^2}. \tag{42b}$$

It is noteworthy^{5,7-9} that, as we are going to see, operator (42a) is nothing but the usual Newton–Wigner operator, while (42b) can be interpreted^{5,31,51-55} as yielding the sizes of the localization-region (an ellipsoid) via its average values over the considered wave-packet.

Let us underline that the previous formalism justifies from the mathematical point of view the treatment presented in papers like Refs. 5, 9, 51–57. We can split^{5,7-9} the operator \hat{z} into two *bilinear* parts as follows:

$$\hat{z} = i \nabla_{\mathbf{p}} = \frac{i}{2} \overleftrightarrow{\nabla}_{\mathbf{p}} + \frac{i}{2} \overleftrightarrow{\nabla}_{\mathbf{p}}^{(+)}, \tag{43}$$

where $\Psi^* \overleftrightarrow{\nabla}_{\mathbf{p}} \Phi \equiv \Psi^* \nabla_{\mathbf{p}} \Phi - \Phi \nabla_{\mathbf{p}} \Psi^*$ and $\Psi^* \overleftrightarrow{\nabla}_{\mathbf{p}}^{(+)} \Phi \equiv \Psi^* \nabla_{\mathbf{p}} \Phi + \Phi \nabla_{\mathbf{p}} \Psi^*$, and where we always referred to a suitable^{5,9} space of wave-packets. Its Hermitian part^{9,5}

$$\hat{x} = \frac{i}{2} \overleftrightarrow{\nabla}_{\mathbf{p}}, \tag{44}$$

which was expected to yield an (ordinary) point-like localization, has been derived also by writing explicitly

$$(\Psi, \hat{x} \Phi) = i \int \frac{d^3 \mathbf{p}}{p_0} \Psi^*(\mathbf{p}) \nabla_{\mathbf{p}} \Phi(\mathbf{p}) \tag{45}$$

and imposing hermiticity, i.e. imposing the reality of the diagonal elements. The calculations yield

$$\Re(\Phi, \hat{x} \Phi) = \frac{i}{2} \int \frac{d^3 \mathbf{p}}{p_0} \Phi^*(\mathbf{p}) \overleftrightarrow{\nabla}_{\mathbf{p}} \Phi(\mathbf{p}), \tag{46}$$

suggesting to adopt just the Lorentz-invariant quantity (44) as a bilinear Hermitian position operator. Then, on integrating by parts (and due to the vanishing of the surface integral), we verify that Eq. (44) *is equivalent to the ordinary Newton–Wigner operator* (NW):

$$\hat{x}_h \equiv \frac{i}{2} \overleftrightarrow{\nabla}_{\mathbf{p}} \equiv i \nabla_{\mathbf{p}} - \frac{i}{2} \frac{\mathbf{p}}{\mathbf{p}^2 + m^2} \equiv \text{NW}. \tag{47}$$

We are left with the (bilinear) anti-Hermitian part

$$\hat{y} = \frac{i}{2} \overleftrightarrow{\nabla}_{\mathbf{p}}^{(+)} \tag{48}$$

whose *average values* over the considered state (wave-packet) can be regarded as yielding^{5,8,9,39,51-57} the sizes of an ellipsoidal localization-region.

After the digression associated with Eqs. (43)–(48), let us go back to our present formalism as expressed by Eqs. (33)–(42).

In general, the extended-type position operator \hat{z} will yield

$$\langle \Psi | \hat{z} | \Psi \rangle = (\boldsymbol{\alpha} + \Delta\boldsymbol{\alpha}) + i(\boldsymbol{\beta} + \Delta\boldsymbol{\beta}), \quad (49)$$

where $\Delta\boldsymbol{\alpha}$ and $\Delta\boldsymbol{\beta}$ are the mean-errors encountered when measuring the point-like position and the sizes of the localization region, respectively. It is interesting to evaluate the commutators ($i, j = 1, 2, 3$):

$$\left(\frac{i}{2} \frac{\overrightarrow{\partial}}{\partial p^i}, \frac{i}{2} \frac{\overrightarrow{\partial}_{(+)}}{\partial p^j} \right) = \frac{i}{2p_0^2} \left(\delta_{ij} - \frac{2p_i p_j}{p_0^2} \right), \quad (50)$$

wherefrom the noticeable “uncertainty correlations” follow:

$$\Delta\alpha_i \Delta\beta_j \geq \frac{1}{4} \left| \left\langle \frac{1}{p_0^2} \left(\delta_{ij} - \frac{2p_i p_j}{p_0^2} \right) \right\rangle_{\Psi} \right|. \quad (51)$$

3.2. Four-position operators

It is tempting to propose as *four-position operator* the quantity $\hat{z}^\mu = \hat{x}^\mu + i\hat{y}^\mu$, whose Hermitian (Lorentz-covariant) part can be written as

$$\hat{x}^\mu = -\frac{i}{2} \frac{\overrightarrow{\partial}}{\partial p_\mu}, \quad (52)$$

in order to be associated with its corresponding “operator” in four-momentum space

$$\hat{p}_h^\mu = +\frac{i}{2} \frac{\overrightarrow{\partial}}{\partial x_\mu}. \quad (53)$$

Let us recall the proportionality between the 4-momentum operator and the 4-current density operator in the chronotopical space, and then underline the canonical correspondence (in the 4-position and 4-momentum spaces, respectively) between the “operators” (cf. the previous subsection):

$$m_0 \hat{\rho} \equiv \hat{p}_0 = \frac{i}{2} \frac{\overrightarrow{\partial}}{\partial t}, \quad (54a)$$

$$m_0 \hat{\mathbf{j}} \equiv \hat{\mathbf{p}} = -\frac{i}{2} \frac{\overrightarrow{\partial}}{\partial \mathbf{r}}, \quad (54b)$$

and the operators

$$\hat{t} \equiv -\frac{i}{2} \frac{\overrightarrow{\partial}}{\partial p_0}, \quad (55a)$$

$$\hat{\mathbf{x}} \equiv \frac{i}{2} \frac{\overrightarrow{\partial}}{\partial \mathbf{p}}, \quad (55b)$$

where the four-position “operator” (55) can be considered as a 4-current density operator in the energy-impulse space. Analogous considerations can be carried on for the anti-Hermitian parts (see Ref. 5).

