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Space-Time Shifts and Cross-Sections in Collisions between Relativistic Wave Packets.

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Summary. — In this work we deal with motion and collision of relativistic wave packets. Starting from space-time definitions of the experimental observables (cross-sections, space-time shifts, etc.), we get their expressions in the energy-momentum representation, by using some Fourier-transformation techniques, thus generalizing some previous (nonrelativistic) results to the case of two relativistic spin-zero interacting particles. We notice that wave-packet spreading is often very large and cannot be neglected, as one usually does; but at the same time we show that the observable quantities do not commonly depend on spreading, blending or « smoothing ». As an application, we calculate the time duration of some interactions at different energies.

1. – Introduction.

It is well known that the various methods, which are commonly used in the quantum theory of relativistic collisions, seem rather artificial, as in general they do not consider time development, and describe monochromatic waves, infinite in space and time.

Many Authors have therefore studied nonrelativistic collisions introducing wave packets into consideration. For instance, in ref. [1-4] it is developed a

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(1) D. BOHM: *Quantum Theory* (London, 1954).
(2) M. L. GOLDBERGER and K. M. WATSON: *Phys. Rev.*, **127**, 2284 (1962).
(3) M. L. GOLDBERGER and K. M. WATSON: *Collision Theory* (New York, 1963).
(4) M. FROISSART, M. L. GOLDBERGER and K. M. WATSON: *Phys. Rev.*, **131**, 2820 (1963).

wave-packet description of nonrelativistic scattering, valid for a very narrow initial energy spread, and the space-time shifts of asymptotic scattered packets are calculated by using the stationary-phase method. Besides, in ref. (5) a rather general wave-packet theory of nonrelativistic scattering is presented, in which the expressions for averaged cross-sections and time delays (or advances) are produced and in detail discussed, and the influence of the *initial* conditions on the observable quantities is emphasized. One of us, in previous letters (6), began to consider the phenomenon of wave-packet *spreading*, and its influence on some experimental observables.

The aim of this work is to generalize to the relativistic case the theory of wave packet collisions (with the aid of some Fourier-integral techniques); we have carefully investigated the problem of defining and calculating cross-sections and space-time shifts; some values of time shifts for concrete examples have been calculated too.

For simplicity, in this paper we limit ourselves to spin-zero particles, so that we can use the solutions of the free Klein-Gordon (K-G) equation to describe their asymptotical motion.

2. – Space-time description of relativistic plane-wave packets.

As is usual in the wave-packet relativistic treatments, let us examine the collision processes, introducing initial plane-wave packets, as prepared *e.g.* by sending entering particles through some collimating slit.

We choose natural units (numerically: $\hbar = c = 1$), and the metric $(+\,-\,-\,-)$.

By definition (owing to the hypothesis of linear superposition of the free K-G solutions), for the initial relativistic plane-wave packets, describing the bombarding (free) particles, we use the Lorentz-invariant expression (being $x \equiv (x, y, z, t)$):

$$\psi^{(\text{in})}(x) = \int \mathrm{d}^4 p\, \delta(p^2 - m^2)\, \Theta(p_0) g(p) \exp[-ip\cdot x] = \int_+ \frac{\mathrm{d}\boldsymbol{p}}{2p_0} g(\boldsymbol{p}) \exp[i(\boldsymbol{p}\cdot\boldsymbol{x} - p_0 t)]\,, \tag{1}$$

where the last integration is confined—as it is understood in the following—to the positive energy range $\left(p_0 = +\sqrt{\boldsymbol{p}^2 + m^2}\right)$, m and $\boldsymbol{p}$ are respectively the incident particle rest-mass and momentum, and $g(\boldsymbol{p})$ is the wave-packet scalar weight function (*e.g.* due to the « monochromator » slit). Usually, for the ex-

(5) T. Ohmura: *Progr. Theor. Phys., Suppl.* **29**, 108 (1964).

(6) V. Olkhovsky: *Nuovo Cimento*, **48** B, 178 (1967); **50** B, 392 (1967).

perimental width of the weight function, one has: $|\Delta \boldsymbol{p}| \ll |\boldsymbol{p}|$. We notice that we do not need to introduce « localized » wave functions, because (instead of using the co-ordinate-operator eigenfunctions) we are considering quadratically-integrable *wave packets*—so that our free particle can be localized, in positive energy states, with any degree of accuracy ([7,8]).

As in the following, we assume the invariant normalization ([8])

$$(2)\qquad (\psi^{(\mathrm{in})}, \psi^{(\mathrm{in})})_t = i\int \mathrm{d}\boldsymbol{x}\, \psi^*_{(\mathrm{in})}(\boldsymbol{x})\, \overleftrightarrow{\partial}_0 \psi_{(\mathrm{in})}(\boldsymbol{x}) = (2\pi)^3 \int \frac{\mathrm{d}\boldsymbol{p}}{2p_0}\, |g(\boldsymbol{p})|^2 = 1\,.$$

The *symmetrized* expression for the corresponding flux-density of the initial K-G particles, in the entering direction z (the z-axis going from the slit-center to the scatterer-center), is

$$(3)\qquad J_z^{(\mathrm{in})}(\boldsymbol{x}) \equiv \frac{1}{2mi}\,(\psi^*_{(\mathrm{in})} \overleftrightarrow{\partial}_z \psi_{(\mathrm{in})}) = \\ = \frac{1}{m}\,\mathrm{Re}\int \frac{\mathrm{d}\boldsymbol{p}}{2p_0}\frac{\mathrm{d}\boldsymbol{p}'}{2p_0'}\, g^*(\boldsymbol{p})\, g(\boldsymbol{p}')\, \frac{p_z + p_z'}{2} \exp\left[i(p-p')\cdot x\right].$$

