TUNNELING TIMES AND "SUPERLUMINAL" TUNNELING: A BRIEF REVIEW.[(*)]

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ABSTRACT – In the First Part of this paper [that was submitted for pub. in 1991 and appeared in print in Phys. Reports 214 (1992) 339] we critically review and analyse the main theoretical definitions and calculations of the sub-barrier tunnelling and reflection *times*. Moreover, we propose a new definition of such durations, on the basis of a recent general formalism of ours (already tested for other types of quantum collisions) within conventional quantum mechanics. At last, we discuss some surprising results regarding the temporal evolution of the tunnelling processes: namely, the fact that QM predicts that tunnelling through opaque barriers takes place with Superluminal group-velocities. Aims of the Second Part [that appeared in print later, in J. de Physique-I 5 (1995) 1351] are: (i) presenting and analysing the results of various numerical calculations (based on our equations) on the penetration and return times $<\tau_{\rm Pen}>$, $<\tau_{\rm Ret}>$, during tunnelling *inside* a rectangular potential barrier, for various penetration depths $x_{\rm f}$; (ii) putting forth and discussing suitable definitions, besides of the mean values, also of the *variances* (or dispersions) $\mathrm{D}\,\tau_{\rm T}$ and $\mathrm{D}\,\tau_{\rm R}$ for the time durations of transmission and reflection processes; (iii) mentioning, moreover, that our definition $<\tau_{\rm T}>$ for the average transmission time results to constitute an *improvement* of the ordinary dwell–time $\overline{\tau}^{\rm Dw}$ formula. The numerical evaluations *confirm* that our approach implied, and implies, the existence of Superluminal tunnelling (that we call "Hartman effect"): an effect that in these days (due to the theoretical connections between tunnelling and evanescent–wave propagation) is receiving —at Cologne, Berkeley, Florence and Vienna— indirect, but quite interesting, experimental verifications. Eventually, we briefly analyze some other definitions of tunnelling times. A more detailed review of the same topics (in Italian, however) has been "published" electronically, as LANL Archives # cond-mat/9802126. At

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last, a brief review on the experimental data, that —in four different sectors of physics— seem to indicate the existence of Superluminal motions, appeared as an Appendix to the paper in LANL Archives # physics/9712051 (to be published in Physica A, 1998).

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FIRST PART

I-1 – INTRODUCTION.

During the last years, *many* attempts were devoted to theoretically defining and calculating the time spent by a sub-barrier–energy particle for tunnelling through a potential barrier; so that critical reviews were already in order. Actually, a first, valuable review article, by Hauge and Stoveneng, did recently appear (at the end of 1989) as ref.[1]. We deem it useful, however, a second review paper since: (i) on one hand, it appears convenient to deepen and extend the criticism to the existing approaches, and (ii) on the other hand, we can show —as a consequence of such a criticism— how a definite proposal can be put forth for the introduction of suitable, self-consistent, physically meaningful definitions of the tunnelling times: a question that was regarded still as an open problem in ref.[1].

The problem of defining the tunnelling times has a long history, since it arose in the fortieth and fiftieth[2−4], simultaneously with the problem of a general definition in quantum mechanics of the collision durations. Let us explicitly recall that the motion inside a potential barrier is a quantum phenomenon without any classical or quasi-classical analogues, so that it lacks an easy intuitive representation; this can explain the birth of a variety of approaches to describe it, sometimes in contradiction with one another.

The advent of high–speed experimental devices, based on tunnelling processes in semiconductors, did revive an interest in the whole question;[5,1] whose relevance has always been apparent in nuclear physics: For instance, tunnelling plays an essential role in the physics of nuclear fission and fusion.

As already mentioned, aims of this paper are a comparative analysis of the various, known definitions for the tunnelling durations; and the introduction of a possible, rigorous way for theoretically evaluating the time spent by a particle *inside* a potential barrier: with the final presentation of some peculiarities of the tunnelling–process time evolution. Our starting point will be the formalism in refs.[6,8].

First of all, in Sect.**I-2**, we critically review the main definitions of the tunnelling times appeared up to now, to the best of our knowledge. Then, Sect.**I-3** is devoted to the application of our own formalism[6,8] to tunnelling. At last, Sect.**I-4** is devoted to the conclusions and to some proposals for further developments.

I-2 – ABOUT THE EXISTING DEFINITIONS OF TUNNELLING TIMES.

I-2.1 – The simplest stationary picture of Tunnelling.

Let us confine ourselves to the ordinary case of particles moving only along the x–direction, and consider a time–independent barrier $V(x)$ in the interval $(0, a)$: see Fig.I-1, in which —for later convenience— a larger interval (x_1, x_2), containing the barrier region, is also indicated. We assume the stationary scattering problem to have been solved exactly for every kinetic energy $E = \hbar^2 k^2/2m$, where k is the wave-number and m the particle mass. The wave function $\psi(x; k)$ will have the general form:

$$\begin{aligned} \psi \equiv \psi_{\rm I} = \psi_{\rm in} + \psi_{\rm R} \qquad & \text{for} \quad x < 0, \\ \psi \equiv \psi_{\rm II} \qquad & \text{for} \quad 0 < x < a, \\ \psi \equiv \psi_{\rm III} = \psi_{\rm T} \qquad & \text{for} \quad x > a; \end{aligned} \tag{1}$$

where: $\psi_{\rm in} \equiv e^{ikx}$; $\psi_{\rm R} \equiv A_{\rm R}\, e^{-ikx}$; $\psi_{\rm T} \equiv A_{\rm T}\, e^{ikx}$; the lower indices R,T staying for "reflected" and "transmitted", respectively. In the simple case of a rectangular barrier $[V(x) = V_0]$, it is $\psi_{\rm II} = \alpha e^{-\kappa x} + \beta e^{\kappa x}$ [with $\kappa \equiv \sqrt{2m(V_0 - E)}/\hbar$], where the coefficients [amplitudes] $A_{\rm R}, A_{\rm T}, \alpha$ and β can be analytically calculated, and are known to be

$$\begin{aligned} \alpha = \frac{2k}{D_+} \frac{k + i\kappa}{(k^2 - \kappa^2)D_-/D_+ \; + \; 2ik\kappa}; \quad \beta = \frac{2k}{D_+} \frac{-k + i\kappa}{(k^2 - \kappa^2)D_-/D_+ \; + \; 2ik\kappa} \; \exp(-2\kappa a); \\ A_{\rm R} = \alpha + \beta - 1; \quad A_{\rm T} = (\alpha e^{\kappa a} + \beta e^{\kappa a})\; e^{-ka}; \quad D_\pm \equiv 1 \pm \exp(-2\kappa a). \end{aligned} \tag{2}$$

The amplitudes $A_{\rm R}$, $A_{\rm T}$ satisfy the probability conservation law

$$|A_{\rm R}|^2 \; + \; |A_{\rm T}|^2 \; = \; 1. \tag{3}$$

The flux density

$$j = {\rm Re}[\frac{i\hbar}{2m}\psi(x)\frac{\partial\psi^*(x)}{\partial x}] \tag{4}$$

does not depend on x. Before the barrier it equals the difference $(1 - |A_{\rm R}|^2)$ between the incoming and reflected wave flux–densities. It is less known that, inside the barrier, the fluxes for the separate components of $\psi_{\rm II}$ (exponentially decreasing and increasing, respectively: $\alpha e^{-\kappa x}$ and $\beta e^{\kappa x}$) do vanish. Only their interference does provide the conservation of j.

I-2.2 – **Construction of the wave–packet**

For later use in the non-stationary description of actually moving wave–packets, let us consider a wave–packet constructed in terms of the solutions $\psi(x;k)$ of the stationary Schroedinger equation: namely, by integrating over E from 0 to ∞ with a weight-amplitude $g(E-\overline{E})$

$$\Psi(x,t) = \int \mathrm{d}E \; g(E-\overline{E}) \; \psi(x;k) \; \exp(-iEt/\hbar) \tag{5}$$

where we introduced the resolution of the time-evolution operator, with the normalization condition $\int \mathrm{d}E \; |g|^2 = 1$, quantity $\overline{E}$ being the average kinetic energy.

In the case of a Gaussian wave–packet it is convenient to pass from the energy to the impulse representation, by recalling that $\mathrm{d}E = (\hbar^2 k/2m)\mathrm{d}k$; when the spread in E of $g(E-\overline{E})$ is much smaller than $\overline{E}$, one easily gets

$$g(E-\overline{E}) \; \mathrm{d}E \approx G(k-\overline{k}) \; \mathrm{d}k \equiv \frac{\hbar^2 \overline{k}}{2m} \;\; g(\frac{\hbar^2}{2m} \; [k^2 - \overline{k}^2]) \; \mathrm{d}k, \tag{6}$$

with $G \equiv C \exp[-b(k-\overline{k})^2]$. Of course, the (initial) wave–packet of the incoming waves will have a Gaussian shape also in the configuration space.

By inserting in the integral of eq.(5) the reflected (ψ_{R}) or transmitted (ψ_{T}) wave, instead of the total wave ψ, we obtain the *final* reflected or transmitted wave–packets, respectively, carrying a time-delay due to the interaction. Notice that one could expect a distortion in the wave–packet form due to the energy dependence of A_{R} and A_{T}; but it has been already shown, for a wide class of weight amplitudes, such a distortion to be negligible.[7] Furthermore, we shall get rid also of the wave components with above-barrier energies by introducing the additional transformation

$$g(E-\overline{E}) \longrightarrow g(E-\overline{E}) \; \Theta(E-V_0), \tag{7}$$

in order to avoid distortions of the sub-barrier penetration (tunnelling).

For simplicity's sake, we shall in general address ourselves to quasi-monochromatic packets, for which the energy spread ΔE is so much smaller than $\overline{E}$ that it is possible to adopt the approximation

$$|g(E-\overline{E})|^2 \simeq \delta(E-\overline{E}) \tag{8}$$

in all the final expressions for our quantities, when averaged over $\rho\mathrm{d}x$ or $J\mathrm{d}t$; where

$$\rho \equiv |\Psi(x,t)|^2; \quad J \equiv \mathrm{Re}[\frac{i\hbar}{2m}\Psi(x,t)\frac{\partial\Psi^*(x,t)}{\partial x}] \tag{9}$$

are the probability density [for a particle to be located in the unitary space–interval centered at x] and the probability–density flux [for a particle to pass through position x during the unitary time–interval centered at t], respectively.

I-2.3 – The ordinary phase–times

Following the usual procedure, introduced in[2−4], it is easy to get the ordinary phase–times for quasi-monochromatic packets, in the stationary–phase approximation

$$\tau_{\mathrm{T}}^{\mathrm{Ph}}(x_{\mathrm{i}}, x_{\mathrm{f}};\ E) = \frac{1}{v}\ (x_{\mathrm{f}} - x_{\mathrm{i}}) + \hbar\ \frac{\mathrm{d}(\arg A_{\mathrm{T}})}{\mathrm{d}E} \tag{10}$$

and

$$\tau_{\mathrm{R}}^{\mathrm{Ph}}(x_{\mathrm{i}}, x_{\mathrm{i}}; E) = \frac{1}{v}\ (-2x_{\mathrm{i}}) + \hbar\ \frac{\mathrm{d}(\arg A_{\mathrm{R}})}{\mathrm{d}E} \tag{11}$$

where $v \equiv \hbar k/m$ is the group–velocity; $x_{\mathrm{i}} \in$ region I, and $x_{\mathrm{f}} \in$ region III. Eqs.(10) and (11) refer to a transmitted [from the initial position x_{i} to the final position x_{f}] particle and to a reflected [from the initial position x_{i} to the same position] particle, respectively; cf., *e.g.*, ref.[1].