Finally, by recalling the properties of the time operator as a maximal Hermitian operator in the nonrelativistic case (Subsec.2.1), one can see that the relativistic time operator (55a) (for the Klein–Gordon case) is also a self-adjoint bilinear operator for the case of continuous energy spectra, and a (maximal) Hermitian linear operator for free particles [due to the presence of the lower limit zero for the kinetic energy, or m_0 for the total energy].

4. Unstable States and Non-Hermitian Hamiltonians

4.1. Introduction

This whole section is based on work performed in collaboration with A. Agodi, M. Baldo and A. Pennisi di Floristella.⁵⁸

In quantum mechanics the “resonance” peaks are generally described as corresponding to unstable states (remember, e.g. Schwinger’s approach⁵⁹). The present attempt proceeds as follows: (i) singling out *one* state $|\varphi\rangle$ in the state space; (ii) finding out the effect of the (internal, virtual) state $|\varphi\rangle$ on the transition-amplitude; (iii) finding, in particular, the necessary conditions for $|\varphi\rangle$ to be connected with a resonance in the cross-section. In this way one can associate the “resonant state” with the eigenvectors of a non-Hermitian Hamiltonian (or, rather, for simplicity, of a “*quasi*-Hermitian” Hamiltonian), such eigenvectors decaying correctly in time. We shall adopt the formalism introduced by Akhiezer and Glazman,⁴² Lifshitz, Galitsky and Migdal,⁶⁰ and Agodi *et al.*^{61–63}

Chosen a state $|\varphi\rangle$, let us define the projectors

$$P \equiv |\varphi\rangle\langle\varphi|, \quad Q = \mathbb{1} - P, \tag{56}$$

where $\mathbb{1}$ is the identity operator.

4.2. Preliminary case: Time-dependent description of potential scattering

Let us preliminarily consider the time-dependent description of potential scattering. Quantity V be the potential operator. In the limiting case of plane-waves, the scattering amplitude writes

$$T(\mathbf{k}, \mathbf{k}') = \langle\mathbf{k}'|V|\mathbf{k}\rangle + \langle\mathbf{k}'|VG(E^+)V|\mathbf{k}\rangle \tag{57}$$

with

$$G(E^+) \equiv (E^+ - H)^{-1}, \quad E^\pm \equiv E \pm i\varepsilon. \tag{58}$$

Choosing the *exploring* vector $|\varphi\rangle$ and using definition (56), we have

$$H = \overset{0}{H} + \overset{1}{H}, \quad \overset{0}{H} = QHQ, \quad \overset{1}{H} = PHP + PHQ + QHP. \tag{59}$$

By introducing the scattering states $|\psi^0\rangle$ due to $\overset{0}{H}$:

$$|\psi_{\mathbf{k}}^{\pm}\rangle = \left\{ 1 + \frac{1}{E^{\pm} - \overset{0}{H}} \left(\overset{0}{H} - E \right) \right\} |\mathbf{k}\rangle, \quad (60)$$

we obtain

$$\begin{aligned} S(\mathbf{k}, \mathbf{k}') &\equiv \langle \psi_{\mathbf{k}'}^{(-)} | \psi_{\mathbf{k}}^{(+)} \rangle = \langle \overset{0}{\psi}_{\mathbf{k}'}^{(-)} | \overset{0}{\psi}_{\mathbf{k}}^{(+)} \rangle - 2\pi\delta(E_{k'} - E_k) \\ &\quad \times \langle \overset{0}{\psi}_{\mathbf{k}'}^{(-)} | HPG(E_k^+)PH | \overset{0}{\psi}_{\mathbf{k}}^{(+)} \rangle, \end{aligned} \quad (61)$$

where the first addendum on the r.h.s. of Eq. (61) (let us call it A) is the contribution coming from processes developing entirely in the subspace onto which Q projects, whilst the second addendum (B) is contributed by processes going through the exploring state $|\varphi\rangle$ onto which P projects. In other words, the processes with $|\varphi\rangle$ as intermediate state correspond to the term

$$(2\pi\delta(E_{k'} - E_k))^{-1} B = -2\pi i \frac{\langle \overset{0}{\psi}_{\mathbf{k}'}^{(-)} | H |\psi\rangle \langle \varphi | H | \overset{0}{\psi}_{\mathbf{k}}^{(+)} \rangle}{E_k^+ - \langle \varphi | H | \varphi \rangle - \langle \varphi | W^\varphi(E_k^+) | \varphi \rangle}, \quad (62)$$

$$W^\varphi(z) = PHQ \frac{1}{z - QHQ} QHP.$$

Our problem is: under what conditions one (or more) resonances are actually associated with the chosen $|\varphi\rangle$? Let us notice in particular that, if $E_\varphi \equiv \langle \varphi | H | \varphi \rangle - \Re \langle \varphi | W^\varphi(E^+) | \varphi \rangle$ and $\Gamma_\varphi \equiv \Im \langle \varphi | W^\varphi(E^+) | \varphi \rangle$ are smooth functions of E , then B gets just the ‘‘Breit and Wigner’’ form:

$$B = -2\pi i \frac{\langle \overset{0}{\psi}_{\mathbf{k}'}^{(-)} | HPH | \overset{0}{\psi}_{\mathbf{k}}^{(+)} \rangle}{E - E_\varphi + i\Gamma_\varphi}. \quad (63)$$