Let us now calculate the mean flux-density of the initial particles (through a detector « window »), averaged on the work-time τ and on the effective work-area S of the « detector ». We have ([9])

$$(4)\qquad \langle J_z^{(\mathrm{in})}(z)\rangle = \int_{S,\tau} J_z^{(\mathrm{in})}\, \frac{\mathrm{d}x\,\mathrm{d}y}{S}\,\frac{\mathrm{d}t}{\tau} = \int_{-\infty}^{\infty} J_z^{(\mathrm{in})}\, \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t}{S\tau}\,,$$

where the second step is justified by the fact that generally S and τ are much greater than the wave-packet cross-section and time-duration respectively. One can easily foresee that expression (4) does not actually depend on the detector position, *i.e.* on its co-ordinate z, owing to total probability conservation. Substituting formula (3) into (4), we get

$$(5)\qquad \langle J_z^{(\mathrm{in})}\rangle = \frac{(2\pi)^3}{2mS\tau}\int \frac{\mathrm{d}\boldsymbol{p}}{2p_0}\,|g(\boldsymbol{p})|^2 = \frac{1}{2mS\tau}\,,$$

([7]) See for instance: D. I. BLOKHINTSEV: *On a Localization of Relativistic Microparticles in Space-Time*, preprint P-2631, J.I.N.R., Dubna, 1966 (in Russian) ([*]).

([*]) *Note added in proofs.* – See also D. I. BLOKHINTSEV: *Macroscopic Causality,* preprint IC/67/36, I.A.E.A. (Trieste, June 1967).

([8]) See *e.g.*,: S. S. SCHWEBER: *An Introduction to Relativistic Quantum Field Theory* (Evanston, Ill., 1961). Alternatively, for convenience, we shall write the labels (in) and (sc) either up or down.

([9]) Analogously, one can define and calculate the other components of $\langle J\rangle$, but they have here little interest.

where we have used the integral representation of the delta « functions » and their properties, in particular the following relation:

$$\delta(p_0 - p_0') = \frac{\delta(p_z - p_z')}{|p_z/p_0|} + \frac{\delta(p_z + p_z')}{|-p_z/p_0|} = \frac{\delta(p_z - p_z')}{v_z} \tag{6}$$

(note that p_z and p_z' are always positive, according to our conventions, for physical reasons).

Let us now follow the motion of a single particle, *i.e.* of a single wave packet. We have to consider its mean space co-ordinates, at fixed time t, and its mean time co-ordinate, at fixed z [10].

For the mean duration of the packet motion from the slit to a distance z, we *define*, following OHMURA,

$$\langle t(z)\rangle_{(\mathrm{in})} = \frac{\int J_z^{(\mathrm{in})}\, t\, \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}y}{\int J_z^{(\mathrm{in})}\, \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}y} = 2m \int J_z^{(\mathrm{in})}\, t\, \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}y\,, \tag{7}$$

where we have averaged *on the time extension of the whole packet in the z-direction* (or on the detector work-time, if this is less than the previous one) and precisely on the different time extensions of our packet corresponding to the different points of its cross-section (or of the counter window area, if it is less than that « cross-section ». For simplicity, we may assume the window to be plane and perpendicular to the z-axis). But here, as in the following, we suppose that the detector work-time and work-area are very large in comparison with the packet dimensions, so that we can integrate over infinite ranges (and use the simplifying mathematical tool of the Fourier-transformations).

Thus, observing firstly that

$$\exp[ip\cdot x]\, t \exp[-ip'\cdot x] \equiv \frac{i}{2}\exp[-i(\boldsymbol{p}-\boldsymbol{p}')\cdot \boldsymbol{x}]\cdot$$
$$\cdot(\exp[ip_0 t]\,\partial_{p_0'}\exp[-ip_0' t] - \exp[-ip_0' t]\,\partial_{p_0}\exp[ip_0 t])\,, \tag{8}$$

integrating secondly by parts, then deriving and simplifying by taking into account that the weight function *and its derivatives* are quadratically integrable (*e.g.* they tend to zero at the integration-domain contour), and remembering the integral representation of the delta functions, their properties and formula (6), we get finally

$$\langle t(z)\rangle_{(\mathrm{in})} = \int \frac{\mathrm{d}\boldsymbol{p}}{2p_0}\, |g(\boldsymbol{p})|^2\, \partial_{p_0}\{\arg g(\boldsymbol{p}) + z p_z\} \equiv \langle \Delta t\rangle_{(\mathrm{in})} + z\langle v_z^{-1}\rangle_{(\mathrm{in})}\,. \tag{9}$$

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[10] In this Section we choose the slit center to be situated at $z=0$.

Evidently, the quantity $\langle \Delta t\rangle_{(\text{in})}$ is the mean time-shift caused by the initial devices.

Let us now consider the wave-packet mean space co-ordinates, at arbitrary time t; we have $(p_z/p_0 \equiv v_z)$

$$\text{(10)}\qquad \langle \boldsymbol{x}(t)\rangle_{(\text{in})} \equiv \frac{\int J_z^{(\text{in})}\boldsymbol{x}\,\mathrm{d}\boldsymbol{x}}{\int J_z^{(\text{in})}\mathrm{d}\boldsymbol{x}} = \\ = \frac{\int (\mathrm{d}\boldsymbol{p}/2p_0)v_z|g(\boldsymbol{p})|^2[\boldsymbol{v}t - \partial_{\boldsymbol{p}}\arg g(\boldsymbol{p})]}{\int (\mathrm{d}\boldsymbol{p}/2p_0)v_z|g(\boldsymbol{p})|^2} \equiv t\langle \boldsymbol{v}\rangle_{(\text{in})} - \langle \Delta\boldsymbol{x}\rangle_{(\text{in})}\,,$$

where it is easy to see that the quantity $\langle \Delta\boldsymbol{x}\rangle_{(\text{in})}$ is the space shift caused by the initial experimental conditions.