For a rectangular barrier with height V_0, the phase–times (10) and (11), when linearly extrapolated[1] to the barrier region ($x_{\mathrm{i}} = 0;\ x_{\mathrm{f}} = a$) would become

$$\tau_{\mathrm{T}}^{\mathrm{Ph}}(0, a; E) = \tau_{\mathrm{R}}^{\mathrm{Ph}}(0, a; E) = \frac{m}{\hbar k\kappa D}\ [2\kappa a k^2(\kappa^2 - k^2) + {k_0}^4 \mathrm{Sinh}(2\kappa a)]\ , \tag{12}$$

which, for $\kappa a >> 1$, would simply yield $2/v\kappa$. In eqs.(12), it is $D \equiv 4\kappa^2 k^2 + {k_0}^4 \mathrm{Sinh}^2(\kappa a)$; and $k_0 \equiv 2mV_0/\hbar$. In other words,[7] for sufficiently wide —*i.e.*, opaque— (or high) barriers, eqs.(12) do not depend on the barrier width a, and the effective tunnelling–velocity a/τ^{Ph} may become arbitrarily large [Hartmann and Fletcher effect[9,10]].

One of the main objections against extrapolations (12) is that they do not describe the actual asymptotic behaviour of the phase–times; since they disregard the fact that both the [magnitude of the] initial packet mean–position, $|x_{\mathrm{i}}|$, and quantity $x_{\mathrm{f}} - a$ (where x_{f} is the transmitted packet mean–position) must be large with respect to the packet spatial extension [of the order of $\hbar v/\Delta E$], in order to avoid “interference” effects between physically quite different processes (*i.e.*, between incident and reflected waves).

Therefore, it is not completely correct to attribute to the extrapolated phase–times the physical meaning of "times spent in the barrier region (= inside the barrier)". Moreover, one cannot separate in $\tau_{\rm T}^{\rm Ph}$ and $\tau_{\rm R}^{\rm Ph}$ the self-interference delays from the time spent inside the barrier.

Before going on, let us clarify the behaviour of the phase–times at the very *top* of the barrier, and check whether there is any continuity —there— between the values of the sub-barrier tunnelling time and those for the above-barrier case. Let us compare eqs.(12) with the following expression for the above-barrier transmission time:

$$\tau_{\rm T}^{\rm Ph}(0,a;E>V_0) = \frac{2m}{\hbar kq}\;\frac{-(k^2-q^2)^2\tan(qa)+4qak^2(k^2+q^2)/\cos^2(qa)}{4k^2q^2+[(k^2+q^2)\tan(qa)]^2} \tag{13}$$

which was obtained(*), by the stationary–phase method, for the case of a rectangular barrier. In such a case, it is $\psi_{\rm II} = \gamma e^{iqx} + \delta e^{-iqx}$ with $q \equiv \sqrt{2m(E-V_0)}/\hbar$, and the coefficients γ and δ are analytically evaluable. By comparing eqs.(12) and (13) one gets

$$\lim_{\kappa\to 0}\tau_{\rm T}^{\rm Ph}(0,a;E>V_0) = \frac{mka^3}{6\hbar(1+k^2a^2/4)} \;\underset{a\to\infty}{\longrightarrow}\; \frac{2ma}{3\hbar k}\;, \tag{12'}$$

$$\lim_{\kappa\to 0}\tau_{\rm T}^{\rm Ph}(0,a;E<V_0) = \frac{mka^3}{6\hbar(1+k^2a^2/4)} \;\underset{a\to\infty}{\longrightarrow}\; \frac{2ma}{3\hbar k}\;. \tag{13'}$$

In other words, we find that that the two limits (12'), (13') do coincide with each other, and linearly depend on a for "opaque" barriers (provided that the condition $\kappa a \to 0$ holds). Notice that such a result does not contradict the Hartmann and Fletcher effect, since the latter takes place only when $\kappa a \to \infty$, while it is absent for finite values of κa.

0(*) These calculations have been explicitly performed by V.S. Sergeyev.

I-2.4 – The dwell time

The total scattering time duration has been defined in[11] as the probability for the particle to be localized in the interval between the initial (maybe, source) position and the final (maybe, detector) position, divided by the incident particle flux density; that is to say, as the time spent by a particle while travelling inside such space–interval: the so–called *dwell time*. In the chosen case of particles moving only along x, the dwell time is therefore defined as:[12]

$$\tau^{\mathrm{Dw}}(x_{\mathrm{i}}, x_{\mathrm{f}}; k) = \frac{1}{v} \int_{x_{\mathrm{i}}}^{x_{\mathrm{f}}} \mathrm{d}x \, |\psi(x;k)|^2. \tag{14}$$

For a rectangular barrier, the (dwell) time spent inside the barrier becomes:[12]

$$\tau^{\mathrm{Dw}}(0, a; k) = \frac{mk}{\hbar \kappa D} \left[2\kappa a(\kappa^2 - k^2) + \frac{2mV_0}{\hbar^2} \mathrm{Sinh}(2\kappa a)\right] \tag{15}$$

which, for $\kappa a >> 1$, would give $\hbar k / \kappa V_0$. The results (12) and (15) are in sharp contrast with each other with regard to the k-dependence. Let us comment on this point.

The stationary definition (14) for the dwell time, according to us, is not self-consistent from its very beginning, and appears to be in contradiction with its physical meaning. In fact, the time variable is firstly discarded (in passing from the time–dependent to the stationary Schroedinger equation), and later on it is re-introduced in an artificial, *ad hoc* way: namely, through the introduction of a localization–probability expressed in terms of time–independent wave functions, instead of actually moving wave–packets.

Moreover, even the modified "dwell time approaches" with time–dependent wave functions[13−15], in which

$$\overline{\tau}^{\mathrm{Dw}}(x_{\mathrm{i}}, x_{\mathrm{f}}; k) = \frac{\int_{-\infty}^{\infty} \mathrm{d}t \int_{x_{\mathrm{i}}}^{x_{\mathrm{f}}} \mathrm{d}x \; \rho(x,t)}{\int_{-\infty}^{\infty} \mathrm{d}t \; J_{\mathrm{in}}(x_{\mathrm{i}}, t)} \quad , \tag{16}$$

do still contain *formal* time–averages, that are *not* actual averages over the physical time (*i.e.*, the time $t(x)$ at which the considered particle passes through the position x). In fact those averages should be obtained —at least when the direction of flux J is time–independent— by integrating $J\mathrm{d}t$, and not $\rho \mathrm{d}t$.[6,16] In eq.(16) quantity J_{in} is defined as in eq.(9), just replacing $\psi(x;k)$ of eq.(5) by $\psi_{\mathrm{in}}(x;k) \equiv e^{ikx}$. At last, the "dwell–time approaches" are unable[6] to define and study the time *distributions* for any kind of collision process.

I-2.5 – The local Larmor times

In[17] a *gedanken experimente* was proposed for measuring the scattering duration as the ratio θ/ω, where θ is the angle by which the magnetic moment μ of the considered particle is rotated due to a small homogeneous magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$ (directed along z) supposedly present in the scattering region, and $\omega \equiv 2\mu B_0/\hbar$ is the Larmor precession frequency. For a magnetic field existing in the interval $(x_{\rm i}, x_{\rm f})$, and for an incident particle (moving along x and) with spin $\frac{1}{2}$ polarized along the x-direction (see Fig.I-2), θ results to be proportional to the average spin component $< s_y >$: namely, $\theta = -2 < s_y >_{\rm T}/\hbar$, or $\theta = -2 < s_y >_{\rm R}/\hbar$, for the transmitted or reflected waves, respectively. In this case, the Larmor times $\tau^{\rm La}_{y{\rm T}}(x_{\rm i}, x_{\rm f}; k)$ and $\tau^{\rm La}_{y{\rm R}}(x_{\rm i}, x_{\rm f}; k)$ become equal to the Phase Times, *plus* —however— terms which do oscillate as $kx_{\rm i}$ and $kx_{\rm f}$ vary.[1] In the particular case of a rectangular barrier one gets

$$\tau^{\rm La}_{y{\rm T}}(0, a; k) = \tau^{\rm La}_{y{\rm R}}(0, a; k) = \tau^{\rm Dw}(0, a; k). \tag{17}$$

In Baz's approach, as it was shown in refs.[6], the expressions for the collision duration [*e.g.*, eqs.(17)] are artificially distorted by the sharp boundaries attributed to the magnetic–field region; in other words, are influenced by the mathematical, rather than physical, assumptions. Actually, the oscillating terms do depend on the kind of boundary [for instance, smoothed] that one adopts. Moreover, both those oscillating terms vanish when $x_{\rm i} \to -\infty$ and $x_{\rm f} \to \infty$, once one does average over the incident particle energy–spread; so that the final expressions do coincide with the Phase Times.

It it also known[1] that the mathematical behaviour assumed for the magnetic–field boundary does not influence only the spin components along y. In fact (see Fig.I-2), the incident particle has finite probabilities of being spin-up or spin-down along the field–direction z. As pointed out in ref.[12], the spin-up components will be preferentially *transmitted* (except when $\mathrm{d}|A_{\rm T}|^2/\mathrm{d}E < 0$): so that one gets the noticeable result that $< s_z >_{\rm T} \;\; >> \;\; < s_y >_{\rm T}$. On the basis of what precedes, Büttiker in ref.[12] introduced the new Larmor times: $\tau^{\rm La}_{z{\rm T}}(x_{\rm i}, x_{\rm f}; k)$ and $\tau^{\rm La}_{z{\rm R}}(x_{\rm i}, x_{\rm f}; k)$, both defined analogously to quantities $\tau^{\rm La}_{y{\rm T}}$ and $\tau^{\rm La}_{y{\rm R}}$, respectively; as well as the *hybrid* Larmor times, defined as follows:

$$(\tau^{\rm La}_{\rm T,R})^2 = (\tau^{\rm La}_{y{\rm T,R}})^2 + (\tau^{\rm La}_{z{\rm T,R}})^2. \tag{18}$$

However, the introduction of so many time durations for a single collision (*e.g.*, transmission and reflection processes) seems to us physically unjustified.

Let us notice that, for an *opaque* rectangular barrier, we obtain

$$\tau^{\rm La}_{z{\rm T}}(0,a;k) \simeq \frac{ma}{\hbar\kappa} \quad , \tag{19}$$

which results to be different from both the extrapolated Phase Times (12') and the Dwell Times (15).

I-2.6 – **A complex time approach**

As it is known, a formal generalization of the classical time spent by a particle inside the barrier can lead, for $E < V_0$ (when the actual presence of the particle, there, is forbidden by classical mechanics), to the introduction of a complex time. More generally, an analogous extension of classical time *to the quantum domain* has been recently proposed in[18] (see also, *e.g.*, refs.[19–21], and refs. therein). For one–dimensional motion, following ref.[18], one of the natural "quantum generalizations" of the classical expression

$$\tau[P(t)] = \int_{t_{\rm i}}^{t_{\rm f}} {\rm d}t \int_V {\rm d}x \;\; \delta[x - P(t)] \tag{20}$$

for the time spent inside a region V —where $P(t)$ is the classical path going from $x_{\rm i}(t_{\rm i})$ to $x_{\rm f}(t_{\rm f})$— is the path–integral average

$$\tau^{\rm Qu}(x_{\rm i}, t_{\rm i}; x_{\rm f}, t_{\rm f}) \; = < \tau[P(\;\;)] >_{\rm paths} \quad , \tag{21}$$

in which $P(\;\;)$ is any arbitrary path between the given end–points.