4.3. Case of central potential and spin-free particles

Let us choose the angular-momentum representation. If $|\varphi\rangle$ is assumed to be in particular invariant under $O(3)$, then both terms in which S was split are diagonal. If $\overset{0}{\delta}_\ell$ are the phase-shifts due to QHQ and μ is the reduced mass, then

$$S_\ell(k) \equiv \exp(2i\delta_\ell(k)) = F_\ell(k) \exp\left(2i\overset{0}{\delta}_\ell(k)\right), \quad (64)$$

with $\ell \equiv l$

$$F_\ell(k) \equiv 1 - \frac{2\pi i \mu}{\hbar^2 k} \frac{\left| \langle \varphi | H | \overset{0}{\psi}_{klm}^{(+)} \rangle \right|^2}{E^+ - \langle \varphi | H | \varphi \rangle - \langle \varphi | W^\varphi(E_k^+) | \varphi \rangle}. \quad (65)$$

Let us observe that the phase-shift of $F_\ell(k)$ crosses the value $\frac{1}{2}\pi$ (with positive slope) when

$$F_\ell(k) = -1. \quad (66)$$

The condition for a resonance to appear are particularly transparent for $l = 0$:

$$F_0(k) = \frac{E - E_\varphi - i\lambda_0(k)}{E - E_\varphi + i\lambda_0(k)}, \tag{67}$$

when

$$\lambda_0(k) \equiv -\Im\langle\varphi|W^\varphi(E_k^+)|\varphi\rangle = \left|\langle\varphi|H\left|\psi_{k00}^{0(+)}\right\rangle\right|^2 \tag{68}$$

is positive-definite. Namely, the condition $F_0(k) = -1$ yields

$$|1 - S_0(k)|^2 = 4\cos^2\delta_0^0, \tag{69}$$

with the supplementary conditions $\lambda_0(k) \neq 0$ and $\cos\delta_0^0 \neq 0$. When $\cos\delta_0^0 \simeq 1$ the scattering due to QHQ is negligible, i.e. the scattering proceeds entirely via the intermediate formation of the (quasibound) state $|\varphi\rangle$; and the possible resonant effects are really related to $|\varphi\rangle$. Of course, $\cos\delta_0^0 \simeq 1$ when, *at the resonance*

$$E = E_\varphi, \quad F(k) = -1, \tag{70}$$

it is $\left|\psi_{klm}^{0(\pm)}\right\rangle = |lm\rangle$.

Notice that with every fixed $|\varphi\rangle$ a *series* of resonances (also for different values of l) may be *a priori* associated, if they are not destroyed by the δ_0^0 behavior.

4.4. Resonance definition

It is essential to recognize that the “resonance condition” $F_l(k) = -1$ may be written⁵⁸ [$l \equiv l$]

$$1 - \alpha(k, l)\langle\varphi_\ell|G(E^+)|\varphi_\ell\rangle = 0, \tag{71}$$

with

$$\alpha(k, l) = \frac{i\pi\mu}{\hbar^2k} \left|\langle\varphi_\ell|H\left|\psi_{klm}^{0(+)}\right\rangle\right|^2. \tag{72}$$

Let us now study the more general equation

$$1 - \lambda\langle\varphi_\ell|G(z)|\varphi_\ell\rangle = 0, \tag{73}$$

where z, λ are complex numbers. Of course, a resonance will appear at $\Re(z)$ if z is near the real axis and if

$$\lambda \simeq \alpha(k, l),$$

both satisfying Eq. (73).

If we introduce at this point the non-Hermitian (“quasi-Hermitian”) Hamiltonian-operator

$$\mathcal{H} \equiv H + \lambda P, \tag{74}$$

where λ is complex and its “resolvent operator” is

$$\mathcal{G}(z) = \frac{1}{z - \mathcal{H}}, \tag{75}$$

then Eq. (73) becomes

$$\frac{\langle \varphi_\ell | \mathcal{G}(z) | \varphi_\ell \rangle}{\langle \varphi_\ell | \mathcal{G}(z) | \varphi_\ell \rangle} = 0. \tag{76}$$

In other words, studying the (necessary) conditions for the resonance appearance is just equivalent to find out the poles in the diagonal elements of the “resolvent” \mathcal{G} -matrix, i.e. the eigenvalues of the *quasi-self-adjoint* operator \mathcal{H} . Notice that, since

$$\mathcal{G} = G + G \frac{\lambda P}{1 - \lambda \langle \varphi_\ell | G(z) | \varphi_\ell \rangle} G, \tag{77}$$

the difference between the spectra of H and \mathcal{H} is just the presence of complex eigenvalues (corresponding to the solution of our “condition” (76)).

Therefore, in our framework the “resonant (decaying) state” $|\psi\rangle$ is expected to be an eigenvector of \mathcal{H} (notice that it does not coincide with the state $|\varphi\rangle$ which is not unstable!), corresponding to the complex energy \mathcal{E} .

4.5. Applications

Let us confine ourselves to the case $l = 0$ and rewrite the non-Hermitian (quasi-self-adjoint) Hamiltonian as

$$\mathcal{H} \equiv H + i\alpha_k |\varphi\rangle \langle \varphi|, \quad \alpha_k = -i\alpha(k, 0), \tag{78}$$

where

$$V_\varphi = i\alpha_k |\varphi\rangle \langle \varphi| \tag{79}$$

is anti-Hermitian. We shall therefore write

$$(H - \mathcal{E})|\psi\rangle = -V_\varphi|\psi\rangle \equiv -|\varphi\rangle i\alpha_k \langle \varphi|\psi\rangle \tag{80}$$

which immediately yields for the eigenvalues the “dispersion-type relation” ($\mathcal{E} \equiv \mathcal{E}_\varphi$):

$$1 + i\langle \varphi| \frac{1}{H - \mathcal{E}} |\varphi\rangle \alpha_k = 0, \tag{81}$$

and for the eigenvectors the explicit expression

$$|\psi\rangle = -\langle \varphi|\psi\rangle i\alpha_k \frac{1}{H - \mathcal{E}} |\varphi\rangle, \tag{82}$$

where $\langle \varphi | \psi \rangle$ is a normalization constant. Notice that to solve eq. (81) we do not need to know α_k , i.e. the scattering states due to QHQ , since fortunately *at the resonances* it is ($E = E_R$):

$$\alpha_k \propto \left| \langle \varphi | H \left| \psi_{k00}^{(+)} \right. \right|^2 = \left| \langle \varphi | \psi_E^{(+)} \rangle - \langle \varphi | k00 \rangle \right|^{-2}.$$

Notice moreover that the present approach, *a priori*, allows distinguishing between true resonances and other effects.