Comparing the results (9) and (10), one can easily observe the fact that in the wave-packet description we cannot consider the packet *average position* and *average motion-time* simultaneously. If we choose as a free parameter the space co-ordinates or the time co-ordinate, the remaining quantity is defined by the value of the first one and by the packet structure (and results as a mean value, with a mean fluctuation of course).

To get formula (10), we have used the relation

$$\text{(11)}\qquad \exp[ip\cdot x]\boldsymbol{x}\exp[-ip'\cdot x] \equiv -\frac{i}{2}\exp[i(p_0-p_0')t]\cdot \\ \cdot\big[\exp[-i\boldsymbol{p}\cdot\boldsymbol{x}]\partial_{\boldsymbol{p}'}\exp[i\boldsymbol{p}'\cdot\boldsymbol{x}] - \exp[i\boldsymbol{p}'\cdot\boldsymbol{x}]\partial_{\boldsymbol{p}}\exp[-i\boldsymbol{p}\cdot\boldsymbol{x}]\big]$$

and have then followed the previous procedure (*i.e.* the one we used to get (9)).

Now we have to calculate the effects of *spreading* on the plane-wave packets, that is to say the dependence on time of the quantity

$$\text{(12)}\qquad \langle(\Delta\boldsymbol{x})^2\rangle_{(\text{in})} = \langle\boldsymbol{x}^2\rangle_{(\text{in})} - \langle\boldsymbol{x}\rangle^2_{(\text{in})}\,.$$

We already know $\langle\boldsymbol{x}\rangle_{(\text{in})}$; to calculate $\langle\boldsymbol{x}^2\rangle_{(\text{in})}$, we use the relation

$$\text{(13)}\qquad \exp[ip\cdot x]\boldsymbol{x}^2\exp[-ip'\cdot x] \equiv -\frac{1}{2}\exp[i(p_0-p_0')t]\cdot \\ \cdot\left[\exp[-i\boldsymbol{p}\cdot\boldsymbol{x}]\frac{\partial^2}{\partial\boldsymbol{p}'^2}\exp[i\boldsymbol{p}'\cdot\boldsymbol{x}] + \exp[i\boldsymbol{p}\cdot\boldsymbol{x}]\frac{\partial^2}{\partial\boldsymbol{p}^2}\exp[-i\boldsymbol{p}\cdot\boldsymbol{x}]\right]$$

and then follow the usual procedure, now however integrating by parts *twice.*

We get ($\boldsymbol{v} \equiv \boldsymbol{p}/p_0$)

$$(14)\qquad \langle \mathbf{x}^2\rangle_{(\mathrm{in})} = \frac{\int J_z^{(\mathrm{in})}\,\mathbf{x}^2\,\mathrm{d}\mathbf{x}}{\int J_z^{(\mathrm{in})}\,\mathrm{d}\mathbf{x}} = $$
$$= \frac{\int (\mathrm{d}\boldsymbol{p}/2p_0)\left\{v_z \boldsymbol{v}^2 |g(\boldsymbol{p})|^2 t[t - (\boldsymbol{v}/v^2)\,\partial_{\boldsymbol{p}} \arg g(\boldsymbol{p})] + p_0^2 v_z \left|\left(\partial_{\boldsymbol{p}}(g(\boldsymbol{p})/p_0)\right)^2\right|\right\}}{\int (\mathrm{d}\boldsymbol{p}/2p_0)\, v_z |g(\boldsymbol{p})|^2}.$$

3. – Space-time description of relativistic scattered wave packets.

Let us now consider the physical case of a flux of particles (with rest mass m) bombarding a target, at rest [11] in the laboratory, and producing some final particles. Let us imagine to detect only final particles of a certain type (with rest mass $\tilde{m}$). We can limit ourselves to examine, long before the interaction, the entering plane-wave packets (1), and, long after the collision (near the detector), the wave packets:

$$(15)\qquad \psi^{(\mathrm{sc})}(\tilde{x}) \simeq \int_+ \frac{\mathrm{d}\boldsymbol{p}}{2p_0}\, g(\boldsymbol{p}) \int_+ \frac{\mathrm{d}\boldsymbol{q}}{2q_0}\, \tilde{g}(\boldsymbol{q}) f(\boldsymbol{p}, \boldsymbol{q}) \exp[iq\cdot\tilde{x}]\,,$$

relative to the considered final (free) particles. In formula (15): $\boldsymbol{q}$ is the final momentum; $f(\boldsymbol{p}, \boldsymbol{q})$ is the transition amplitude [12] from initial *plane waves* with momentum $\boldsymbol{p}$ to final *plane waves* with momentum $\boldsymbol{q}$; and $\tilde{g}(\boldsymbol{q})$ is the final-particle « detector weight », including angular and energetical detector resolutions [13].

We have chosen the c.m. frame of reference, and the space axes $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ so that $\tilde{z}$ goes from the collision center ($\tilde{z} = 0$) to the detector-window center; we have now: $\tilde{x} = (\tilde{x}, \tilde{y}, \tilde{z}, t)$. Obviously, using final plane-wave packets is justified by the fact that we are interested only in the waves reaching the detector window (at distances r much bigger than the interaction radius); in particular after the « window » we would meet a situation similar to the one after the monochromator slit.

