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Time Operator in Quantum Mechanics.

I: Nonrelativistic Case.

V. S. OLKHOVSKY (*) and E. RECAMI (**)

Institute of Theoretical Physics, Ukrainian Academy of Sciences - Kiev
Istituto di Fisica Teorica dell'Università - Catania

A. J. GERASIMCHUK

Kiev State University - Kiev

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Summary. — Within a *space-time description* of nonrelativistic quantum objects in terms of wave packets, one may simply consider (for every fixed spatial point $\bar{\boldsymbol{x}}$: see eq. (5)) the « wave-packets » $F(t, \bar{\boldsymbol{x}}) = \int \mathrm{d}E\, f(E, \bar{\boldsymbol{x}}) \cdot \exp[-iEt]$, that we shall assume to have as weight functions the vectors of the functional space $\mathscr{P}$ defined as follows. The space $\mathscr{P}$ is the space of continuous L^2-functions i) defined over the (total) energy interval $0 \leqslant E < \infty$, ii) with square-integrable first derivatives and iii) for which a Hermitian energy operator exists. Such a space $\mathscr{P}$ is *dense* in the Hilbert space of L^2-functions. It is then shown that a « good » time operator exists, $\hat{t} = -(i/2)(\overleftrightarrow{\partial}/\partial E)$, that acts onto $\mathscr{P}$ and i) is « symmetric » (but not self-adjoint), ii) is canonically conjugate to the (total) energy, and iii) satisfies the Ehrenfest principle and Galilei invariance. The old, known objection by Pauli is recognized to point out merely that our operator $\hat{t}$ cannot be hypermaximal, as was clarified by von Neumann. But even nonhypermaximal operators may be given a physical meaning and may represent observables in quantum mechanics. As already emphasized by previous authors, confining one's attention only to self-adjoint operators in quantum mechanics is too restrictive a postulate. Notwithstanding that $\hat{t}$ has no true eigenfunctions, nevertheless we succeed in calculating the *average values* of our time operator over our « wave packets » (and over the physical states corresponding to them). The case of wave packets moving freely is first considered. Secondly, the nonfree cases of *scattering by a potential* are investigated.

(*) Permanent address: Kiev State University, Kiev.
(**) On leave from the Istituto di Fisica Teorica dell'Università, Catania, under an exchange program supported in part by the Istituto Nazionale di Fisica Nucleare, Sezione di Torino, and the Ukrainian Academy of Sciences. Permanent address: Istituto di Fisica Teorica dell'Università, Catania.

1. – Introduction.

Since every experiment is performed in space-time, all the observed quantities may always be considered as functions of space ($\boldsymbol{x}$) and of time (t). In such a context, $\boldsymbol{x}$ and t enter as parameters.

But every experiment implies also the determination of a certain space region and a certain time duration; we may be precisely interested in measuring position or time. In such a context, $\boldsymbol{x}$ and t must enter as *operators* into the framework of quantum mechanics. We shall first confine ourselves to the nonrelativistic case.

The fact that the operator « time » seems to have peculiar (even if not exceptional) features (*) led to its unjustified neglect. As a consequence, the Heisenberg uncertainty correlations for energy and time got particular obscurity as compared to the other ones. But the meaning of

$$\Delta E \cdot \Delta t \geqslant \frac{\hbar}{2} \tag{1}$$

is standard, according to us: the uncertainty ΔE that we meet when *measuring* the energy E of a particle is linked to the time duration Δt of the *measurement interaction* itself through eq. (1). For instance, to have a very good energy determination it takes an experiment very prolonged in time. This is also the essential meaning of the example forwarded in ref. (1).

For example, let us suppose we are *measuring* the energy of a particle by observing its track in a bubble chamber. If we examine (through a photograph) a *long* track segment, we can have good « statistics » in counting bubbles, and therefore a good determination of the (average) energy of the particle when producing that track; but the time at which the particle possessed that energy will have a large uncertainty. On the contrary, if we examine a *short* track segment, we shall have a good measure of time (**) but at the price of poor statistics in bubbles. In this example, the « experiment » (or better the *measurement*) is the track segment examination.

Sometimes, we are told that one is not entitled to claim for a time operator

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(*) We shall see that it does *not* admit a spectral decomposition, in nonrelativistic quantum mechanics, according to ref. (8).

(1) L. D. LANDAU and E. M. LIFSHITZ: *Kvantovaya Mekhanika* (Moscow, 1963). Since Heisenberg's uncertainty relations hold for one and the same experiment, the example worked out by these authors for eq. (1) must be understood as a *unique* measurement process.

(**) By the way, the motion of a particle (even if not periodical) *may* serve as a clock for small time intervals. See ref. (2).