For the process relative to Fig.I-1 and to eq.(1), one has:[5,19]

$$\tau^{\rm Qu}_{\rm T} = i\hbar \int_V {\rm d}x \, \frac{\delta \log A_{\rm T}}{\delta\Omega(x)} \; ; \qquad \tau^{\rm Qu}_{\rm R} = i\hbar \int_V {\rm d}x \, \frac{\delta \log A_{\rm R}}{\delta P(x)} \quad , \tag{22}$$

where V is nothing but the interval $(x_{\rm i}, x_{\rm f})$ [or, in particular, $(0,a)$]; and $\delta / \delta P(x)$ is a functional derivative. In general, quantities $\tau^{\rm Qu}_{\rm T,R}$ are complex; and are connected with the Larmor Times by the relations

$$\text{Re}\; \tau^{\rm Qu}_{\rm T,R} = \tau^{\rm La}_{y{\rm T,R}} \; ; \qquad \text{Im}\; \tau^{\rm Qu}_{\rm T,R} = -\tau^{\rm La}_{z{\rm T,R}} \quad . \tag{23}$$

Of course, complex time is a useful theoretical tool; even if the ordinary tunnelling–times should be real. The physical meaning of the imaginary part is still controversial.[22]

I-2.7 – The Büttiker–Landauer time

In refs.[23] the tunnelling times were studied via a new kind of "gedanken experimente", namely by supposing the barrier to possess, besides the ordinary (time–independent) part, an additional part oscillating in time:

$$V(t) = V_0 + V_1 \cos \omega t \ . \tag{24}$$

Since the potential V varies with time, the incident particles —if endowed with electric charge (or magnetic moment)— can absorb or emit "modulation quanta" $\hbar\omega$ during the tunnelling, which leads to the appearance of *sidebands* with energies $E + n\hbar\omega$; $[n = \pm 1, \pm 2, ...]$. In the first–order approximation in V_1, it is enough to consider only the neighboring sidebands with energies $E \pm \hbar\nu$. Büttiker and Landauer did obtain the following expressions for the relative sideband intensities

$$\begin{aligned} I_{\mathrm{T}}^{(\pm 1)}(\omega) &= | \frac{A_{\mathrm{T}}^{(\pm 1)}(\omega)}{A_{\mathrm{T}}^{(0)}} |^2 \simeq [\frac{V_1}{2\hbar\omega} \ \exp(\pm\omega\tau_{\mathrm{T}}^{\mathrm{BL}}) - 1]^2 \ , \\ I_{\mathrm{R}}^{(\pm 1)}(\omega) &= | \frac{A_{\mathrm{R}}^{(\pm 1)}(\omega)}{A_{\mathrm{R}}^{(0)}} |^2 \simeq (\frac{V_1 \tau_{\mathrm{R}}^{\mathrm{BL}}}{2\hbar})^2 (1 \pm \omega\tau_\kappa) \ , \end{aligned} \tag{25}$$

where $A_{\mathrm{T}}^{(\pm 1)}$ and $A_{\mathrm{T}}^{(0)}$ are the perturbed (sideband) and unperturbed transmission–amplitudes, respectively; and similarly for the reflection–amplitudes $A_{\mathrm{R}}^{(\pm 1)}$ and $A_{\mathrm{R}}^{(0)}$. In eqs.(25) the last equalities ($\simeq$) hold only for the case of *opaque* [rectangular] barriers and not too high frequencies: *i.e.*, for $\hbar\omega$ small with respect to both E and $V_0 - E$. Moreover, $\tau_{\mathrm{T}}^{\mathrm{BL}} \equiv ma/\hbar\kappa$; $\tau_{\mathrm{R}}^{\mathrm{BL}} \equiv 2mk/[\hbar\kappa(\kappa^2 + k^2)]$; and $\tau_\kappa \equiv m/\hbar\kappa^2$. One can see that $\tau_{\mathrm{T}}^{\mathrm{BL}}$ is identical to the Larmor time $\tau_{z\mathrm{T}}^{\mathrm{La}}$ as given in eq.(19).

In our opinion, it is not worthwhile to report about the discussions originated by Büttiker–Landauer's approach, since they seem to us as being too technical and insufficiently justified; let us only quote, here, the refs.[24–27] However, two results should be mentioned.

First, Hauge and Stovneng[1] did find a simple connection between, on one side, the $\omega \to 0$ limits of $A_{\mathrm{T}}^{(\pm 1)}(\omega)/A_{\mathrm{T}}^{(0)}$ and $A_{\mathrm{R}}^{(\pm 1)}(\omega)/A_{\mathrm{R}}^{(0)}$, and, on the other side, the complex times of eqs.(23):

$$\frac{A_{\rm R}^{(\pm 1)}(\omega)}{A_{\rm R}^{(0)}} = -i\,\frac{V_1}{2\hbar}\;\tau_{\rm T}^{\rm Qu}\;; \qquad \frac{A_{\rm T}^{(\pm 1)}(\omega)}{A_{\rm T}^{(0)}} = -i\,\frac{V_1}{2\hbar}\;\tau_{\rm R}^{\rm Qu}\;, \tag{26}$$

even if the physical meaning of such a connection is not yet very clear.

Second, it is interesting to recall that Bruinsma and Bak[28] (see also ref.[1]) proposed the characteristic frequency $(\tau_{\rm T}^{\rm BL})^{-1} \equiv \hbar\kappa/ma$ to give information about the *coupling* between tunnelling and other accompanying channels, rather than about the intrinsic tunnelling times.

I-3 – ABOUT THE POSSIBILITY OF INTRODUCING CLEAR DEFINITIONS OF $\tau_{\rm T}$ AND $\tau_{\rm R}$.

I-3.1 – A comment on Hauge and Stovneng's conclusions.

After having reviewed the main definitions and evaluations of the tunnelling times (which we also have presented, and criticized, in Sect.**I-2**), the authors of ref.[1] concluded that no definite, acceptable approach still exists to calculating such tunnelling durations. As a necessary but not sufficient condition, to be obeyed by any physically acceptable expression of the tunnelling and reflection times $\tau_{\rm T}$ and $\tau_{\rm R}$, those authors did propose the following relation [T = Transmitted, in this case!]

$$\tau^{\rm Dw} = |A_{\rm T}|^2 \tau_{\rm T} \; + \; |A_{\rm R}|^2 \tau_{\rm R}\;, \tag{27}$$

which they required to be satisfied by the durations calculated via *any* method (except the dwell one, of course, which *ab initio* does not separate transmission from reflection time). Let us observe that the negative conclusion of ref.[1], which is actually the main conclusion of that review, is based not only on a criticism of all the previously existing approaches (a criticism that we made more complete and even stronger), but also on the fact that none of them satisfies condition (27).

However, relation (27) is unacceptable as a general criterion, since it attributes a special role (and meaning) to the Dwell Time $\tau^{\rm Dw}$, which on the contrary does *not* possess —in general— the physical meaning of global collision–duration, as we showed in Sect.**I-2.**4.

In the following Section, we are going to show that it is possible to define (and calculate) —in a physically meaningful and self-consistent way— those durations $\tau_{\rm T}$

and $\tau_{\rm R}$.

I-3.2 – **A general definition of the collision durations; and Applications to Tunnelling**

A direct, general definition of the collision durations was put forth first by Ohmura[29], and then improved —and generalized for finite distances— by us.[6,8] Following refs.[6], the transmission and reflection durations $<\tau_{\rm T}>$, $<\tau_{\rm R}>$ (averaged over the corresponding flux densities) can be defined, in the considered case of one–dimensional motion in presence of a barrier, as follows [T = Traversal]:

$$\begin{aligned} <\tau_{\rm T}> \;&\equiv\; <t(x_{\rm f})>^{\rm III}_{\rm T} - <t(x_{\rm i})>^{\rm I}_{\rm in} = \\ &= \frac{\int_{-\infty}^{\infty} {\rm d}t\; t\; J_{\rm T}^{\rm III}(x_{\rm f},t)}{\int_{-\infty}^{\infty} {\rm d}t\; J_{\rm T}^{\rm III}(x_{\rm f},t)} - \frac{\int_{-\infty}^{\infty} {\rm d}t\; t\; J_{\rm in}(x_{\rm i},t)}{\int_{-\infty}^{\infty} {\rm d}t\; J_{\rm in}(x_{\rm i},t)} = \\ &= \frac{\int_0^{\infty} {\rm d}E\, v\, |gA_{\rm T}|^2\; \tau_{\rm T}^{\rm Ph}(x_{\rm i},x_{\rm f},\; E)}{\int_0^{\infty} {\rm d}E\, v\, |gA_{\rm T}|^2} \equiv (x_{\rm f}-x_{\rm i}) <v^{-1}> + <\Delta\tau_{\rm T}> \,; \quad (28) \end{aligned}$$

$$\begin{aligned} <\tau_{\rm R}> \;&\equiv\; <t(x_{\rm i})>^{\rm II}_{\rm R} - <t(x_{\rm i})>_{\rm in} = \\ &= \frac{\int_{-\infty}^{\infty} {\rm d}t\; t\; J_{\rm R}^{\rm II}(x_{\rm i},t)}{\int_{-\infty}^{\infty} {\rm d}t\; J_{\rm R}^{\rm II}(x_{\rm i},t)} - \frac{\int_{-\infty}^{\infty} {\rm d}t\; t\; J_{\rm in}(x_{\rm i},t)}{\int_{-\infty}^{\infty} {\rm d}t\; J_{\rm in}(x_{\rm i},t)} = \\ &= \frac{\int_0^{\infty} {\rm d}E\, v\, |gA_{\rm R}|^2\; \tau_{\rm R}^{\rm Ph}(x_{\rm i},x_{\rm i},\; E)}{\int_0^{\infty} {\rm d}E\, v\, |gA_{\rm R}|^2} \equiv 2|x_{\rm i}| \; <v^{-1}> + <\Delta\tau_{\rm R}> \;, \quad (29) \end{aligned}$$

which hold when the incoming, reflected and transmitted wave–packets do not interfere: *i.e.*, are totally separated in space–time. Quantity $g \equiv g(E-\overline{E})$ was defined in eqs.(5), (6); while the ordinary Phase Times $\tau_{\rm T}^{\rm Ph}$, $\tau_{\rm R}^{\rm Ph}$ have been defined in eqs.(10) and (11). Moreover, quantities $J_{\rm R}^{\rm II}$ and $J_{\rm T}^{\rm III}$ are defined as in eq.(9), just replacing $\psi(x,k)$ of eq.(5) by $\psi_{\rm R} \equiv A_{\rm R}\; e^{-ikx}$ and $\psi_{\rm III} \equiv \psi_{\rm T} \equiv A_{\rm T}\; e^{ikx}$, respectively. Let us stress that our equations (28), (29) do implicitly *define* also the time delays $<\Delta\tau_{\rm T}>$, $<\Delta\tau_{\rm R}>$ due to transmission and reflection, respectively; as well as the "average" instants $<t(x_{\rm f})>^{\rm III}_{\rm T}$, $<t(x_{\rm i})>^{\rm II}_{\rm R}$, $<t(x_{\rm i})>_{\rm in}$ at which the corresponding wave–packets [transmitted, reflected and initial, respectively] pass through point $x_{\rm f}$ or $x_{\rm i}$.

Notice that for quasi–monochromatic wave packets, *i.e.*, when approximation (8) holds, eqs.(28), (29) do directly yield the ordinary Phase–Times $\tau_{\rm T}^{\rm Ph}$, $\tau_{\rm R}^{\rm Ph}$, given in eqs.(10), (11).

However, when $x_{\rm i}$, $x_{\rm f}$ are not far from the barrier, then it happens that the incoming, reflected and transmitted wave–packets *can* interfere. Moreover, the flux density $J(x,t)$ does in general change its sign with time; for example, the sign of $J(0,t)$ does change from + into − approximately a time $\hbar \, {\rm d}(\arg A_{\rm R})/{\rm d}E$ after the arrival at $x=0$ of the initial wave–packet. Therefore, the integrals $\int_{-\infty}^{\infty} {\rm d}t \; t \; J(x,t)$ do represent in general the algebraic sum of positive and negative quantities, so that the probability densities

$$\frac{J(x,t) \; {\rm d}t}{\int_{-\infty}^{\infty} {\rm d}t \; J(x,t)}$$

are not positive definite and do not possess a direct physical sense.