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(11) Were it not at rest, we ought to consider also the momentum distribution of these second particles.

(12) As usually, we do not consider the final-state particles with $q=p$.

(13) We can neglect the possible impulse-energy spread, due to eventual interactions between scattered particles and « window » edges. Such wave-diffraction may be neglected *e.g.* if the detector energy resolution is *not better* than the monochromator one: in this case the weight $\tilde{g}$ does not depend on $|\boldsymbol{q}|$. The detector weight function could be normalized in analogy to formula (2).

We have now to define the cross-section for unit solid angle (mb/sr), averaged for outgoing particles inside the small solid-angle defined by the detector angular resolution [14]:

$$\langle\sigma(\boldsymbol{p},\boldsymbol{q})\rangle = \frac{\int J_{\tilde{z}}^{(\mathrm{sc})}(\boldsymbol{r},t)\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y}\,\mathrm{d}t}{\int J_{z}^{(\mathrm{in})}(\boldsymbol{0},t)\,\mathrm{d}t}\,, \tag{16}$$

where $J_z^{(\mathrm{in})}(\boldsymbol{0},t)$ is the initial flux-density in the interaction-region center [4], and where in the numerator and in the denominator we integrate on the space and work-time extensions respectively of the detector and of the « source ». Easily (here $p_z'^2 = p_0^2 - p_x'^2 - p_y'^2 - m^2$)

$$\int J_z^{(\mathrm{in})}(\boldsymbol{0},t)\,\mathrm{d}t = \frac{\pi}{m}\,\mathrm{Re}\int\frac{\mathrm{d}p_0}{2\boldsymbol{p}^2}\,\mathrm{d}p_x\mathrm{d}p_y\mathrm{d}p_x'\mathrm{d}p_y'\,\frac{p_z+p_z'}{2}\,g^*(\boldsymbol{p})\,g(\boldsymbol{p}')\,.$$

Again the integrations may be extended from $-\infty$ to $+\infty$, as generally the space and time extensions of the wave packets are not larger than the detector or source ones. We may resort to Fourier transformations and follow the same procedure we used for the initial packets. For instance we would get

$$J_{\tilde{z}}^{(\mathrm{sc})}(\tilde{x},\tilde{y},\tilde{z}) = \frac{1}{\tilde{m}}\,\mathrm{Re}\int\frac{\mathrm{d}\boldsymbol{q}}{2q_0}\frac{\mathrm{d}\boldsymbol{q}'}{2q_0'}\frac{\mathrm{d}\boldsymbol{p}}{2p_0}\frac{\mathrm{d}\boldsymbol{p}'}{2p_0'}\,F^*(\boldsymbol{p},\boldsymbol{q})F(\boldsymbol{p}',\boldsymbol{q}')\,\frac{q_{\tilde{z}}+q_{\tilde{z}}'}{2}\exp[i(q-q')\cdot\tilde{x}]\,, \tag{17}$$

where

$$F(\boldsymbol{p},\boldsymbol{q}) \equiv g(\boldsymbol{p})\,\tilde{g}(\boldsymbol{q})\,f(\boldsymbol{p},\boldsymbol{q})\,. \tag{18}$$

Finally we have, in the impulse representation,

$$\langle\sigma(\boldsymbol{p},\boldsymbol{q})\rangle = \frac{m}{\tilde{m}}\,\frac{(2\pi)^2\,\mathrm{Re}\int(\mathrm{d}\boldsymbol{q}/2q_0)(\mathrm{d}\boldsymbol{p}/2p_0)(\mathrm{d}\boldsymbol{p}'/2p_0')F^*(\boldsymbol{p},\boldsymbol{q})F(\boldsymbol{p}',\boldsymbol{q})}{\mathrm{Re}\int(\mathrm{d}p_0/2\boldsymbol{p}^2)\,\mathrm{d}p_x\mathrm{d}p_y\mathrm{d}p_x'\mathrm{d}p_y'((p_z+p_z')/2)\,g^*(\boldsymbol{p})g(\boldsymbol{p}')|_{p_0'=p_0}}\,. \tag{19}$$

In the *particular case* of elastic scattering with broad initial packets, when

$$g(\boldsymbol{p}) = \sqrt{2p_0}\,g(p_z)\,\delta(p_x)\,\delta(p_y)\,, \tag{20}$$

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[14] Without limiting the generality, we assume the space and work-time extensions of the detector and of the source to be the same. We might replace the denominator of formula (16) by the expression: $(1/\mathscr{S}_{(\mathrm{in})})\int J_z^{(\mathrm{in})}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$, $\mathscr{S}_{(\mathrm{in})}$ being the cross-section of the initial packet near the interaction region. Analogously in formula (19).

and when [13]

$$(20') \qquad \tilde{g}(\boldsymbol{q}) = \frac{\sqrt{4p_0q_0}}{|\boldsymbol{q}|}\,\delta(|\boldsymbol{p}| - |\boldsymbol{q}|)\,\tilde{g}(\Omega_{\boldsymbol{q}})\,,$$

$$|\tilde{g}(\boldsymbol{q})|^2 = \frac{4p_0q_0}{\boldsymbol{q}^2}\,\delta(|\boldsymbol{p}| - |\boldsymbol{q}|)\,\delta(p'_z - p_z)\cdot\left|\frac{\boldsymbol{p}}{p_z}\right|\cdot|\tilde{g}(\Omega_{\boldsymbol{q}})|^2\,,$$

whe have (here $|\boldsymbol{p}| = p_z$ and $p_0 = q_0$)

$$(19') \qquad \langle\sigma_0(\boldsymbol{p}, \boldsymbol{q})\rangle = \frac{m}{\tilde{m}}\cdot\frac{\int \mathrm{d}p_z\,\mathrm{d}\Omega_{\boldsymbol{q}}|g(p_z)\,\tilde{g}(\Omega_{\boldsymbol{q}})\,f(\boldsymbol{p}, \boldsymbol{q})|^2}{(2\pi)^{-2}\int \mathrm{d}p_z|g(p_z)|^2}\,,$$

where the denominator is a constant.