In Ref. 58 the application was considered for the case of scattering by a spherical-well potential $U(r) = U_0 \theta(a - r)$, and as exploring states the class was adopted of the normalized Laurentian wave-packets (good for low energies)

$$\langle k00 | \varphi \rangle = \sqrt{2b} \frac{1}{k^2 + b^2} \Leftrightarrow \langle \mathbf{r} | \varphi \rangle = \sqrt{\frac{b}{2\pi}} \frac{\exp(-br)}{r}.$$

By integration, for low energies ($k^2 \ll 2mU_0$) one gets one equation whose real and imaginary parts forward a system of two equations. The latter ones individuate $|\varphi\rangle$, i.e. the parameter b , for which a series of (true) resonances arises. These resonances are expected to appear for ($k^2 = 2mE$, $K^2 = 2m(E + U_0)$)

$$\cos Ka = 0 \rightarrow K = \left(n + \frac{1}{2} \right) \pi.$$

This system of equations is rather complicated (even when the resonance width is $\gamma < k_0$). But the first equation does not contain γ and yields b . For instance, for $n = 0$ one gets a single solution ($ab \simeq 0.69$).

4.6. Decay of the unstable state

We are more interested in the decay in time of the unstable state $|\psi\rangle$:

$$\langle \psi | \psi_t \rangle \equiv \langle \psi | U_t | \psi \rangle \equiv \langle \psi | \exp(-iOt) | \psi \rangle. \tag{83}$$

If we assume, as usual, $O = H$, then

$$\langle \psi | \psi_t \rangle = \int_0^\infty \left| \langle \psi | \psi_E^{(+)} \rangle \right|^2 \exp(-iEt) dE \tag{84}$$

since the bound-states do not contribute for large t . Moreover, let us remember that

$$|\psi\rangle = -i\alpha_k \langle \varphi | \psi \rangle \frac{1}{H - \mathcal{E}} |\varphi\rangle.$$

Therefore,

$$\left| \langle \psi_E^{(+)} | \psi \rangle \right|^2 = C \frac{|\alpha_k|^2}{(\Re(\mathcal{E}) - E)^2 - (\Im(\mathcal{E}))^2}, \quad C \equiv \left| \langle \psi_E^{(+)} | P | \psi \rangle \right|^2.$$

The integral (84) can be evaluated following Ref. 60. The expression C contains denominators that — analytically extended — produce *one* pole in $E = \mathcal{E}$. If in the strip $\Im(\mathcal{E}) < \Im(E) < 0$ no other singularities arise from the remaining factors, then we obtain the exponential-type decay

$$\langle \psi | \psi_t \rangle = (C + Dt) \exp(-iE_0 t - \gamma_0 t) \quad (85)$$

with $E_0 \equiv \Re(\mathcal{E})$, $\gamma_0 \equiv \Im(\mathcal{E})$, C and D *constants*.

More interesting appears, however, the assumption

$$O = \mathcal{H} \quad (86)$$

since in this case our approach does surely possess a “Lie-admissible” structure⁶⁴ (due to the fact that the time-evolution operator with \mathcal{H} is not unitary). In such case one would simply get

$$\langle \psi | \psi_t \rangle = \bar{K} \exp(iE_0 t + \gamma_0 t) \quad (87)$$

with $\bar{K} \equiv \langle \psi | \psi \rangle$. But in this case the whole approach ought to be carefully rephrased in Lie-admissible terms (otherwise, e.g., *all* states would seem to be decaying).

For instance, following Ref. 64 and with the choice (86), we may look for a Hermitian operator \tilde{H} such that the operator evolution law becomes

$$A(t) = e^{it\mathcal{H}} A(0) e^{-it\mathcal{H}^\dagger} = e^{it\tilde{H}S} A(0) e^{-itR\tilde{H}},$$

where $\tilde{H}^\dagger = \tilde{H}$; $S^\dagger = R$. In particular, let us identify \tilde{H} with H . One can verify, first of all, that it does *not* seem possible to come down to the Lie-isotopic case,⁶⁴ in which $R = S \equiv g$ with $g^\dagger = g$. In fact, in that case the operator g ought to be

$$g = \mathbb{1} + i\alpha H^{-1} |\varphi\rangle\langle\varphi| = \mathbb{1} - i\alpha |\varphi\rangle\langle\varphi| H^{-1}$$

and this would imply $\langle\varphi|H|\varphi\rangle = \langle\varphi|H^{-1}|\varphi\rangle = 0$.

In the more general (“Lie-admissible”) case, however, the relations ($S^\dagger \neq S$)

$$\mathcal{H} = H + i\alpha |\varphi\rangle\langle\varphi| = HS, \quad \mathcal{H}^\dagger = H - i\alpha |\varphi\rangle\langle\varphi| = S^\dagger H$$

allow us to write immediately

$$R = S^\dagger = \mathbb{1} - i\alpha |\varphi\rangle\langle\varphi| H^{-1}.$$

5. Further Examples of Non-Hermitian Hamiltonians:

The Cases of the Nuclear Optical Model, and of Microscopic Quantum Dissipation

5.1. Nuclear optical model

Since the fifties, the so-called optical model has been frequently used for describing the experimental data on nucleon–nucleus elastic scattering, and, not less, on more general nuclear collisions: see, e.g. Refs. 65–68; while for a generalized optical model — namely, for the coupled-channel method with an optical model in

any channel of the nucleon–nucleus (elastic or inelastic) scattering—, one can see Ref. 69 and references therein.