Each probability density acquires a physical meaning only during those (partial) time–intervals in which the corresponding flux–density $J(x,t)$ does *not* change its direction. As a consequence, the previous integrals are to be split into various integrals, each one carried over a partial time–interval such that during it the sign of $J(x,t)$ is only positive, or only negative. Afterwards, one will sum over all such contributions.

In other words, we have to deal only with the *positive definite* probability densities

$$\frac{{\rm d}t \; J_{+}(x,t)}{\int_{-\infty}^{\infty} {\rm d}t \; J_{+}(x,t)} \quad \text{and} \quad \frac{{\rm d}t \; J_{-}(x,t)}{\int_{-\infty}^{\infty} {\rm d}t \; J_{-}(x,t)} \; ,$$

where J_{+} and J_{-} represent the positive and negative values of $J(x,t)$, respectively.

Therefore, we do *propose* as physically adequate definitions for the average *transmission time* and the average *reflection time* the following expressions:

$$< \tau_{\rm T} > \; \equiv \; < t(x_{\rm f}) >_{+} - < t(x_{\rm i}) >_{+} =$$

$$= \frac{\int_{-\infty}^{\infty} {\rm d}t \; t \; J_{+}(x_{\rm f},t)}{\int_{-\infty}^{\infty} {\rm d}t \; J_{+}(x_{\rm f},t)} - \frac{\int_{-\infty}^{\infty} {\rm d}t \; t \; J_{+}(x_{\rm i},t)}{\int_{-\infty}^{\infty} {\rm d}t \; J_{+}(x_{\rm i},t)} \; ; \qquad (30)$$

$$< \tau_{\rm R} > \; \equiv \; < t(x_{\rm i}) >_{-} - < t(x_{\rm i}) >_{+} =$$

$$= \frac{\int_{-\infty}^{\infty} {\rm d}t \; t \; J_{-}(x_{\rm i},t)}{\int_{-\infty}^{\infty} {\rm d}t \; J_{-}(x_{\rm i},t)} - \frac{\int_{-\infty}^{\infty} {\rm d}t \; t \; J_{+}(x_{\rm i},t)}{\int_{-\infty}^{\infty} {\rm d}t \; J_{+}(x_{\rm i},t)} \; . \qquad (31)$$

Let us notice that, when $x_{\rm f} \geq a$ and $x_{\rm i} \rightarrow -\infty$, equation (30) goes into equation (28) since in that case $J_{+}(x_{\rm f},t) = J_{\rm T}(x_{\rm f},t) \equiv J(x_{\rm f},t)$ and $J_{+}(x_{\rm i},t) = J_{\rm in}(x_{\rm i},t)$. Analogously, when $x_{\rm i} \rightarrow -\infty$, equation (31) goes into equation (29) since in such a case $J_{+}(x_{\rm i},t) = J_{\rm in}(x_{\rm i},t)$ and $J_{-}(x_{\rm i},t) = J_{\rm R}(x_{\rm i},t)$.

What precedes, and in particular eqs.(30), (31), lead us to adopt as suitable, strict definitions for the very *Tunnelling Time* and the Reflection Time at the barrier–front (or *To-and-Fro Time*) the following ones:

$$< \tau_{\rm tun} > \ \equiv \ < t(a) >_+ - < t(0) >_+ \ , \tag{32}$$

$$< \tau_{\rm to-fro} > \ \equiv \ < \tau_{\rm R}(x_{\rm i} = 0) > \ \equiv \ < t(0) >_- - < t(0) >_+ \ , \tag{33}$$

where one should recall that the barrier starts at the point $x = 0$.

According to us, eqs.(32) and (33) are the correct definitions for the "Tunnelling time" $< \tau_{\rm tun} >$ and the "Reflection-due-to-the-whole-barrier time" (*i.e.*, the Reflection time at the barrier front wall) $< \tau_{\rm to-fro} >$. In conclusion, at variance with the authors of review[1], we think that a positive answer *can* be given to their question about the possibility of a precise, meaningful, univocal definition of the Tunnelling and Reflection times; such an answer being provided by our equations (30)–(33).

Unfortunately, simple analytical expressions for those time–durations *in the energy representation* exist only in particular, limiting cases. In general, even for Gaussian or quasi–monochromatic wave packets, calculations can be performed only numerically. Anyway, eqs.(30)–(33) can be qualitatively tested in an easy way.

We are left with the question of the time evolution of wave–packets **inside the barrier**: a problem which till now was paid attention to only in ref.[7]. We shall examine it in the coming Section.

Before going on, let here mention —however— that In the second part of this paper we shall show that our definition $< \tau_{\rm T} >$ for the average transmission time results to constitute an *improvement* with respect to the ordinary dwell–time $\overline{\tau}^{\rm Dw}$ formula.

I-3.3 – **Time evolution of the tunnelling wave–packets** *inside* **the barrier**

In ref.[7] calculations were performed of $\rho(x,t)$ and $J(x,t)$, at different points x inside the barrier, for a Gaussian wave–packet with an energy spread $\Delta E = 0.025 \ E$. The results of those calculations are presented in Fig.I-3, for $E = \frac{1}{2} V_0$ and $\kappa a = 5/\sqrt{2}$. From it, one can see that the times $\tau_\rho(x)$, $\tau_{J+}(x)$ and $\tau_{J-}(x)$, taken by the maximum of $\rho(x,t)$, $J_+(x,t)$ and $|J_-(x,t)|$, respectively, to penetrate the barrier till the depth $\Delta x = x$, do *not* depend *linearly* on x; and that $J(x,t)$ —inside the barrier— *does change* its sign with time, not very far from the barrier forward wall ($0 \leq x < 0.6 \ a$).

It is worthwhile to notice that: (i) although the continuity equation $\partial\rho/\partial t + \partial J/\partial x = 0$ goes on holding inside the barrier, nevertheless the equality $J = v\rho$ (which is valid for quasi-monochromatic wave packets outside the barrier) is *not* valid

—not even approximately— inside the barrier; and that: (ii) the effective velocities $v_\rho \equiv (\mathrm{d}\tau_\rho/\mathrm{d}x)^{-1}$, $v_{J+} \equiv (\mathrm{d}\tau_{J+}/\mathrm{d}x)^{-1}$ and $v_{J-} \equiv (\mathrm{d}\tau_{J-}/\mathrm{d}x)^{-1}$ of the maximum of ρ, J_+ and $|J_-|$, respectively, not only are *non-constant* as x varies, but also *do not coincide* with each other.

Passing to the *mean velocity* of the wave–packet while tunnelling through *the whole* barrier, it can be defined in a natural way as follows:

$$\overline{v_{\rm tun}} \;\equiv\; \frac{a}{< \tau_{\rm tun} >} \, . \tag{34}$$

Let us explicitly notice that, if $< \tau_{\rm tun} >$ does not increase with a, the "effective" speed $\overline{v_{\rm tun}}$ may become arbitrarily large. This would actually happen when the tunnelling time can be expressed by the ordinary Phase Time: $< \tau_{\rm tun} >= \tau_{\rm T}^{\rm Ph}(0, a; E)$; cf. Sect.**I-2.**3.

Moreover, we can show that even *in general*, for rectangular barriers and large values of a, quantity $< \tau_{\rm tun} >$ does not depend practically on a. In fact, it is obviously $< t(a) >_{\rm T}^{\rm III} = < t(a) >_+$; so that the corresponding terms, for $x_{\rm f} = a$, become equal in eq.(32) and in eqs.(28). Therefore, the same will happen —for quasi-monochromatic wave packets— for the corresponding term included in eqs.(12). As a consequence, the term $< t(a) >_+$ does not depend on a for *opaque* barriers. As to the second term, $< t(0) >_+$, of eq.(32), it differs from $< t(0) >_{\rm in}$ of eqs.(28) [owing to the effect of interference between incoming and reflected waves in the flux $J_+(x_{\rm i} = 0, t)$] by a quantity that depends on the reflection time $\tau_{\rm R}^{\rm Ph}(0, a; E)$, on the wave packet time–extension [of the order of $\hbar/\Delta E$], and on the form of the wave–packet. Consequently, also such a "second" term does not depend on a for opaque barriers. **We can conclude that the Hartmann–Fletcher's effect is valid even for our definition (32), so that our approach by wave-packets confirms that Q.M. predicts Superluminal tunnelling through opaque barriers**.

Let us mention, at this point, that another example of barrier —the inverted oscillator potential— was carefully investigated by Barton,[30] through a slightly different formalism. In that paper some interesting new results (which are partially similar to our ones) have been put forth in connection with the time evolution of the tunnelling wave–packets. In particular, Barton met cases in which, under the barrier, wave–groups with lower energy travel faster [cf. also our eq.(28), as well as eq.(12)].

It is perhaps worthwhile to add the following observations. The arriving, initial wave–packet does interfere with the reflected waves, that start to be generated as soon as the packet forward tail reaches the barrier edge; in such a way that (when considering the profiles of fluxes $J(x, t)$ before the barrier) the backward tail of $J_{\rm in}$ decreases —for destructive interference with $J_{\rm R}$— in a larger degree than the forward one. This simulates an increase of the average speed of the entering–flux profile, $J_+(x, t)$. Hence, the term $< t(x) >_+^{\rm I}$ decreases for negative $x \approx 0$. In other words, the effective (average) flight–time of the approaching packet from the source to the barrier does decrease. Let us now consider what happens inside the barrier, for positive $x \approx 0$. An analogous interference effect leads to expect an *increase* of the (effective) tunnelling time $< \tau_{\rm tun} >$; which consequently will not coincide with the Phase Time $\tau_{\rm T}^{\rm Ph}(0, a; E)$: not even for quasi-monochromatic packets. Finally, it is interesting to note, and easy to recognize, that the time–flight decrease before the barrier, and the tunnelling–time increase inside the barrier, do exactly cancel each other out, so that the total effect vanishes. In any case, let us stress that the fact that the entrance time–instant $< t(0) >_+$ is decreased by the mentioned "distortion" does *not* obscure the physical sense of our definition of the tunnelling time $< \tau_{\rm tun} >$, eq.(32).

Coming back to the time–evolution of the wave packet inside the barrier, we cannot describe it as the quasi–classical motion of a particle. We can visually describe it, on the contrary, in terms of the "motions" of the three densities $\rho^{\rm II}(x, t)$, $J_+^{\rm II}(x, t)$ and $J_-^{\rm II}(x, t)$.

In particular, we can pictorially say that both profiles [bumps] of the "incoming" and the "reflected" flux–densities $J_+(0, t)$, $J_-(0, t)$ (depicted for $x = 0$ as a function of time: see Fig.I-3) do repeat themselves —with distortion— for increasing values of the penetration depth x. Such a "transmission" takes place rather rapidly (when considering the velocities v_{J+}, v_{J-} of the bump maxima) in comparison with the initial speed v, non-uniformly, and with a gradual decreasing of the bumps [the *non-uniformity* of such "motions" being immediately evident from the *non-linear* dependence on x of the stationary wave-function phases inside the barrier: cf. eqs.(1)]. In particular, such decreasing is so strong for the reflected waves, that they *disappear* at the end of the barrier. At last, we can qualitatively say that the probability density $\rho(x, t)$ does conserve the form of its temporal shape at every position x inside the barrier, but does exponentially decreases as the depths increases, and quickly, non-uniformly moves towards the final barrier wall.

I-4 – CONCLUSIONS OF PART I AND PERSPECTIVES

In the Fist Part of this work, besides a critical review of the previous definitions of the tunnelling times and of their consequences, we proposed for them —on the basis of our general formalism[6,8]— *new* definitions, eqs.(30)–(33), that we regard as physically acceptable. We do share with the authors of review[1], however, the opinion that none of the previously known definitions were generally acceptable.

In the First Part above we began analyzing —for the first time— also the time–evolution of the tunnelling process **inside** the barrier.