At this point we have to generalize expression (7) for the mean motion time for the scattered packets. We get, following the usual procedure,

$$(21) \qquad \langle t\rangle_{(\mathrm{sc})} \equiv \frac{\int J_{\tilde{z}}^{(\mathrm{sc})}\,t\,\mathrm{d}t\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y}}{\int J_{\tilde{z}}^{(\mathrm{sc})}\,\mathrm{d}t\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y}} = \\ = \frac{\mathrm{Re}\int(\mathrm{d}\boldsymbol{p}/2p_0)(\mathrm{d}\boldsymbol{p}'/2p'_0)(\mathrm{d}\boldsymbol{q}/2q_0)\{F^*F'(\tilde{z}/v_{\tilde{z}}) + (1/2i)F^*\overleftrightarrow{\partial}_{q_0}F'\}}{\mathrm{Re}\int(\mathrm{d}\boldsymbol{p}/2p_0)(\mathrm{d}\boldsymbol{p}'/2p'_0)(\mathrm{d}\boldsymbol{q}/2q_0)F^*F'}\,,$$

$\boldsymbol{v} = |\boldsymbol{q}|/q_0$ being now (in this Section) the phase velocity corresponding to the final momentum $\boldsymbol{q}$. Further

$$(22) \qquad F \equiv F(\boldsymbol{p}, \boldsymbol{q})\,; \qquad F' \equiv F(\boldsymbol{p}', \boldsymbol{q})\,.$$

Formula (21) may be rewritten more compactly as follows:

$$(23) \qquad \langle t\rangle_{(\mathrm{sc})} = \tilde{z}\,\langle v_{\tilde{z}}^{-1}\rangle + \langle\Delta t\rangle\,,$$

where the term $\langle\Delta t\rangle$ has the clear physical meaning of *time shift* (delay or advance) produced by the collision and by the initial and final experimental devices.

In the *particular case* (elastic scattering) represented by (20) and (20′), expression (21) transforms into

$$(24) \qquad \langle t\rangle_{(\mathrm{sc})} = \frac{\int \mathrm{d}p_z\,\mathrm{d}\Omega_{\boldsymbol{q}}|g(p_{\tilde{z}})\,\tilde{g}(\Omega_{\boldsymbol{q}})f(\boldsymbol{p}, \boldsymbol{q})|^2\{\tilde{z}/v_{\tilde{z}} + \partial_{q_0}[\arg g(p_z) + \arg f(\boldsymbol{p}, \boldsymbol{q})]\}}{\int \mathrm{d}p_z\mathrm{d}\Omega_{\boldsymbol{q}}|g(p_z)\,\tilde{g}(\Omega_{\boldsymbol{q}})f(\boldsymbol{p}, \boldsymbol{q})|^2}\,,$$

where the time shift *produced only by the interaction* is now

$$\langle\Delta t\rangle_{(\mathrm{int})} = \left\langle\frac{\partial \arg f(\boldsymbol{p}, \boldsymbol{q})}{\partial p_0}\right\rangle\,.$$

Then, we generalize expression (10) of the mean wave-packet spatial coordinates; again with the usual procedure we get

$$\text{(25)}\qquad \langle \boldsymbol{x}\rangle_{(\mathrm{sc})} \equiv \frac{\int J_{\tilde{\boldsymbol{x}}}^{(\mathrm{sc})}\boldsymbol{x}\,\mathrm{d}\boldsymbol{x}}{\int J_{\tilde{\boldsymbol{x}}}^{(\mathrm{sc})}\,\mathrm{d}\boldsymbol{x}} = \\ = \frac{\operatorname{Re}\int (\mathrm{d}\boldsymbol{p}/2p_0)(\mathrm{d}\boldsymbol{p}'/2p_0')(\mathrm{d}\boldsymbol{q}/2q_0)\,v_{\tilde{\boldsymbol{x}}}\,[F^*F'\boldsymbol{v}t-(1/2i)F^*\overleftrightarrow{\partial}_{\boldsymbol{q}}F']}{\operatorname{Re}\int (\mathrm{d}\boldsymbol{p}/2p_0)(\mathrm{d}\boldsymbol{p}'/2p_0')(\mathrm{d}\boldsymbol{q}/2q_0)\,v_{\tilde{\boldsymbol{x}}}F^*F'}\,.$$

Expression (25) may be rewritten briefly as

$$\text{(26)}\qquad \langle \boldsymbol{x}\rangle_{(\mathrm{sc})} = \langle \boldsymbol{v}t\rangle - \langle \boldsymbol{v}\Delta t\rangle + \langle \delta\mathscr{D}_\perp\rangle\,,$$

where $\perp$ means perpendicular to the direction of $\boldsymbol{v}$, and

$$\langle \delta\mathscr{D}_\perp\rangle = \left\langle -\frac{1}{2i}\,F^*\overleftrightarrow{\partial}_{\boldsymbol{q}}F'\right\rangle + \langle \boldsymbol{v}\Delta t\rangle\,.$$

The last two quantities of formula (26) give the *space shifts*, produced by the collision with influence of the initial and final experimental devices.