In the previous cases, the Hamiltonian contains a complex potential, its imaginary part describing the absorption processes that take place via compound-nucleus formation and subsequent decay. As to the Hamiltonian with complex potential, here we just refer the reader to work of ours already published, where it was studied the nonunitarity and analytical structure of the S -matrix, the completeness of the wave-functions, and so on: see Ref. 70, and also Refs. 71, 72.

5.2. Microscopic quantum dissipation

Various different approaches are known, aimed at getting dissipation — and possibly decoherence — within quantum mechanics. First of all, the simple introduction of a “chronon” (see, e.g. Refs. 73–79) allows one to go on from differential to finite-difference equations, and in particular to write down the quantum theoretical equations (Schrödinger’s, Liouville–von Neumann’s, etc.) in three different ways: symmetrical, retarded and advanced. The retarded “Schrödinger” equation describes in a rather simple and natural way a dissipative system, which exchanges (loses) energy with the environment. The corresponding nonunitary time-evolution operator obeys a semigroup law and refers to irreversible processes. The retarded approach furnishes, moreover, an interesting way for solving the “measurement problem” in quantum mechanics without any need for a wave-function collapse (and without having recourse to any statistical approach): see Refs. 78–82. The chronon theory can be regarded as a peculiar “coarse grained” description of the time evolution.

Let us stress that it has been shown that the mentioned discrete approach can be replaced with a continuous one at the price of introducing a *non-Hermitian* Hamiltonian: see, e.g. Ref. 83.

Further relevant work can be found, for instance, in papers like Refs. 84–94 and references therein.

Let us add at this point that much work is still needed for the description of time irreversibility at the microscopic level. Indeed, various approaches have been proposed, in which new parameters are introduced (regulation or dissipation) into the microscopic dynamics (building a bridge, in a sense, between microscopic structure and macroscopic characteristics). Besides the Caldirola–Kanai^{95,96} Hamiltonian

$$\hat{H}_{\text{CK}}(t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} e^{-\gamma t} + V(x)e^{\gamma t} \tag{88}$$

(which has been used, e.g. in Ref. 97), other rather simple approaches, based of course on the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \Psi(x, t) = \hat{H} \Psi(x, t), \tag{89}$$

and adopting a microscopic dissipation defined via a coefficient of extinction γ , are for instance the following ones:

(a) *Nonlinear (non-Hermitian) Hamiltonians*

$$\hat{H}_{nl} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) + \hat{W}, \quad (90)$$

with “potential” operators \hat{W} of the type:

(i) Kostin’s operator (see Ref. 98):

$$\hat{W}_K = -\frac{i\hbar}{2m} \left\{ \frac{\ln \Psi}{\Psi^*} - \left\langle \frac{\ln \Psi}{\Psi^*} \right\rangle \right\}; \quad (91)$$

(ii) Albrecht’s operator (see Ref. 99):

$$\hat{W}_A(x) = \langle p \rangle (x - \langle x \rangle), \quad (92)$$

where $\langle \rangle$ is average over $|\Psi(x)|^2$;

(iii) Ref. 100:

$$\hat{W}_H(x) = \frac{1}{4} [x - \langle x \rangle, p + \langle p \rangle]_+, \quad (93)$$

where $[A, B]_+$ is the anticommutator: $[A, B]_+ = AB + BA$.

(b) *Linear (non-Hermitian) Hamiltonians:*

(i) Ref. 101:

$$\hat{H}_G = (1 - i\gamma)\hat{H} + i\gamma\langle\hat{H}\rangle; \quad (94)$$

(ii) Ref. 102:

$$\hat{H}_E = \hat{H} + i\hat{W}(x) - i\langle\hat{W}(x)\rangle. \quad (95)$$

One might recall also the important, so-called “microscopic models”,¹⁰³ even if they are not based on the Schrödinger equation.

All such proposals are to be further investigated and completed since they have not been apparently exploited enough till now. Let us remark just as an example that it would be desirable to take into deeper consideration other related phenomena like the ones associated with the “Hartman effect” (and “generalized Hartman effect”),^{24–26,104–107} in the case of tunneling with dissipation: a topic faced in few further papers like Refs. 108 and 109.

As a small contribution of ours, in the Appendix we present a scheme of iterations (successive approximations) as a possible tool for explicit calculations of wavefunctions in the presence of dissipation, by using as an example the simple Albrecht’s potential. Our scheme may be useful, in any case, for the investigation of possible violations of the Hartman effect,^{104–107} as well as for analyzing a few irreversible phenomena. See the Appendix.

At last, let us incidentally recall that two generalized Schrödinger equations introduced by Caldirola^{85,110–112} in order to describe two different dissipative processes (behavior of open systems, and the radiation of a charged particle) have been shown — see, e.g. Ref. 113 — to possess the same algebraic structure of the Lie-admissible type.¹¹⁴

6. Some Conclusions

- (i) We have shown that the time *operator* (1), Hermitian even if non-self-adjoint, works for any quantum collisions or motions, in the case of a continuum energy spectrum, in nonrelativistic quantum mechanics and in 1D quantum electrodynamics. The uniqueness of the (maximal) Hermitian time operator (1) directly follows from the uniqueness of the Fourier-transformations from the time to the energy representation. The time operator (1) has been fruitfully used in the case, for instance, of tunneling times (see Refs. 24–28), and of nuclear reactions and decays (see Refs. 10–13 and also Ref. 115). We have discussed the advantages of such an approach with respect to POVM's, which is not applicable for 3D particle collisions, within a wide class of Hamiltonians.

The mathematical properties of the present time operator have actually demonstrated — without introducing any new physical postulates — that *time* can be regarded as a quantum-mechanical observable, at the same degree of other physical quantities (energy, momentum, spatial coordinates, ...).