Let us recall that our formalism is based on the introduction and application of a [non–selfadjoint, but *hermitian*] operator for Time as well as on suitable definitions for the observables' time–averages [cf. eqs.(28)–(31)]. The physical self-consistency of such a formalism may be regarded as supported by results as: (i) the validity of a correspondence principle between the time–energy QM commutation relation and the CM Poisson brackets; (ii) the validity of an Ehrenfest principle for the average time–durations;[6,8] (iii) the coincidence of the quasi-classical limit of our own QM definitions for time durations (when such a limit exists: *i.e.*, for above–barrier energies) with the analogous, well-known expressions of classical mechanics [see in particular the second and third refs.[6], and refs. therein]. Moreover, definitions similar to eqs.(28)–(29) have been already applied and tested in the analysis of experimental data on nuclear reaction durations, in the range $10^{-21} - 10^{-15}$s, obtained by blocking–effect, X–ray spectroscopy and brehmsstrahlung experiments [see in particular the first two refs.[6], and refs. therein]. Let us stress that for completely extracting the time–duration values from experimental data on the above-mentioned (non–stationary) processes, which are always non–linearly depending on the nuclear reaction durations, it is necessary to have recourse not only to equations of the type (28)–(29), but also to correct definitions for the duration variances, and for the duration distribution higher–order central moments:[6] which is provided by our formalism. At last, let us mention that such a formalism did provide also useful tools for the resolution of some long–standing problems related to the time–energy uncertainty relation.[6, 8]

As to the future, let us before all recall that analytical expressions in the energy representation for $< \tau_{\rm tun} >$ and $< \tau_{\rm to-fro} >$ do not exist, even in the case of the simplest barriers. However, numerical calculations can be straightforwardly performed about the time evolution of the considered wave–packet inside the barrier. On the basis of such one-particle (and one-dimensional) calculations, it will be possible to start developing a *kinematical theory* for the tunnelling of bound or metastable many–particle systems, and of unbound aggregates, through various barriers.

Within the field of nuclear physics, it will be interesting —on the basis of the peculiarities of the tunnelling–process temporal evolution (Sect.**I-3.**3 above; and ref.[7])— to investigate the possibility of *observing* effects due to the change in form, in volume, in orientation and in life–time of many–particle systems (nuclei, fragments,...) during their tunnelling. As well as effects due to "collisions", inside the barrier (*e.g.*, the ion–ion barrier), between different, successive, penetrating particles: for example, the role of those possible collisions in enhancing two–proton sub-barrier transfer in heavy ion interactions. Another future task will be developing a multi-dimensional description, in terms of tunnelling processes, of the sub-barrier fusion of two nuclei: taking account of nuclear deformation, formation of noses or a neck, and of dissipation phenomena.[31]

At last, we should comment on the fact that —when $< \tau_{\rm tun} >$ does not practically depend on the barrier width a, as we have seen to happen for opaque or high barriers (Sects.**I-3.**3 and **I-2.**3)— then one apparently meets, in connection with eq.(34), speeds that (*inside the barrier*) can assume arbitrarily large values. This does not violate any postulate, *as far as* we deal with non-relativistic (quantum) physics; and in fact such a phenomenon has already been frequently met within "quantum systems".[32]

It is easy to check that the same would happen, in our case, *even* when replacing the Schroedinger equation by the Klein–Gordon or Dirac equations, at least for barriers that do not depend explicitly on time. More interesting is the occurrence of this fact in such "quantum field theories" (which deal, however, with semi–classical potentials).

One could think that those infinitely large speeds might disappear in a *self-consistent* relativistic quantum theoretical treatment. But such an expectation seems to be wrong, for reasons that come also from Special Relativity itself.[33] The Hartman (Superluminal tunnelling) effect should then be added to the already known results that suggest the existence of Superluminal motions: cf., *e.g.*, refs.[34]. Results that have been predicted, inside relativistic theories like QFT, by many authors, like Sudarshan[35], Van Dam and Wigner[36], Recami[33], Ne'eman[37] and others.

I-5 – Note added in the 1992 proofs:

In two recent papers[38,39], Büttiker and Landauer published some other critical comments about the various approaches examined by us in Sect.**I-2**. Let us mention, among them, a further criticism of Hauge and Stovneng's relationship [eq.(27), Sect.**I-3**.1], appeared in ref.[39]. Moreover, an interesting, brief review of the first experimental measurements of tunnelling times did appear in ref.[38].

CAPTIONS OF THE FIGURES OF PART I:

Fig.I-1 – The case of *stationary* scattering and tunnelling. In this figure we depict a generic potential–barrier $V(x)$ (which does not depend explicitly on time), with the three regions generated by its presence; and the incoming, reflected and transmitted plane waves.

Fig.I-2 – Orientations of spin and magnetic field for the case of the "Larmor clock" (see the text).

Fig.I-3 – A pictorial view of the time dependence of $J(x,t)$ and $\rho(x,t)$, for various values of the penetration depth x inside a rectangular barrier (for Gaussian wave–packets). Actually, as x increases, all the "bumps" suffer an exponential damping; but, for convenience, we neglected in this figure the exponential factor $\exp(-\kappa x)$.

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SECOND PART

II-1. – Introduction

In the First Part[II−1] of this paper, we have put forth an analysis of the main theoretical definitions of the sub-barrier tunnelling and reflection times, and proposed new definitions for such durations which seem to be self-consistent within *conventional* quantum mechanics.#1 In particular, the "prediction" by our theory[II−1] of the reality of the *Hartman effect*[II−2] in tunnelling processes has recently received —due to the analogy[II−3] between tunnelling electrons and evanescent waves— quite interesting, even if indirect, experimental verifications at Cologne,[II−4] Berkeley,[II−5] Florence[II−6] and Vienna.[II−6]

Main aims of this Second Part [that appeared in print in J. de Physique-I 5 (1995) 1351] are: presenting and analysing the results of several numerical calculations of the penetration and return times *inside* a rectangular potential barrier during tunnelling (Sect.**II-3**); and proposing new suitable formulae for the distribution variances of the transmission and reflection times (Sect.**II-2**). The results of our numerical evaluations seem to confirm that our approach is physically acceptable, and that it moreover implied, and implies, the existence of the so-called *"Hartman effect"* (i.e., of tunnelling with Superluminal group-velocities) even for *non*–quasi-monochromatic packets.

Before all, let us add here —however— some brief comments about a few furthet papers, appeared recently:

(i) First, let us mention that in Part I above we have overlooked a new expression for the dwell–time $\overline{\tau}^{\rm Dw}$ derived by Jaworsky and Wardlaw[II−7,8]

$$\overline{\tau}^{\rm Dw}(x_{\rm i}, x_{\rm f}; k) \ = \ \left(\int_{-\infty}^{\infty} {\rm d}t \, t \, J(x_{\rm f}, t) \ - \ \int_{-\infty}^{\infty} {\rm d}t \, t \, J(x_{\rm i}, t)\right) \left(\int_{-\infty}^{\infty} {\rm d}t \ J_{\rm in}(x_{\rm i}, t)\right)^{-1} \quad , \qquad \text{(II-1)}$$

which is indeed equivalent[II−7] to our eq.(16) of ref.[II-1] (all notations being defined therein):

$$\overline{\tau}^{\rm Dw}(x_{\rm i}, x_{\rm f}; k) \ = \ \left(\int_{-\infty}^{\infty} {\rm d}t \int_{x_{\rm i}}^{x_{\rm f}} {\rm d}x \ \rho(x, t)\right) \left(\int_{-\infty}^{\infty} {\rm d}t \ J_{\rm in}(x_{\rm i}, t)\right)^{-1} \quad . \qquad \text{(II-2)}$$

This equivalence *reduces* the difference, between our definition $< \tau_{\rm T} >$ of the average transmission time —under our assumptions— and quantity $\overline{\tau}^{\rm Dw}$, *to* the dif-

0#1 Let us take advantage of the present opportunity for pointing out that a misprint entered our eq.(10) in ref.[II-1], whose last term ka ought to be eliminated. Moreover, due to an *editorial* error, in the footnote at page 32 of our ref.[II-16] the dependence of G on Δk disappeared, whilst in that paper we had assumed $G(k - \overline{k}) \equiv C \exp[-(k - \overline{k})^2/(\Delta k)^2]$.

ference between the average made by using the positive–definite probability density $\mathrm{d}t\, J_+(x,t)/\int_{-\infty}^{\infty}\mathrm{d}t\, J_+(x,t)$ and the average made by using the ordinary "probability density" $\mathrm{d}t\, J(x,t)/\int_{-\infty}^{\infty}\mathrm{d}t\, J(x,t)$. Generally speaking, the last expression is *not* always positive definite, as it was explained at page 350 of ref.[II-1], and hence does not possess any direct physical meaning.

(ii) In ref.[II-9] an attempt was made to analyze the evolution of the wave packet mean position $< x(t) >$ ("center of gravity"), averaged over $\rho \mathrm{d}x$, during its tunnelling through a potential barrier. Let us here observe that the conclusion to be found therein, about the absence of a causal relation between the incident space centroid and its transmitted equivalent, holds *only* when the contribution coming from the barrier region to the space integral is negligible.

(iii) Let us also add that in ref.[II-10] it was analyzed the *distribution* of the transmission time τ_{T} in a rather sophisticated way, which is very similar to the dwell–time approach, however with an *artificial*, abrupt switching on of the initial wave packet. We are going to propose, on the contrary, and in analogy with our eqs.(30)-(31) in ref.[II-1], the following expressions, as physically adequate definitions for the *variances* (or dispersions) $\mathrm{D}\,\tau_{\mathrm{T}}$ and $\mathrm{D}\,\tau_{\mathrm{R}}$ of the transmission and reflection time [see Sect.**II-2**], respectively:

$$\mathrm{D}\,\tau_{\mathrm{T}} \;\equiv\; \mathrm{D}\,t_+(x_{\mathrm{f}}) + \mathrm{D}\,t_+(x_{\mathrm{i}}) \tag{II-3}$$

and

$$\mathrm{D}\,\tau_{\mathrm{R}} \;\equiv\; \mathrm{D}\,t_-(x_{\mathrm{i}}) + \mathrm{D}\,t_-(x_{\mathrm{i}}) \;, \tag{II-4}$$

where

$$\mathrm{D}\,t_{\pm}(x) \;\equiv\; \frac{\int_{-\infty}^{\infty}\mathrm{d}t\; t^2\, J_{\pm}(x,t)}{\int_{-\infty}^{\infty}\mathrm{d}t\, J_{\pm}(x,t)} \;-\; \left(\frac{\int_{-\infty}^{\infty}\mathrm{d}t\; t\; J_{\pm}(x,t)}{\int_{-\infty}^{\infty}\mathrm{d}t\, J_{\pm}(x,t)}\right)^2 . \tag{II-5}$$

Equations (I-3)–(I-5) are based on the formalism expounded in refs.[II-11], as well as on our definitions for $J_{\pm}(x,t)$ in ref.[II-1]. Of course, we are supposing that the integrations over $J_+(x_{\mathrm{f}})\,\mathrm{d}t$, $J_+(x_{\mathrm{i}})\,\mathrm{d}t$ and $J_-(x_{\mathrm{i}})\,\mathrm{d}t$ are independent of one another. We shall devote Sect.**II-2**, below, to these problems, i.e., to the problem of suitably defining mean values and variances of durations, for various transmission and reflection processes during tunnelling.

(iv) Below, in Sect.**II-4**, we shall briefly re-analyse some other definitions of tunnelling durations.

Before going on, let us recall that several reasons "justify" the existence of different approaches to the definition of tunnelling times: (a) the problem of defining tunnelling durations is closely connected with that of defining a time operator, i.e., of introducing *time* as a (non-selfadjoint) quantum mechanical observable, and subsequently of adopting a general definition for collision durations in quantum mechanics. Such preliminary problems did receive some clarification in recent times (see, for example, ref.[II-1] and citations [8] and [22] therein); (b) the motion of a particle tunnelling inside a potential barrier is a purely quantum phenomenon, devoid of any classical, intuitive limit; (c) the various theoretical approaches may differ in the choice of the boundary conditions or in the modelling of the experimental situations.