In the *particular case* (elastic scattering) represented by (20) and (20′), we have

$$\text{(27)}\qquad \langle \boldsymbol{x}\rangle_{(\mathrm{sc})} = \frac{\int \mathrm{d}p_z\,\mathrm{d}\Omega_{\boldsymbol{q}}|g(p_z)\,\tilde{g}(\Omega_{\boldsymbol{q}})\,f(\boldsymbol{p},\boldsymbol{q})|^2\{\boldsymbol{v}t-\boldsymbol{v}\cdot\Delta t_{(\mathrm{int})}-\partial_{\boldsymbol{q}}\arg g\tilde{g}+\delta\mathscr{D}_{\perp(\mathrm{int})}\}}{\int \mathrm{d}p_z\,\mathrm{d}\Omega_{\boldsymbol{q}}|g(p_z)\,\tilde{g}(\Omega_{\boldsymbol{q}})\,f(\boldsymbol{p},\boldsymbol{q})|^2}\,,$$

where the trasversal space shift produced by the interaction is now

$$\text{(28)}\qquad \langle \delta\mathscr{D}_\perp\rangle_{(\mathrm{int})} = \left\langle -\frac{\partial \arg f(\boldsymbol{p},\boldsymbol{q})}{\partial \boldsymbol{q}}\right\rangle + \langle \boldsymbol{v}\cdot\Delta t_{(\mathrm{int})}\rangle\,.$$

At this point, we calculate the effects of *spreading* on scattered wave packets. Following the same method we used in the initial case (formulas (12) and followings), it is easy to generalize the last results of Sect. 2. For instance

$$\text{(29)}\qquad \langle(\Delta\boldsymbol{x})^2\rangle_{(\mathrm{sc})} = \\ = \frac{\operatorname{Re}\int (\mathrm{d}\boldsymbol{p}/2p_0)(\mathrm{d}\boldsymbol{p}'/2p_0')(\mathrm{d}\boldsymbol{q}/2q_0)\,v_{\tilde{\boldsymbol{x}}}\{v^2[tF^*F'+(i/2)(F_0^*\overleftrightarrow{\partial}_{\boldsymbol{q}}F_0')]+q_0^2\,\partial_{\boldsymbol{q}}F_0^*\cdot\partial_{\boldsymbol{q}}F_0'\}}{\operatorname{Re}\int (\mathrm{d}\boldsymbol{p}/2p_0)(\mathrm{d}\boldsymbol{p}'/2p_0')(\mathrm{d}\boldsymbol{q}/2q_0)\,v_{\tilde{\boldsymbol{x}}}F^*F'} - \langle \boldsymbol{x}\rangle_{(\mathrm{sc})}^2\,,$$

where

$$\text{(30)}\qquad F_0^* \equiv F^*(\boldsymbol{p},\boldsymbol{q})\exp[-iq_0t]\,,\qquad F_0' \equiv F(\boldsymbol{p}',\boldsymbol{q})\exp[+iq_0t]\,.$$

4. – Conclusions and applications.

First, we can see that—besides the interaction dynamics—also the initial and final experimental conditions (which are defined by the weight factors g and $\tilde{g}$) influence the physical observables (cross-sections, mean space-time shifts, etc.). Of course, in most practical cases, the interaction-dynamics role is very important. But it is possible to imagine cases when the energetical resolution is $\Delta E \gg \Gamma$, where Γ is an *energy interval* corresponding to a noticeable amplitude variation, or when on the contrary $\Delta E \ll \Gamma$. In these cases, information about cross-section energy dependence or about collision lifetimes is insufficient, as in the first case the experimental cross-sections, averaged on ΔE, fail to reveal possible resonances of other strong energy dependences, while in the second case the information about time shifts is lost in the great packet time extension.

Another important point we want to clarify is the role of *spreading* in the real experimental situations. For example, if one considers a weight function of Gaussian type, it is possible to show ([3]) that, if the parameter $e = \hbar t/2mw^2 \gg 1$ (where w is the initial length of the plane-wave packet), then spreading is very large. And, examining for instance the collision of two nucleons, with an entering laboratory momentum of 1 GeV/c, energy resolution $\Delta E = 10^{-2}$ GeV, and a laboratory distance of 10 cm between the target and the detector, we find in the c.m.s. the (dimensionless) « spreading parameter »:

$$e = \frac{t}{2\hbar m}\left(\frac{\Delta E}{v}\right)^2 \approx 1.7\times 10^{10} \gg 1\,, \tag{31}$$

v being the packet c.m. group velocity. Thus, in contrast with the usual assumptions, one can often consider spreading *practically* almost infinite. *But, in spite of this*, the spreading in the recent day's experimental situations does not influence the observables, as can be seen from their expressions in the impulse representation.

In fact, during the motion, a packet space-redistribution occurs, due to spreading and smoothing with time ([15]). *For example*, if we have two packets with almost no space overlapping ([16]), one can write initially (supposing we

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([15]) If the same scattering reaction can proceed via *various* prompt and delayed processes, also the interference between them (depending on the initial superposition region too) will cause a new space-time redistribution of the wave packets (*i.e.* their spectral redistribution with time).