(2) See, *e.g.*, Y. AHARONOV and D. BOHM: *Phys. Rev.*, **122**, 1649 (1961).

as well as for the position operator, since the time measurement is a « peculiar » one. The clock itself is said to be essentially different from a length rod, since a clock implies movement: movement *in space*. But they forget that the instrument for measuring length is not the mere rod, but an apparatus able to superpose the rod to the measurable length (*); and such a superposition implies a movement too, which happens *in time*. Such concepts are clearer in relativity, where we can get *e.g.* the length of a collinearly moving rod also from the *time* taken by it to pass « before our eyes », and the Lorentz-transformed value of the time unit also from the space travelled (in our frame) by a lighting « lamp » which is switched on for a unitary time (in the comoving frame).

In the relativistic framework we know that for a particle with a *constant* mass m_0, moving freely, in the tetraimpulse space we have

$$p_0^2 = \boldsymbol{p}^2 + m_0^2 . \tag{2}$$

Such a Lorentz-invariant bound does not have any counterpart in the configuration space, since the particle world-lines may fill the light-cone interior. But, *with reference to one* and the same *observer*, that particle moves along a straight line with *constant* velocity, so that in *partial* analogy to eq. (2) also t depends on $|\boldsymbol{x}|$. In the simplest case (3)

$$t^2 = \frac{1}{v_0^2} \cdot \boldsymbol{x}^2 . \tag{2 bis}$$

Therefore, when we pass to operators, for free wave packets we have $\hat{t}$ linked to $\hat{\boldsymbol{x}}$ as well as $\hat{E}$ linked to $\hat{\boldsymbol{p}}$. And, as we make use of $\hat{E}$ besides $\hat{\boldsymbol{p}}$, so we shall use also $\hat{t}$ besides $\hat{\boldsymbol{x}}$.

At last, we want to call attention to the fact that any phenomenon can be consistently described by one (and the same) observer *at a time*; whilst mixing together various descriptions by different observers may lead to contradictions (**). And every observer uses essentially « one clock » (or better many *synchronized* clocks, *i.e. one time*) and « *one space* ». Therefore, when dealing with many particles, we ought to refer (time by time) to one and the same frame. That is to say, we should disregard the « many-times formulation » as not comparable with experience, as well as we would consider unphysical the recourse to a « many-proper-positions formulation ». In conclusion, using *contemporary* (*e.g.*) various proper times seems incorrect.

(*) Such an apparatus may well consist of « our arms *plus* the rod ».

(3) For a more general case, see, *e.g.*, M. RAZAVY: *Am. Journ. Phys.*, **35**, 955 (1967). See also ref. (4).

(4) A. J. KÁLNAY: *Boletín del I.M.A.F.* (*Córdoba*), **2**, 41 (1966); J. A. GALLARDO, A. J. KÁLNAY, B. A. STEC and B. P. TOLEDO: *Nuovo Cimento*, **48** A, 393 (1967). See also ref. (7).

(**) *Note added in proofs.* – Cf., *e.g.*, E. RECAMI and R. MIGNANI: *Riv. Nuovo Cimento*, **4**, 259 (1974).

Of course, we may have different observers, each one using only one time. (For the moment we shall not care about Lorentz covariance, since in this paper we are essentially involved in the nonrelativistic case.)

2. – Definition of time operator.

In the previous Section we have seen that it is desirable to have an operator for time in quantum mechanics even in the nonrelativistic case (*). It is *possible* to pursue such an aim (*e.g.*, the famous objection by PAULI [5] will be answered in the next Section). We shall first confine ourselves to the simple case of *wave packets* [6] *moving freely*, and to studying their motion time as a function of position (see ref. [6]).

We require for our time operator $\hat{t}$ the following necessary properties:

i) to reduce to the mere multiplication by t in the « space t », *i.e.* in a suitable space of functions of time (see the following, and the Summary);

ii) to be canonically conjugate to the (total) energy;

iii) to satisfy the Ehrenfest principle (besides being Galilei invariant).

In ref. [6,7] the quantities

$$\hat{t}_1 = -i\frac{\partial}{\partial E}\,, \qquad \hat{t}_2 = -\frac{i}{2}\frac{\overleftrightarrow{\partial}}{\partial E} \tag{3}$$

have been considered, both satisfying [7] the commutation relation

$$[\hat{t}, \hat{E}] = -i\hbar\,. \tag{4}$$

We want now to investigate if these quantities do indeed satisfy all the mentioned necessary requirements.