II-2. – Mean values and Variances for various Penetration and Return Times during tunnelling

In our previous papers, and in Part I above, we proposed for the *transmission* and *reflection* times some formulae which imply —as functions of the penetration depth— integrations over time of $J_+(x,t)$ and $J_-(x,t)$, respectively. Let us recall that the total flux $J(x,t)$ inside a barrier consists of two components, J_+ and J_-, associated with motion along the positive and the negative x-direction, respectively. Work in similar directions did recently appear in ref.[II-12].

Let us refer ourselves —here— to tunnelling and reflection processes of a particle by a potential barrier, confining ourselves to one space dimension. Namely, let us study the evolution of a wave packet $\Psi(x,t)$, starting from the initial state $\Psi_{\rm in}(x,t)$; and follow the notation introduced in ref.[II-1]. In the case of uni-directional motions it is already known[II−13] that the flux density $J(x,t) \equiv {\rm Re}[(i\hbar/m)\,\Psi(x,t)\;\partial\Psi^*(x,t)/\partial x]$ can be actually interpreted as the probability that the particle (wave packet) passes through position x during a unitary time–interval centered at t, **as it easily follows from the continuity equation and from the fact that quantity $\rho(x,t) \equiv |\Psi(x,t)|^2$ is the probability density** for our "particle" to be located, at time t, inside a unitary space–interval centered at x. Thus, in order to determine the *mean* instant at which a moving wave packet $\Psi(x,t)$ passes through position x, we have to take the average of the time variable t with respect to the weight $w(x,t) \equiv J(x,t)/\int_{-\infty}^{\infty} J(x,t)\,{\rm d}t$.

However, if the motion direction can *vary*, then quantity $w(x,t)$ is no longer positive definite, and moreover is not bounded, because of the variability of the $J(x,t)$ sign. In such a case, one can introduce the two weights:

$$w_+(x,t) = J_+(x,t)\;\left[\int_{-\infty}^{\infty} J_+(x,t)\,{\rm d}t\right]^{-1} \qquad ({\rm II}-6)$$

$$w_-(x,t) = J_-(x,t)\;\left[\int_{-\infty}^{\infty} J_-(x,t)\,{\rm d}t\right]^{-1}\;, \qquad ({\rm II}-7)$$

where $J_+(x,t)$ and $J_-(x,t)$ represent the positive and negative parts of $J(x,t)$, respectively, which are bounded, positive–definite functions, normalized to 1. Let us show that, from the ordinary probabilistic interpretation of $\rho(x,t)$ and from the well-known continuity equation

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial J(x,t)}{\partial x} = 0 \ , \tag{II – 8}$$

it follows *also in this (more general) case* that quantities w_+ and w_-, represented by eqs.(II-6), (II-7), can be regarded as the probabilities that our "particle" passes through position x during a unit time–interval centered at t (in the case of forward and backward motion, respectively).

Actually, for those time intervals for which $J = J_+$ or $J = J_-$, one can rewrite eq.(II-8) as follows:

$$\frac{\partial \rho_>(x,t)}{\partial t} = -\frac{\partial J_+(x,t)}{\partial x} \tag{II – 9.a}$$

$$\frac{\partial \rho_<(x,t)}{\partial t} = -\frac{\partial J_-(x,t)}{\partial x} \ , \tag{II – 9.b}$$

respectively. Relations (9.a) and (9.b) can be considered as formal definitions of $\partial\rho_>/\partial t$ and $\partial\rho_</\partial t$. Let us now integrate eqs.(II-9.a), (II-9.b) over time from $-\infty$ to t; we obtain:

$$\rho_>(x,t) = -\int_{-\infty}^{t} \frac{\partial J_+(x,t')}{\partial x}\, \mathrm{d}t' \tag{II – 10.a}$$

$$\rho_<(x,t) = -\int_{-\infty}^{t} \frac{\partial J_-(x,t')}{\partial x}\, \mathrm{d}t' \tag{II – 10.b}$$

with the initial conditions $\rho_>(x,-\infty) = \rho_<(x,-\infty) = 0$. Then, let us introduce the quantities

$$N_>(x,\infty;t) \equiv \int_{x}^{\infty} \rho_>(x',t)\, \mathrm{d}x' = \int_{-\infty}^{t} J_+(x,t')\, \mathrm{d}t' \ > 0 \tag{II – 11.a}$$

$$N_<(-\infty,x;t) \equiv \int_{-\infty}^{x} \rho_<(x',t)\, \mathrm{d}x' = -\int_{-\infty}^{t} J_-(x,t')\, \mathrm{d}t' \ > 0 \ , \tag{II – 11.b}$$

which have the meaning of probabilities for our "particle" to be located at time t on the semi-axis (x, ∞) or $(-\infty, x)$ respectively, as functions of the flux densities $J_+(x,t)$ or $J_-(x,t)$, provided that the normalization condition $\int_{-\infty}^{\infty} \rho(x,t) \mathrm{d}x = 1$ is fulfilled. The r.h.s.'s of eqs.(II-11.a) and (II-11.b) have been obtained by integrating the r.h.s.'s of eqs.(II-10.a) and (II-10.b) and adopting the boundary conditions $J_+(-\infty, t) = J_-(-\infty, t) = 0$. Now, by differenciating eqs.(II-11.a) and (II-11.b) with respect to t, one obtains:

$$\frac{\partial N_>(x, \infty, t)}{\partial t} = J_+(x,t) > 0 \tag{II – 12.a}$$

$$\frac{\partial N_<(x, -\infty, t)}{\partial t} = -\, J_-(x,t) > 0 \; . \tag{II – 12.b}$$

Finally, from eqs.(II-11.a), (II-11.b), (II-12.a) and (II-12.b), one can infer that:

$$w_+(x,t) = \frac{\partial N_>(x, \infty; t)/\partial t}{N_>(x, -\infty; \infty)} \tag{II – 13.a}$$

$$w_-(x,t) = \frac{\partial N_<(x, -\infty; t)/\partial t}{N_<(-\infty, x; \infty)} \; , \tag{II – 13.b}$$

which justify the abovementioned probabilistic interpretation of $w_+(x,t)$ and $w_-(x,t)$. Let us notice, incidentally, that our approach does *not* assume any ad hoc postulate, contrarily to what believed by the author of ref.[II-14].

At this point, we can eventually define the *mean value* of the time at which our "particle" passes through position x, travelling in the positive or negative direction of the x axis, respectively, as:

$$< t_+(x) > \equiv \frac{\int_{-\infty}^{\infty} t \, J_+(x,t) \, \mathrm{d}t}{\int_{-\infty}^{\infty} J_+(x,t) \, \mathrm{d}t} \tag{II – 14.a}$$

$$< t_-(x) > \equiv \frac{\int_{-\infty}^{\infty} t J_-(x,t) \, \mathrm{d}t}{\int_{-\infty}^{\infty} J_-(x,t) \, \mathrm{d}t} \tag{II – 14.b}$$

and, moreover, the *variances* of the distributions of these times as:

$$\mathrm{D} \, t_+(x) \; \equiv \; \frac{\int_{-\infty}^{\infty} t^2 J_+(x,t) \mathrm{d}t}{\int_{-\infty}^{\infty} J_+(x,t) \mathrm{d}t} - [< t_+(x) >]^2 \tag{II – 15.a}$$

$$\mathrm{D} \, t_-(x) \; \equiv \; \frac{\int_{-\infty}^{\infty} t^2 J_-(x,t) \mathrm{d}t}{\int_{-\infty}^{\infty} J_-(x,t) \mathrm{d}t} - [< t_-(x) >]^2 \; , \tag{II – 15.b}$$

in accordance with the proposal presented in refs.[II-1,15].

Thus, we have a formalism for defining mean values, variances (and other central moments) related to the duration *distributions* of all possible processes for a particle, tunnelling through a potential barrier located in the interval $(0, a)$ along the x axis; and not only for tunnelling, but also for all possible kinds of collisions, with arbitrary energies and potentials. For instance, we have that

$$< \tau_{\rm T}(x_{\rm i}, x_{\rm f}) > \equiv < t_+(x_{\rm f}) > - < t_+(x_{\rm i}) > \qquad ({\rm II}-16)$$

with $-\infty < x_{\rm i} < 0$ and $a < x_{\rm f} < \infty$; and therefore (as anticipated in eq.(II-3)) that

$$\mathrm{D}\ \tau_{\rm T}(x_{\rm i}, x_{\rm f}) \equiv \mathrm{D}\, t_+(x_{\rm f}) + \mathrm{D}\, t_+(x_{\rm i}) \ ,$$

for *transmissions* from region $(-\infty, 0)$ to region (a, ∞) which we called[II-1] regions I and III, respectively. Analogously, for the pure (complete) tunnelling process one has:

$$< \tau_{\rm Tun}(0, a) > \equiv < t_+(a) > - < t_+(0) > \qquad ({\rm II}-17)$$

and

$$\mathrm{D}\ \tau_{\rm Tun}(0, a) \equiv \mathrm{D}\, t_+(a) + \mathrm{D}\, t_+(0) \ ; \qquad ({\rm II}-18)$$

while one has

$$< \tau_{\rm Pen}(0, x_{\rm f}) > \equiv < t_+(x_{\rm f}) > - < t_+(0) > \qquad ({\rm II}-19)$$

and

$$\mathrm{D}\ \tau_{\rm Pen}(0, x_{\rm f}) \equiv \mathrm{D}\, t_+(x_{\rm f}) + \mathrm{D}\, t_+(0) \qquad ({\rm II}-20)$$

[with $0 < x_{\rm f} < a$] for *penetration* inside the barrier region (which we called region II). Moreover:

$$< \tau_{\rm Ret}(x, x) > \equiv < t_-(x) > - < t_+(x) > \qquad ({\rm II}-21)$$

$$\mathrm{D}\ \tau_{\rm Ret}(x, x) \equiv \mathrm{D}\, t_-(x) + \mathrm{D}\, t_+(x) \qquad ({\rm II}-22)$$

[with $0 < x < a$] for "*return processes*" inside the barrier. At last, for *reflections* in region I, we have that:

$$< \tau_{\rm R}(x_{\rm i}, x_{\rm i}) > \equiv < t_-(x_{\rm i}) > - < t_+(x_{\rm i}) > \qquad ({\rm II}-23)$$

[with $-\infty < x_{\rm i} < a$], and (as anticipated in eq.(II-4)) that $\mathrm{D}\,\tau_{\rm R}(x_{\rm i}, x_{\rm i}) \equiv \mathrm{D}\,t_{-}(x_{\rm i}) + \mathrm{D}\,t_{+}(x_{\rm i})$.

Let us stress that our definitions hold within the framework of conventional quantum mechanics, without the introduction of any new postulates, and with the single measure expressed by weights (13.*a*), (13.*b*) for all time averages (both in the initial and in the final conditions).

According to our definition, the tunnelling phase time (or, rather, the transmission duration), defined by the stationary phase approximation for quasi-monochromatic particles, is meaningful *only* in the limit $x_{\rm i} \rightarrow \infty$ when $J_{+}(x,t)$ is the flux density of the initial packet $J_{\rm in}$ of *incoming* waves (*in absence of reflected waves*, therefore).