([16]) This case in particular seems to be realistic in the field of nuclear physics, where sometimes after a collision one has two wave packets, corresponding to a direct collision process and to a compound-nucleus process respectively.

may use formula (19′))

$$(32) \qquad \langle\sigma_0\rangle \propto \frac{m}{\tilde{m}}\cdot\int \mathrm{d}p_z\,\mathrm{d}\Omega_q(|f_1|^2+|f_2|^2)|g\tilde{g}|^2\,,$$

where $f=f_1+f_2$, and $\int \mathrm{d}p_z\,\mathrm{d}\Omega_q|g\tilde{g}|^2 f_1^* f_2=0$. With time, their spreading produces a blending of the two packets, but, owing to total probability conservation, $\int J^{(\mathrm{sc})}(\tilde{\mathbf{x}}, t)\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y}\,\mathrm{d}t=$ const, the value $\langle\sigma_0\rangle$ will remain the same.

Thus, even if the packet space redistribution with time corresponds obviously to a new spectral redistribution of the reaction amplitudes, the general values of $\langle\sigma\rangle$—and analogously of the other observables, like $\langle\Delta x\rangle$ and $\langle\Delta t\rangle$—will not change.

From the point of view of the collision lifetimes, we can observe the following analogy between nuclear physics and elementary-particle physics, neglecting electromagnetic interactions. In nuclear physics, processes can roughly be divided in two classes: the prompt processes, corresponding to « direct interactions » (with lifetimes of the order of $(10^{-24}\div10^{-21})$ s) and the delayed processes, corresponding to formation and subsequent decays of a « compound nucleus » (with lifetimes of the order of $(10^{-20}\div10^{-14})$ s, and sometimes even much more). The direct processes are interpreted as strong nuclear interactions, with participation of few degrees of freedom; while the compound-nucleus ones are interpreted as a consequence of the partecipation of very many degrees of freedom.

In elementary-particle physics, we have « strong processes » (with lifetimes $(10^{-24}\div10^{-22})$ s) and « weak interactions » (with lifetimes $\approx 10^{-10}$ s, and sometimes even much more). If we accept the elementary-particle composite model ([17]), we are led, by the analogy, to suppose that in strong interactions only few internal freedom degrees are involved, while in the weak ones a lot of internal degrees of freedom participate to the reactions. We add that, in the fields both of elementary particles and of nuclear physics, the relative probability of strong (direct) interactions with respect to weak (compound-nucleus) reactions increases with energy.

For the future, it seems interesting to investigate for instance the role of the relativistic wave-packet formalism in the description of some final-state interactions (such as the ones with triangle graphs), with the additional aim to obtain information about the lifetimes of intermediate unstable « particles » or resonances.

At last, let us calculate—for exemplification—some values of time shifts, as an application of what precedes. Let us remember the Lorentz-invariant

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([17]) For instance the Sakata model or the quark models.

expression of the amplitude for *elastic scattering* of spin-zero particles:

$$f(\theta) = \frac{1}{|\boldsymbol{p}|}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos\theta)\,, \tag{33}$$

where δ_l is the phase-shift for the l-th partial wave. If only one partial wave contributes (and the inelastic channels are closed), we have [18]:

$$\arg f = \operatorname{arctg}\operatorname{tg}\delta = \delta\,; \tag{34}$$

and if the packet-spectrum is narrow, we conclude that the time shift produced by the interactions is

$$\langle\Delta t\rangle_{(\text{int})} \approx \frac{\hbar}{c}\frac{\partial \arg f}{\partial p_0} = \hbar\frac{\partial\delta}{\partial E}\,, \tag{35}$$

where $E = cp_0$.

To illustrate some examples, as this work is concerned with spin-zero particles, let us refer only to s-waves and to their *partial* time shifts. In fact, in elastic processes with $l = 0$, spin plays no role. It is evident that our following calculations have a realistic interest only when the elastic partial s-wave dominates *all the reactions*.

TABLE I. – *Time shifts (advances) in the c.m.s. for the s-wave elastic* K^+p *scattering* [19], *at different* K^+ *laboratory momenta* [20].

lab p_K (MeV/c)	c.m. $\langle\Delta t\rangle_{(\text{int})}$ (10^{-23} s)
~ 160	-4.7
190	-1.9
215	-2.0
250	-0.9
310	-2.1
440	-1.5
580	-1.3

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(18) If the inelastic channels are not closed, or the elastic scattering does not dominate all the reactions, the phase shifts are complex (as is well known): $\delta_l = \alpha_l + i\beta_l$. Then formula (34) reads: $\arg f_l = \arg(1 - S_l)$, where: $S_l = \exp[2i\delta_l] = \exp[-2\beta_l + 2i\alpha_l]$. Analogously in formulas (35) and (36).

(19) S. GOLDHABER, W. CHINOWSKY, G. GOLDHABER, W. LEE, T. O'HALLORAN, T. F. STUBBS, G. M. PJERROU, D. H. STORK and H. K. TICHO: *Phys. Rev. Lett.*, **9**, 135 (1962).

(20) The data on particles are taken from: A. H. ROSENFELD, A. BARBARO-GUALTIERI, W. J. PODOLSKY, L. R. PRICE, M. ROOS, P. SODING, W. J. WILLS and C. G. WOHL: *Rev. Mod. Phys.*, **39**, 1 (1967).