The commutation relations (Eqs. (8), (22), (31)) here analyzed, and the uncertainty relations (9), result to be analogous to those known for other pairs of canonically conjugate observables (as for coordinate \hat{x} and momentum \hat{p}_x , in the case of Eq. (9)). Of course, our new relations do not replace, but merely extend the meaning of the classic time and energy uncertainties, given e.g. in Ref. 40. In Subsec. 2.6, we have also studied the properties of time, as an observable, for quantum-mechanical systems with *discrete* energy spectra.

- (ii) Let us recall that the time operator (1) and relations (2)–(4), (15), (16), have been used for the temporal analysis of nuclear reactions and decays in Refs. 10–13; as well as of new phenomena, about time delays–advances in nuclear physics and about time resonances or explosions of highly excited compound nuclei in Refs. 115–118. Let us also recall that, besides the time operator, other quantities, to which (maximal) Hermitian operators correspond, can be analogously regarded as quantum-physical observables: for example, von Neumann himself^{8,9,44} considered as physical the case of the momentum operator $-i\partial/\partial x$ in a semispace with a rigid wall orthogonal to the x -axis at $x = 0$, or of the radial momentum $-i\partial/\partial r$, even if both act on packets defined only over the positive x or r axis, respectively, and are not therefore self-adjoint.

Subsection 2.5 has been devoted to a new “Hamiltonian approach”: namely, to the important, interesting introduction of the analogue of the “Hamiltonian” for the case of the time operator.

- (iii) In Sec. 3, we have proposed a suitable generalization for the time operator (or, rather, for a *space–time operator*) in relativistic quantum mechanics. For instance, for the Klein–Gordon case, we have shown that the Hermitian part of the three-position operator \hat{x} is nothing but the Newton–Wigner operator, and corresponds to a point-like position; while its anti-Hermitian part can be regarded as yielding the sizes of an extended-type (ellipsoidal) localization. When dealing with a 4-position operator, one finds that the time operator is self-adjoint for unbounded energy spectra, while it is a (maximal) Hermitian operator when the kinetic energy, and the total energy, are bounded from below, as for a free particle. We have extensively made recourse, in the latter case, to *bilinear* forms, which dispense with the necessity of eliminating the lower point — corresponding to zero velocity — of the spectra. It would be interesting to proceed to further generalizations of the 3- and 4-position operator for other relativistic cases, and analyze the localization problems associated with Dirac particles, or in 2D and 3D quantum electrodynamics, etc. Work is in progress on time analyses in 2D quantum electrodynamic, for application, e.g., to frustrated (almost total) internal reflections. Further work has still to be done also about the joint consideration of particles and antiparticles.
- (iv) Section 4 has been devoted to the association of unstable states (decaying “resonances”) with the eigenvectors of quasi-Hermitian^{39,58,115} Hamiltonians.
- (v) Non-Hermitian Hamiltonians, and nonunitary time-evolution operators, can play an important role also in microscopic quantum dissipation:^{73–94} namely, in getting decoherence through interaction with the environment.^{78,79} This topic is touched in Sec. 5; together with questions related with collisions in absorbing media. In particular, in Sec.5 we mention also the case of the optical model in nuclear physics; without forgetting that non-Hermitian operators show up even in the case of tunneling — e.g. in fission phenomena — with quantum dissipation, and of quantum friction. As to the former topic of microscopic quantum dissipation, among the many approaches to quantum irreversibility we have discussed in Subsec. 5.2 a possible solution of the quantum measurement problem (via interaction with the environment) by the introduction of finite-difference equations (e.g. in terms of a “chronon”).
- (vi) Let us eventually observe that the “dual equations” (26) and (27) seem to be promising also for the study the initial stage of our cosmos, when tunnelings can take place through the barriers which appear in quantum gravity in the limiting case of quasi-Schrödinger equations.

Acknowledgments

Part of this paper is based on work performed by one of us in cooperation with P. Smrz, and with A. Agodi, M. Baldo and A. Pennisi di Floristella. Thanks are moreover due for stimulating discussions, or kind collaboration, to Y. Aharonov,

G. Battistoni, R. H. A. Farias, L. Fraietta, A. S. Holevo, V. L. Lyuboshitz, R. Mignani, V. Petrillo, P. Pizzochero, G. Salesi, B. N. Zakhariev and M. Zamboni-Rached.

Appendix. Time-Dependent Schrödinger Equation with Dissipative Terms

A.1. Introduction

Let us consider the time-dependent Schrödinger equation

$$i \frac{\partial}{\partial t} \Psi(x, t) = \left(-\frac{\partial^2}{\partial x^2} + V(x, t) \right) \Psi(x, t), \tag{A.1}$$

where we put $\hbar = 1$. Let us rewrite the time-dependent wave function (WF), $\Psi(x, t)$ (which can be considered as a wave-packet (WP)), in the form of a Fourier integral

$$\Psi(x, t) = \int_0^{E_0} g(E) e^{-iEt} \varphi(E, x) dE, \tag{A.2}$$

where $\varphi(E, x)$ is the WF component independent of time and $g(E)$ is a weight factor. One can choose the function $g(E)$ to be, e.g., a *Gaussian*:

$$g(E) = A e^{-a^2(k-\bar{k})^2}. \tag{A.3}$$

Here, A and a are constants, and \bar{k} is the selected value for the impulse, constituting the center of the WP; we shall call \bar{E} the energy corresponding to \bar{k} . We substitute the Fourier-expansion (A.2) of WF into Eq. (A.1). Thus, the l.h.s. of this equation transforms into

$$i \frac{\partial}{\partial t} \Psi(x, t) = \int_0^{E_0} g(E) e^{-iEt} \varphi(E, x) E dE. \tag{A.4}$$

Afterwards, the r.h.s. of Eq. (A.1) gets transformed into

$$\begin{aligned} \left(-\frac{\partial^2}{\partial x^2} + V(x, t) \right) \Psi(x, t) &= - \int_0^{E_0} g(E) e^{-iEt} \frac{\partial^2 \varphi(E, x)}{\partial x^2} dE \\ &+ \int_0^{E_0} g(E) V(x, \bar{E}, t) e^{-iEt} \varphi(E, x) dE. \end{aligned} \tag{A.5}$$