Analogously, the dwell time, which can be represented (cf. eqs.(II-1),(II-2)) by the expression[II−7,8,16]

$$\overline{\tau}^{\rm Dw}(x_{\rm i}, x_{\rm f}) = \left[\int_{-\infty}^{\infty} t\ J(x_{\rm f}, t)\ {\rm d}t - \int_{-\infty}^{\infty} t\ J(x_{\rm i}, t)\ {\rm d}t\right] \left[\int_{-\infty}^{\infty} J_{\rm in}(x_{\rm i}, t)\ {\rm d}t\right]^{-1} ,$$

with $-\infty < x_{\rm i} < 0$, and $x_{\rm f} > a$, is not acceptable, generally speaking. In fact, the weight in the time averages is meaningful, positive definite and normalized to 1 *only* in the rare cases when $x_{\rm i} \longrightarrow -\infty$ and $J_{\rm in} = J_{\rm III}$ (i.e., when the barrier is transparent).

II-3. – Penetration and Return process durations, inside a rectangular barrier, for tunnelling gaussian wave packets: Numerical results

We put forth here the results of our calculations of mean durations for various penetration (and return) processes, *inside* a rectangular barrier, for tunnelling gaussian wave packets; one of our aims being to investigate the *tunnelling speeds*. In our calculations, the initial wave packet is

$$\Psi_{\rm in}(x,t) = \int_{0}^{\infty} G(k-\overline{k})\ \exp[ikx - iEt/\hbar]\ {\rm d}k \qquad ({\rm II}-24)$$

with

$$G(k-\overline{k}) \equiv C \exp[-(k-\overline{k})^2/(2\,\Delta k)^2]\ , \qquad ({\rm II}-25)$$

exactly as in ref.[II-8]; and with $E = \hbar^2 k^2/2m$; quantity C being the normalization constant, and m the electron mass. Our procedure of integration was described in ref.[II-16].

Let us express the penetration depth in ångstroms, and the penetration time in seconds. In Fig.II-1 we show the plots corresponding to $a = 5$ Å, for $\Delta k = 0.02$ and

0.01 Å^{-1}, respectively. The penetration time $< \tau_{\rm Pen} >$ always tends to a *saturation* value.

In Fig.II-2 we show, for the case $\Delta k = 0.01$ Å^{-1}, the plot corresponding to $a =$ 10 Å. It is interesting that $< \tau_{\rm Pen} >$ is practically *the same* (for the same Δk) for $a = 5$ and $a = 10$ Å, a result that confirm, let us repeat, the existence[II−1] of the so-called *Hartman effect*.[II−2] Let us add that, when varying the parameter Δk between 0.005 and 0.15 Å^{-1} and letting a to assume values even larger than 10 Å, analogous results have been always gotten. Similar calculations have been performed (with quite reasonable results) also for various energies $\overline{E}$ in the range 1 to 10 eV. #2

In Figs.II-3, 4 and 5 we show the behaviour of the mean penetration and return durations as function of the penetration depth (with $x_{\rm i} = 0$ and $0 \leq x_{\rm f} \equiv x \leq a$), for barriers with height $V_0 = 10$ eV and width $a = 5$ Å or 10 Å. In Fig.II-3 we present the plots of $< \tau_{\rm Pen}(0, x) >$ corresponding to different values of the mean kinetic energy: $\overline{E}$ = 2.5 eV, 5 eV and 7.5 eV (plots 1, 2 and 3, respectively) with $\Delta k = 0.02$Å; and $\overline{E} = 5$ eV with $\Delta k = 0.04\text{Å}^{-1}$ (plot 4), always with $a = 5$Å. In Fig.II-4 we show the plots of $< \tau_{\rm Pen}(0, x) >$, corresponding to $a = 5$Å, with $\Delta k = 0.02\text{Å}^{-1}$ and 0.04 Å^{-1} (plots 1 and 2, respectively); and to $a = 10$Å, with $\Delta k = 0.02\text{Å}^{-1}$ and 0.04 Å^{-1} (plots 3 and 4, respectively), the mean kinetic energy $\overline{E}$ being 5 eV, i.e., one half of V_0. In Fig.II-5 the plots are shown of $< \tau_{\rm Ret}(x, x) >$. The curves 1, 2 and 3 correspond to $\overline{E} = 2.5$ eV, 5 eV and 7.5 eV, respectively, for $\Delta k = 0.02\text{Å}^{-1}$ and $a = 5$Å; the curves 4, 5 and 6 correspond to $\overline{E} = 2.5$ eV, 5 eV and 7.5 eV, respectively, for $\Delta k = 0.04\text{Å}^{-1}$ and $a = 5$Å; while the curves 7, 8 and 9 correspond to $\Delta k = 0.02\text{Å}^{-1}$ and 0.04 Å^{-1}, respectively, for $\overline{E} = 5$ eV and $a = 10$Å.

Also from the new Figs.II-3–5 one can see that: 1) at variance with ref.[II-8], no plot considered by us for the mean penetration duration $< \tau_{\rm Pen}(0, x) >$ of our wave packets presents any interval with negative values, nor with a decreasing $< \tau_{\rm Pen}(0, x) >$ for increasing x; and, moreover, that 2) the mean tunnelling duration $< \tau_{\rm Tun}(0, a) >$ does not depend on the barrier width a ("Hartman effect"); and finally that 3) quantity $< \tau_{\rm Tun}(0, a) >$ decreases when the energy increases. Furthermore, it is noticeable that also from Figs.II-3–5 we observe: 4) a rapid increase for the value of the electron penetration time in the initial part of the barrier region (near $x = 0$); and 5) a tendency of $< \tau_{\rm Pen}(0, x) >$ to a saturation value in the final part of the barrier, near $x = a$.

0#2 For the interested reader, let us recall that, when integrating over dt, we used the interval -10^{-13} s to $+10^{-13}$ s (symmetrical with respect to $t = 0$), very much larger than the temporal wave packet extension. [Recall that the extension in time of a wave packets is of the order of $1/(\overline{v}\,\Delta k) = (\Delta k \sqrt{2\overline{E}/m})^{-1} \simeq 10^{-16}$ s]. Our "centroid" has been always $t_0 = 0$; $x_0 = 0$. For clarity's sake, let us underline again that in our approach the initial wave packet $\Psi_{\rm in}(x, t)$ is not regarded as prepared at a certain instant of time, but it is expected to flow through any (initial) point $x_{\rm i}$ during the infinite time interval $(-\infty, +\infty)$, even if with a *finite* time–centroid t_0. The value of such centroid t_0 is essentially defined by the phase of the weight amplitude $G(k - \overline{k})$, and in our case is equal to 0 when $G(k - \overline{k})$ is real.

Feature 2), firstly observed for quasi-monochromatic particles,[II−2] does evidently agree with the predictions made in ref.[II-1] for arbitrary wave packets. Feature 3) is also in agreement with previous evaluations performed for quasi-monochromatic particles and presented, for instance, in refs.[II-1,2,15]. Features 4) and 5) can be apparently explained by interference between those initial penetrating and returning waves inside the barrier, whose superposition yields the resulting fluxes J_+ and J_-. In particular, if in the initial part of the barrier the returning–wave packet is comparatively large, it does essentially extinguish the leading edge of the incoming–wave packet. By contrast, if for growing x the returning–wave packet quickly vanishes, then the contribution of the leading edge of the incoming–wave packet to the mean penetration duration $< \tau_{\rm Pen}(0, x) >$ does initially (quickly) grow, while in the final barrier region its increase does rapidly slow down.

Furthermore, the larger is the barrier width a, the larger is the part of the back edge of the incoming–wave packet which is extinguished by interference with the returning–wave packet. Quantitatively, these phenomena will be studied elsewhere. Finally, in connection with the plots of $< \tau_{\rm Ret}(x, x) >$ as a function of x, presented in Fig.II-5, let us observe that: (i) the mean reflection duration $< \tau_{\rm R}(0, 0) > \equiv < \tau_{\rm Ret}(0, 0) >$ does not depend on the barrier width a; (ii) in correspondence with the barrier region betwen 0 and approximately 0.6 a, the value of $< \tau_{\rm Ret}(0, x) >$ is almost constant; while (iii) its value increases with x only in the barrier region near $x = a$ (even if it should be pointed out that our calculations near $x = a$ are not so good, due to the very small values assumed by $\int_{-\infty}^{\infty} J_-(x, t) {\rm d}t$ therein). Let us notice that point (i), also observed firstly for quasi-monochromatic particles,[II−2] is as well in accordance with the results obtained in ref.[II-1] for arbitrary wave packets. Moreover, also points (ii) and (iii) can be explained by interference phenomena inside the barrier: if, near $x = a$, the initial returning–wave packet is almost totally quenched by the initial incoming–wave packet, then only a negligibly small piece of its back edge (consisting of the components with the smallest velocities) does remain. With decreasing x $(x \to 0)$, the unquenched part of the returning–wave packet seems to become more and more large (containing more and more rapid components), thus making the difference $< \tau_{\rm Ret}(0, x) > - < \tau_{\rm Pen}(0, x) >$ almost constant. And the interference between incoming and reflected waves at points $x \le 0$ does effectively constitute a retarding phenomenon [so that $< t_-(x = 0) >$ is larger than $< \tau_{\rm R}(x = 0)$], which can explain the larger values of $< \tau_{\rm R}(x = 0, x = 0) >$ in comparison with $< \tau_{\rm Tun}(x = 0, x = a) >$.

Therefore our evaluations, in all the cases considered above, appear to confirm our previous analysis at page 352 of ref.[II-1], and our conclusions therein concerning in particular the validity of the Hartman effect also for *non*–quasi-monochromatic wave packets. Even more, since the interference between incoming and reflected waves before the barrier (or between penetrating and returning waves, inside the barrier, near the entrance wall) does just *increase* the tunnelling time as well as the transmission times, we can expect that our non-relativistic formulae for $< \tau_{\rm Tun}(0, a) >$ and $< \tau_{\rm T}(x_{\rm i} < 0, x_{\rm f} > a) >$ will always forward positive values.#3

At this point, it is necessary —however— to observe the following. *Even if our non-relativistic equations are not expected (as we have just seen) to yield negative times*, **nevertheless one ought to bear in mind that (whenever it is met an object**, $\mathcal{O}$, **travelling at Superluminal speed) negative contributions should be expected to the tunnelling times: and this ought not to be regarded as unphysical**. In fact, whenever an "object" $\mathcal{O}$ *overcomes* the infinite speed[II−18] with respect to a certain observer, it will afterwards appear to the same observer as its "*anti*-object" $\overline{\mathcal{O}}$ travelling in the opposite *space* direction[II−18]. For instance, when passing from the lab to a frame $\mathcal{F}$ moving in the *same* direction as the particles or waves entering the barrier region, the objects $\mathcal{O}$ penetrating through the final part of the barrier (with almost infinite speeds, like in Figs.II-1–5) will appear in the frame $\mathcal{F}$ as anti-objects $\overline{\mathcal{O}}$ crossing that portion of the barrier *in the opposite space–direction*[II−18]. In the new frame $\mathcal{F}$, therefore, such anti-objects $\overline{\mathcal{O}}$ would yield a *negative* contribution to the tunnelling time: which could even result, in total, to be negative. For any clarifications, see refs.[II-18]. So, we have no objections a priori against the fact that Leavens can find, in certain cases, negative values[II−8,17]: e.g, when applying our formulae to wave packets with suitable initial conditions. What we want to stress here is that the appearance of negative[II−2] times (it being predicted by Relativity itself,[II−18] when in presence of *anything* travelling faster than c) is not a valid reason to rule out a theoretical approach.