TABLE II. – *Time shifts (advances and delays) in the c.m.s. for the s-wave elastic $\pi\mathcal{N}$ scattering* ([21]), *at different laboratory kinetic energies of the entering pions. T means the (total) $\pi\mathcal{N}$-isospin* ([20]).

lab E_π (MeV)	c.m. $\langle\Delta t\rangle_{(\text{int})}$ (10^{-23} s) $T=\frac{3}{2}$	lab E_π (MeV)	c.m. $\langle\Delta t\rangle_{(\text{int})}$ (10^{-23} s) $T=\frac{1}{2}$	
~28	− 1.3	~36	+ 0.4	
34	+ 0.3	51	− 0.2	
39	− 1.8	80	+ 0.4	
51	− 0.7	109	− 0.3	
72	− 1.5	135	+ 0.7	
90	+ 1.0	158	− 0.3	
109	− 0.9	195	+ 0.2	
135	− 0.5	236	+ 1.7	
158	− 5.1	259	+ 0.6	
168	+ 2.8	290	+ 0.7	
173	+ 1.3	340	+ 0.7	
188	+ 4.0	390	+ 0.1	
212	− 4.0	430	+ 2.5	
236	− 0.8	470	− 0.8	
259	− 0.1	512	+ 0.7	
290	− 1.1	542	+11.5	
340	− 0.8	565	+ 1.4	
390	− 0.5	591	−11.3	
430	− 0.6	625	− 6.2	
470	− 0.2	674	− 4.2	
511	− 0.1	722 (*)	−11.1	+ 7.6
542	− 1.4	771 (*)	+12.9	+ 6.4
565	+ 0.6	821 (*)	−32.2	+22.2
590	+ 0.03	858 (*)	+ 6.4	− 5.9
625	+ 0.8	885 (*)	+13.7	+ 7.1
674	+ 0.6	925 (*)	− 1.9	+ 2.5
723	− 3.4	970 (*)	+ 0.2	+ 0.6
771	− 5.8	1069 (*)	+ 1.0	+ 1.0
821	+ 1.5	1188 (*)	− 7.7	− 7.7
858	−12.5	1269 (*)	− 2.0	− 2.0
885	+ 1.6	—	—	
925	+ 5.3	—	—	
970	− 5.6	—	—	
1019	− 3.1	—	—	
1138	+ 0.7	—	—	
1270	+ 0.7	—	—	

(*) The two time-shift values correspond to the solutions 1 and 2; see ref. ([21]).

([21]) A DONNACHIE: *Pion-Nucleon Phase Shift Analysis*, in *Particles Interactions at High Energies* (Scottish Universities' 1966 Summer School), edited by T. W. PREIST and L. L. J. VICK (Edinburg, 1967), p. 330.

Firstly, we shall consider the elastic K^+p scattering, which from 140 to 642 MeV/c has been interpreted as an s-wave process ([19]). The c.m. time shifts (advances), calculated at different K^+ laboratory momenta by formula (35), or better from its approximate version

$$\langle \Delta t \rangle_{(\text{int})} \approx \hbar \frac{\Delta \delta}{\Delta E}, \tag{36}$$

are reported in Table I. The data are taken from ref. ([19]).

As a second example, we want to consider the « classical » elastic $\pi\mathcal{N}$ scattering, taking the phase-shift data from ref. ([21]). The c.m. time shifts (advances and delays), calculated by formula (36) at different laboratory kinetic energies of the entering pions, are reported in Table II and refer to partial s-wave scattering. In Table II, T means the (total) $\pi\mathcal{N}$-isospin. Owing to the unknown approximation introduced by substituting the phase-shift derivatives with « incremental ratios » (formula (36)), it is not possible to add the main errors in Tables I and II. But we emphasize that sometimes the phase-shift standard deviations too are enormously large ([21]). In the (c.m.s.) energy region of a *narrow* Breit-Wigner « resonance », with width Γ, when $f_l = A\Gamma/2(E_R - E - i\Gamma/2)$ and $\delta_l = \operatorname{arctg}\left(\Gamma/2(E_R - E)\right)$, we should find, in particular, the known formulas

$$\Delta t_l(E) = \frac{\hbar\Gamma/2}{(E - E_R)^2 + \Gamma^2/4}, \qquad \Delta t_l(E_R) = \frac{2\hbar}{\Gamma},$$

and, if the experimental resolution is much bigger than Γ,

$$\langle \Delta t \rangle_l = \hbar/\Gamma .$$

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RIASSUNTO

In questo lavoro si considerano moto e collisione di pacchetti d'onda relativistici. Partendo da alcune definizioni spazio-temporali delle osservabili sperimentali (space-time shifts, sezioni d'urto, ecc.), si ottengono le loro espressioni nella rappresentazione

degli impulsi, con l'ausilio di tecniche delle trasformazioni di Fourier. Si generalizzano così alcuni precedenti risultati (non relativistici) al caso dell'interazione di due particelle relativistiche a spin zero. Si osserva che non è lecito trascurare — come di solito si fa — lo *spreading* dei pacchetti d'onda nel loro moto, dato che al contrario esso è sovente molto grande. Nel contempo però si dimostra che, ciononostante, le quantità osservabili nelle usuali odierne situazioni sperimentali non dipendono né dallo spreading, né dai «blending» e «smoothing» dei pacchetti. Come applicazione, si calcolano le durate temporali di alcuni processi d'interazione a varie energie.

О пространственно-временных смещениях и сечениях в случае столкновений релятивистских волновых пакетов.

Резюме. — В настоящей работе исследуется движение и столкновения релятивистских волновых пакетов. Исходя из пространственно-временных определений наблюдаемых (пространственно-временных смещений и сечений), мы получили с помощью простой техники фурье-преобразований выражения для этих величин в представлении энергии-импульса, обобщая, таким образом, известные нерелятивистские результаты на случай релятивистских частиц со спином *О*. Следует отметить, что пренебрежение расплыванием волновых пакетов за время их движения на расстояниях поряка лабораторных не является всегда оправданным. Напротив, оно довольно часто очень велико. В то же время показано, что численные значения наблюдаемых величин не зависят от расплывания, смешивания и «сглаживания» пакетов. В качестве приложения мы вычислили временные задержки (опережения) для некоторых типов столкновений.