Therefore, the whole Eq. (A.1) has been transformed into

$$\begin{aligned} \int_0^{E_0} g(E) e^{-iEt} \varphi(E, x) E dE &= - \int_0^{E_0} g(E) e^{-iEt} \frac{\partial^2 \varphi(E, x)}{\partial x^2} dE \\ &+ \int_0^{E_0} g(E) V(x, \bar{E}, t) e^{-iEt} \varphi(E, x) dE. \end{aligned} \tag{A.6}$$

Let us now apply the inverse Fourier-transformation to this equation. Its left part becomes

$$\begin{aligned} & \frac{1}{2\pi} \int dt e^{iE't} \int_0^{E_0} g(E) e^{-iEt} \varphi(E, x) E dE \\ &= \frac{1}{2\pi} \int_0^{E_0} dE E g(E) \varphi(E, x) \int e^{i(E'-E)t} dt \\ &= \int_0^{E_0} g(E) \varphi(E, x) \delta(E' - E) E dE = g(E') E' \varphi(E', x), \end{aligned} \tag{A.7}$$

while its right part becomes

$$\begin{aligned} & -\frac{1}{2\pi} \int dt e^{iE't} \int_0^{E_0} g(E) e^{-iEt} \frac{\partial^2 \varphi(E, x)}{\partial x^2} dE \\ &+ \frac{1}{2\pi} \int dt e^{iE't} \int_0^{E_0} g(E) V(x, \bar{E}, t) e^{-iEt} \varphi(E, x) dE \\ &= -\frac{1}{2\pi} \int_0^{E_0} dE g(E) \frac{\partial^2 \varphi(E, x)}{\partial x^2} \int e^{i(E'-E)t} dt \\ &+ \frac{1}{2\pi} \int_0^{E_0} dE g(E) \varphi(E, x) \int V(x, \bar{E}, t) e^{i(E'-E)-t} dt \\ &= -g(E') \frac{\partial^2 \varphi(E', x)}{\partial x^2} + \frac{1}{2\pi} \int_0^{E_0} dE g(E) \varphi(E, x) \int V(x, \bar{E}, t) e^{i(E'-E)t} dt. \end{aligned} \tag{A.8}$$

As a result, we obtain Eq. (A.6) in the form

$$\begin{aligned} g(E') E' \varphi(E', x) &= -g(E') \frac{\partial^2 \varphi(E', x)}{\partial x^2} \\ &+ \frac{1}{2\pi} \int_0^{E_0} dE g(E) \varphi(E, x) \int V(x, \bar{E}, t) e^{i(E'-E)t} dt. \end{aligned} \tag{A.9}$$

A.2. The case of the simple Albreht's potential

Just as an example of a possible potential $V(x, t)$, let us choose

$$V(x, t) = V_0(x) + \gamma W_A(x), \tag{A.10}$$

where $W_A(x)$ is the simple Albreht's dissipation term. Here, γ is a constant, $V_0(x)$ is the usual stationary component of $V(x)$, and the dissipative component of $V(x)$ has the form

$$W_A(x) = \langle p \rangle (x - \langle x \rangle), \tag{A.11}$$

where the averages are fulfilled by integrating over x by means of the functions $\Psi^*(x, t)$ and $\Psi(x, t)$. For the right part of Eq. (A.11) one gets

$$\begin{aligned} \langle p \rangle &= -i \int dx \int_0^{E_0} dE_1 \int_0^{E_0} dE_2 g(E_1)g(E_2) \\ &\quad \times e^{i(E_1-E_2)t} \varphi^*(E_1, x) \frac{\partial \varphi(E_2, x)}{\partial x}, \end{aligned} \tag{A.12}$$

$$\begin{aligned} \langle x \rangle &= \int dx \int_0^{E_0} dE_3 \int_0^{E_0} dE_4 g(E_3)g(E_4) \\ &\quad \times e^{i(E_3-E_4)t} x \varphi^*(E_3, x) \varphi(E_4, x), \end{aligned}$$

so that the total potential $V(x, t)$ becomes

$$\begin{aligned} V(x, t) &= V_0(x) - i\gamma \int dx_1 \int dx_2 \int_0^{E_0} dE_1 \int_0^{E_0} dE_2 \\ &\quad \times \int_0^{E_0} dE_3 \int_0^{E_0} dE_4 g(E_1)g(E_2)g(E_3)g(E_4) \\ &\quad \times e^{i(E_1-E_2+E_3-E_4)t} (x - x_2) \varphi^*(E_1, x_1) \\ &\quad \times \frac{\partial \varphi(E_2, x_1)}{\partial x_1} \varphi^*(E_3, x_2) \varphi(E_4, x_2). \end{aligned} \tag{A.13}$$

Taking into account this, we find the second term, on the r.h.s. of Eq. (A.9), to be

$$\begin{aligned} &\frac{1}{2\pi} \int_0^{E_0} dE g(E) \varphi(E, x) \int V(x, \bar{E}, t) e^{i(E'-E)t} dt \\ &= \int_0^{E_0} dE g(E) \varphi(E, x) V_0(x) \delta(E' - E) - i\gamma \int_0^{E_0} dE g(E) \varphi(E, x) \int dx_1 \int dx_2 \\ &\quad \times \int_0^{E_0} dE_1 \int_0^{E_0} dE_2 \int_0^{E_0} dE_3 \int_0^{E_0} dE_4 g(E_1)g(E_2)g(E_3)g(E_4) (x - x_2) \varphi^*(E_1, x_1) \\ &\quad \times \frac{\partial \varphi(E_2, x_1)}{\partial x_1} \varphi^*(E_3, x_2) \varphi(E_4, x_2) \delta(E' - E + E_1 - E_2 + E_3 - E_4) \\ &= g(E') \varphi(E', x) V_0(x) - i\gamma \int dx_1 \int dx_2 \\ &\quad \times \int_0^{E_0} dE_1 \int_0^{E_0} dE_2 \int_0^{E_0} dE_3 \int_0^{E_0} dE_4 g(E_1)g(E_2)g(E_3)g(E_4) g(E'') \\ &\quad \times (x - x_2) \varphi^*(E_1, x_1) \frac{\partial \varphi(E_2, x_1)}{\partial x_1} \varphi^*(E_3, x_2) \varphi(E_4, x_2) \varphi(E'', x), \end{aligned} \tag{A.14}$$