At last, let us —incidentally— recall and mention the following fact. Some preliminary calculations of penetration times (inside a rectangular barrier) for tunnelling gaussian wave packets had been presented by us in 1994 in ref.[II-16]. Later on —looking for any possible explanations for the disagreement between the results in ref.[II-8] and in our ref.[II-16]— we discovered, however, that an exponential factor was missing in a term of one of the fundamental formulae on which the numerical computations (performed by our group in Kiev) were based: a mistake that could not be detected, of course, by our careful checks about the computing process. Therefore, the new results of ours appearing here in Figs.II-1–2 [and appeared in J. de Physique-I 5 (1995) 1351] should *replace* Figs.1–3 of ref.[II-16]. One may observe that, by using the same parameters as (or parameters very near to) the ones adopted by for the Figs.3 and 4 of ref.[II-8], our new, corrected figures II-1 and 2 result to be more similar to Leavens' than the uncorrected ones (and

0#3 A different claim by Delgado, Brouard and Muga[II−17] does not seem to be relevant to our calculations, since it is based once more, like ref.[II-8], not on our but on different wave packets (and over–barier components are also retained in ref.[II-17], at variance with us). Moreover, in their classical example, they overlook the fact that the mean entrance time $< t_+(0) >$ gets contribution mainly by the rapid components of the wave packet; they forget, in fact, that the slow components are (almost) totally reflected by the initial wall, causing a quantum–mechanical reshaping that contributes to the initial "time decrease" discussed by us already in the last few paragraphs of page 352 in ref.[II-1]. All such phenomena *reduce* the value of $< t_+(0) >$, and we expect it to be (in our non-relativistic treatment) less than $< t_+(a) >$.

this is of course a welcome step towards the solution of the problem). One can verify once more, however, that our theory appears to yield *for those parameters* non-negative results for $< \tau_{\rm Pen}(x_{\rm f}) >$, contrarily to a claim in ref.[II-8]. Actually, our previous general conclusions have not been apparently affected by the mentioned mistake. In particular, the value of $< \tau_{\rm Pen}(x_{\rm f}) >$ increases with increasing $x_{\rm f}$, and tends to saturation for $x_{\rm f} \rightarrow a$. We acknowledge, however, that the difference in the adopted integration ranges ($-\infty$ to $+\infty$ for us, and 0 to $+\infty$ for ref.[II-8]) does not play an important role, contrarily to our previous belief,[II−16] in explaining the remaining discrepancy between our results and those in ref.[II-8]. Such a discrepancy *might* perhaps depend on the fact that the functions to be integrated do fluctuate heavily#4 (anyway, we did carefully check that our own elementary integration step in the integration over dk was small enough in order to guarantee the stability of the numerical result, and, in particular, of their sign, for strongly oscillating functions in the integrand). More probably, the persisting disagreement can be merely due to the fact —as recently claimed also by Delgado et al.[II−17]— that *different* initial conditions for the wave packets were actually chosen in ref.[II-8] and in ref.[II-1]. Anyway, our approach seems to get support, at least in some particular cases, also by a recent article by Brouard et al., which "generalizes" —even if starting from a totally different point of view— some of our results.[II−12,15].

Let us take advantage of the present opportunity for answering other criticisms appeared in ref.[II-8], where it has been furthermore commented about our way of performing actual averages over the physical time. We cannot agree with those comments: let us re-emphasize in fact that, within conventional quantum mechanics, the time $t(x)$ at which our particle (wave packet) passes through the position x is "statistically distributed" with the probability densities $\mathrm{d}t J_{\pm}(x,t) / \int_{-\infty}^{\infty} \mathrm{d}t J_{\pm}(x,t)$, as we explained at page 350 of ref.[II-1]. This distribution meets the requirements of the time–energy uncertainty relation.

We also answered in Sect.**II-1** the comments in ref.[II-8] about our analysis[II−1] of the dwell–time approaches.[II−19]

The last object of the criticism in ref.[II-8] refers to the impossibility, in our approach, of distinguishing between "to be transmitted" and "to be reflected" wave packets at the leading edge of the barrier. Actually, we do *distinguish* between them; only, we cannot —of course— *separate* them, due to the obvious *superposition* (and interference) of both wave functions in $\rho(x,t)$, in $J(x,t)$ and even in $J_{\pm}(x,t)$. This is known to be an unavoidable consequence of the superposition principle, valid for *wave functions* in conventional quantum mechanics. That last objection, therefore, should be addressed to quantum mechanics, rather then to us. Nevertheless, Leavens' aim of comparing the definitions proposed by us for the tunnelling times not only with conventional, but also with non–standard quantum mechanics *might* be regarded a priori as stimulating and

0#4 We can *only* say that we succeeded in reproducing results of the type put forth in ref.[II-8] by using larger steps; whilst the *"non-causal"* results disappeared —*in the considered cases*— when adopting small enough integration steps.

possibly worth of further investigation.

II-4. – Further remarks

In connection with the question of "causality" for relativistic tunnelling particles, let us stress that the Hartman-Fletcher phenomenon (very small tunnelling durations), with the consequence of Superluminal velocities for sufficiently wide barriers, was found theoretically also in QFT for Klein–Gordon and Dirac equations,[II−1] and experimentally for electromagnetic evanescent–mode wave packets[II−4−6] (tunnelling photons). It should be recalled that the problem of Superluminal velocities for electromagnetic wave packets in media with anomalous dispersion, with absorption, or behaving as a barrier for photons (such as regions with frustrated internal reflection) has been present in the scientific literature since long (see, for instance, quotations [2,1,18], and refs. therein); even if a complete settlement of the causal problem —already available for point particles[II−18]— does not seem to be yet available for the case of relativistic waves. Apparently, it is not sufficient to pay attention only to group velocity and mean duration for a "particle" passing through a medium; on the contrary, it is important taking into account and studying ab initio the *variances* (and the higher order central moments) of the duration distributions, as well as the wave packet *reshaping* in presence of a barrier, or inside anomalous media (even if reshaping does *not* play *always* an essential role).

Passing to the approaches *alternative* to the direct description of tunnelling processes in terms of wave packets, let us here recall those ones which are based on averaging over the set of all dynamical paths (through the Feynman path integral formulation, the Wigner distribution method, and the non-conventional Bohm approach), and others that use additional degrees of freedom which can be used as "clocks". General analyses of all such alternative approaches can be found in refs.[II-1,20–24] from different points of view.

If one confines himself within the framework of conventional quantum mechanics, then the Feynman path integral formulation seems to be adequate.[II−24] But it is not clear what procedure is needed to calculate physical quantities within the Feynman–type approach[II−23], and usually such calculations result in complex tunnelling durations. The Feynman approach seems to need further modifications if one wants to apply it to the time analysis of tunnelling processes, and its results obtaind up to now cannot be considered as final.

As to the approaches based on introducing additional degrees of freedom as "clocks", one can often realize that the tunnelling time happens to be noticeably distorted by the presence of such degrees of freedom. For example, the Büttiker-Landauer time is connected with absorption or emission of modulation quanta (caused by the time–dependent oscillating part of the barrier potential) during tunnelling, rather than with the tunnelling process itself.[II−1,15] And, with reference to the Larmor precession time, it has been shown[II−11,20] that this time definition is connected not only with the intrinsic tunnelling process, but also with the geometric boundaries of the magnetic field introduced as a part of the clock: for instance, if the magnetic field region is infinite, one ends up with the phase tunnelling time, after an average over the (small) energy spread of the wave packet. Actually, those "clock" approaches, when applied to tunnelling wave packets, seem to lead —after eliminating the distortion caused by the additional degrees of freedom— to the same results as the direct wave packet approach, whatever be the weight function adopted in the time integration.

A more pedagogical, and detailed, review on the same subject (in Italian, however) has been "published" electronically, as LANL Archive # cond-mat/9802126.

Before closing this paper, in which we met Superluminal motions, we would like to put forth and comment the following information. Since the pioneering work by Bateman, it is known that the relativistic wave equations —scalar, electromagnetic and spinor— admit solutions with subluminal ($v < c$) group velocities[II-25]. More recently, also Superluminal ($v > c$) solutions have been constructed for those homogeneous wave equations, in refs.[II-26] and quite independently in refs.[II-27]: in some cases just by applying a Superluminal Lorentz transformation[II-18,28]. Exactly the same happens in the case of acoustic waves, with the presence of "sub-sonic" and "Super-sonic" solutions[II-29]; so that one can expect they to exist, e.g., also for seismic wave equations. Or, rather, we might expect the same to be true even in the case of gravitational waves. At last, it is interesting to remark that some, at least, of the Super-sonic (and Super-luminal) solutions, when experimentally realized[II-30], appear to be X-shaped, so as predicted in 1982 in ref.[II-31].

A brief review on the experimental data, that —in four different sectors of physics— seem to indicate the existence of Superluminal motions, appeared as an Appendix to the paper in LANL Archives # physics/9712051 (to be published in Physica A, 1998).

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Captions of the Figures of Part II

Fig.II-1 – Behaviour of the average penetration time $< \tau_{\rm Pen}(0,x) >$ (expressed in seconds) as a function of the penetration depth $x_{\rm f} \equiv x$ (expressed in ångstroms) through a rectangular barrier with width $a = 5$ Å, for $\Delta k = 0.02$ Å^{-1} (dashed line) and $\Delta k = 0.01$ Å^{-1} (continuous line), respectively. The other parameters are listed in footnote #1. It is worthwhile to notice that $< \tau_{\rm Pen} >$ rapidly increases for the first, few initial ångstroms (~ 2.5 Å), tending afterwards to a saturation value. This seems to confirm the existence of the so-called "Hartman effect".[II−2,1,15]

Fig.II-2 – The same plot as in Fig.II-1, for $\Delta k = 0.01$ Å^{-1}, except that now the barrier width is $a = 10$ Å. Let us observe that the numerical values of the (total) tunnelling time $< \tau_{\rm T} >$ practically does not change when passing from $a = 5$ Å to $a = 10$ Å, again in agreement with the characteristic features[1] of the Hartman effect. Figures II-1 and 2 do improve (and correct) the corresponding ones, preliminarily presented by us in ref.[II-16].

Fig.II-3 – Behaviour of $< \tau_{\rm Pen}(0, x) >$ (expressed in seconds) as a function of x (expressed in ångstroms), for tunnelling through a rectangular barrier with width $a = 5$ Å and for different values of $\overline{E}$ and of Δk:
curve 1: $\Delta k = 0.02$Å^{-1} and $\overline{E} = 2.5$ eV; curve 2: $\Delta k = 0.02$Å^{-1} and $\overline{E} = 5.0$ eV; curve 3: $\Delta k = 0.02$Å^{-1} and $\overline{E} = 7.5$ eV; curve 4: $\Delta k = 0.04$Å^{-1} and $\overline{E} = 5.0$ eV.

Fig.II-4 – Behaviour of $< \tau_{\rm Pen}(0, x) >$ (in seconds) as a function of x (in ångstroms) for $\overline{E} = 5$ eV and different values of a and Δk:
curve 1: $a = 5$ Å and $\Delta k = 0.02$Å^{-1}; curve 2: $a = 5$ Å and $\Delta k = 0.04$Å^{-1}; curve 3: $a = 10$ Å and $\Delta k = 0.02$Å^{-1}; curve 4: $a = 10$ Å and $\Delta k = 0.04$Å^{-1}.

Fig.II-5 –Behaviour of $< \tau_{\rm Ret}(x, x) >$ (in seconds) as a function of x (in ångstroms) for different values of a, $\overline{E}$ and Δk:
curve 1: $a = 5$ Å, $\overline{E} = 2.5$ eV and $\Delta k = 0.02$Å^{-1}; curve 2: $a = 5$ Å, $\overline{E} = 5.0$ eV and $\Delta k = 0.02$Å^{-1}; curve 3: $a = 5$ Å, $\overline{E} = 7.5$ eV and $\Delta k = 0.02$Å^{-1}; curve 4: $a = 5$ Å, $\overline{E} = 2.5$ eV and $\Delta k = 0.02$Å^{-1}; curve 5: $a = 5$ Å, $\overline{E} = 5.0$ eV and $\Delta k = 0.02$Å^{-1}; curve 6: $a = 5$ Å, $\overline{E} = 7.5$ eV and $\Delta k = 0.02$Å^{-1}; curve 7: $a = 5$ Å, $\overline{E} = 5.0$ eV and $\Delta k = 0.02$Å^{-1}; curve 8: $a = 5$ Å, $\overline{E} = 5.0$ eV and $\Delta k = 0.02$Å^{-1}.

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