where

$$E'' = E' + E_1 - E_2 + E_3 - E_4. \tag{A.15}$$

As a consequence, the whole Eq. (A.9) gets transformed into

$$\begin{aligned}
 g(E')E'\varphi(E', x) &= -g(E')\frac{\partial^2\varphi(E', x)}{\partial x^2} + g(E')\varphi(E', x)V_0(x) \\
 &\quad - i\gamma \int dx_1 \int dx_2 \int_0^{E_0} dE_1 \int_0^{E_0} dE_2 \int_0^{E_0} dE_3 \\
 &\quad \times \int_0^{E_0} dE_4 g(E_1)g(E_2)g(E_3)g(E_4)g(E'') \\
 &\quad \times (x - x_2)\varphi^*(E_1, x_1)\frac{\partial\varphi(E_2, x_1)}{\partial x_1}\varphi^*(E_3, x_2)\varphi(E_4, x_2)\varphi(E'', x)
 \end{aligned}$$

or (with the change of variables $E' \rightarrow E$):

$$\begin{aligned}
 &\left(-\frac{\partial^2}{\partial x^2} + V_0(x) - E\right)\varphi(E, x) \\
 &= i\gamma \int dx_1 \int dx_2 \int_0^{E_0} dE_1 \int_0^{E_0} dE_2 \int_0^{E_0} dE_3 \\
 &\quad \times \int_0^{E_0} dE_4 \frac{g(E_1)g(E_2)g(E_3)g(E_4)g(E'')}{g(E)}(x - x_2)\varphi^*(E_1, x_1) \\
 &\quad \times \frac{\partial\varphi(E_2, x_1)}{\partial x_1}\varphi^*(E_3, x_2)\varphi(E_4, x_2)\varphi(E'', x). \tag{A.16}
 \end{aligned}$$

We have thus obtained for this case the time-independent Schrödinger equation, by taking however into account dissipation via the parameter γ . Of course, when γ tends to zero, one goes back to the stationary Schrödinger equation.

A.3. Method of the successive approximations

Assuming the coefficient γ to be small, one can find the unknown function $\varphi(x)$ in the simplified form

$$\varphi(x) = \varphi_0(x) + \gamma\varphi_1(x), \tag{A.17}$$

where as function $\varphi_0(x)$ it has been used the standard WF of the time-independent Schrödinger equation with potential $V_0(x)$ and energy E_0 :

$$\left(-\frac{\partial^2}{\partial x^2} + V_0(x)\right)\varphi_0(x) = E_0\varphi_0(x). \tag{A.18}$$

Substituting solution (A.17) into Eq. (A.16), we obtain a new equation containing all the powers n of γ , namely, the γ^n . Let us confine ourselves, however, to write

down this equation with accuracy up to γ^1 only:

$$\begin{aligned} & \left(-\frac{\partial^2}{\partial x^2} + V_0(x) - E \right) (\varphi_0(E, x) + \gamma\varphi_1(E, x)) \\ &= i\gamma \int dx_1 \int dx_2 \int_0^{E_0} dE_1 \int_0^{E_0} dE_2 \int_0^{E_0} dE_3 \\ & \quad \times \int_0^{E_0} dE_4 \frac{g(E_1)g(E_2)g(E_3)g(E_4)g(E'')}{g(E)} (x - x_2)\varphi_0^*(E_1, x_1) \\ & \quad \times \frac{\partial\varphi_0(E_2, x_1)}{\partial x_1} \varphi_0^*(E_3, x_2)\varphi_0(E_4, x_2)\varphi_0(E'', x), \end{aligned} \tag{A.19}$$

where the unknown $\varphi_1(x)$ does not appear any longer, of course, into the r.h.s. of this equation.

Taking

$$E = E_0, \tag{A.20}$$

we can rewrite in Eq. (A.19), separately, the various terms with different powers of γ . When limiting ourselves to $n = 0, 1$, we obtain

$$\begin{aligned} \gamma^0 & : \left(-\frac{\partial^2}{\partial x^2} + V_0(x) - E_0 \right) \varphi_0(E_0, x) = 0, \\ \gamma^1 & : \left(-\frac{\partial^2}{\partial x^2} + V_0(x) - E_0 \right) \varphi_1(E_0, x) \\ &= i\gamma \int dx_1 \int dx_2 \int_0^{E_0} dE_1 \int_0^{E_0} dE_2 \int_0^{E_0} dE_3 \int_0^{E_0} dE_4 \tag{A.21} \\ & \quad \times \frac{g(E_1)g(E_2)g(E_3)g(E_4)g(E'')}{g(E_0)} (x - x_2)\varphi_0^*(E_1, x_1) \\ & \quad \times \frac{\partial\varphi_0(E_2, x_1)}{\partial x_1} \varphi_0^*(E_3, x_2)\varphi_0(E_4, x_2)\varphi_0(E'', x), \end{aligned}$$

where

$$E'' = E_0 + E_1 - E_2 + E_3 - E_4. \tag{A.22}$$

The first equation holds when dissipation is absent. The second equation determines the unknown function φ_1 in terms of the given φ_0 : it results to be an ordinary differential equation of the second order, that can be solved by the ordinary numerical methods; but we deem convenient, here, to skip any numerical evaluations.

Let us just observe that, of course, one can go on iteratively to higher values of n .

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