Let us consider, for example, the simplest case of a free, *unidimensional*, nonrelativistic wave packet [6]:

$$\mathscr{F}(t, x) = \int_0^\infty \mathrm{d}p\cdot\tilde{\mathscr{F}}(E, p)\cdot\exp\left[i(px - Et)\right]$$

(*) In quantum mechanics, an operator must correspond to every observable.

(5) W. PAULI: *Handbook der Physik*, edited by S. FLÜGGE, Vol. **5**/1 (Berlin, 1926), p. 60.

(6) See: V. S. OLKHOVSKY and E. RECAMI: *Nuovo Cimento*, **53** A, 610 (1968); **63** A, 814 (1969), and references therein. See also: V. S. OLKHOVSKY: *Nuovo Cimento*, **48** B, 170 (1967); E. RECAMI: *Acc. Naz. Lincei, Rendic. Sci.*, **49**, 77 (1970).

(7) See: M. BALDO and E. RECAMI: *Lett. Nuovo Cimento*, **2**, 643 (1969), and references therein; V. S. OLKHOVSKY and E. RECAMI: *Lett. Nuovo Cimento*, **4**, 1165 (1970). See also E. PAPP: *Nuovo Cimento*, **5** B, 119 (1971); **10** B, 69, 471 (1972).

with

$$\hbar = 1 , \qquad E = \frac{p^2}{2m_0} .$$

It is important that, for our purposes, we may simplify our problem by considering a *fixed* (*) value $x = \bar{x}$ and therefore only study (for every fixed $\bar{x}$) the « packets » (**)

$$(5) \qquad F(t, \bar{x}) = \int_0^\infty \mathrm{d}p \cdot f'(p, \bar{x}) \cdot \exp[-iEt] = \int_{\substack{0 \\ (+)}}^\infty \mathrm{d}E \cdot f(E, \bar{x}) \cdot \exp[-iEt] ,$$

where

$$E = E_{\text{tot}} = E_{\text{kin}} = \frac{p^2}{2m_0} , \qquad f'(p, \bar{x}) = f(E, \bar{x}) \cdot \frac{\mathrm{d}E}{\mathrm{d}|p|} .$$

Briefly, we shall write also

$$F = F(t, \bar{x}) , \qquad f = f(E, \bar{x}) .$$

It is easy to go back from the functions F, or f, to « physical » wave packets, so as to have a one-to-one correspondence between these wave packets and our functions.

The functional space of the functions $F(t, \bar{x})$, endowed with the mathematical conditions that we are going to specify, will be called « space t ». Analogously, the functional space of the transformed functions $f(E, \bar{x})$ will be called « space E ».

In these spaces the norms will be respectively

$$\|F\| = \int |F(t, \bar{x})|^2 \, \mathrm{d}t , \qquad \|f\| = \int |f(E, \bar{x})|^2 \, \mathrm{d}E .$$

Due to eqs. (5), however, space t and space E are representations of the same « abstract space » $\mathscr{P}$. In this space $\mathscr{P}$ we shall as usual indicate

$$F(t, \bar{x}) \to |F\rangle , \qquad f(E, \bar{x}) \to |f\rangle ,$$

where of course $|F\rangle = |f\rangle$.

(*) In fact, given a wave-packet, its mean space–co-ordinates are obviously defined at fixed t, and its mean time–co-ordinate is defined at fixed x. By the way, we cannot consider (6) the packet average position and average motion-time simultaneously.
(**) The following functions $F(t, \bar{x})$ and $f(E, \bar{x})$, being functions only of t or of E respectively, are *not* wave functions (satisfying Schrödinger's equation), and do *not* represent states in the chronotopical or four-momentum space! They are (vectors in the functional space of) energy functions defined in the following, or (of) the *transformed* time functions, respectively.

As we shall explain in the following, the weight functions entering eq. (5) are supposed to be continuous, differentiable, and such that

$$\int_0^\infty |f(E,\bar{x})|^2\,\mathrm{d}E < \infty\,, \tag{6a}$$

$$\int_0^\infty \left|\frac{\partial f(E,\bar{x})}{\partial E}\right|^2 \mathrm{d}E < \infty\,, \tag{6b}$$

$$\int_0^\infty |f(E,\bar{x})|^2\,E^2\,\mathrm{d}E < \infty\,. \tag{6c}$$

Therefore, we are assuming space $\mathscr{P}$ to be the space of continuous, differentiable L^2-functions that satisfy the above conditions. Of course, our « physical space » will be on the contrary the space of the physical (wave packet) states corresponding to our space $\mathscr{P}$ in the one-to-one correspondence mentioned before.

Our (physical) choice of considering only positive momenta in eq. (5) is due to the *boundary conditions imposed by the initial and final experimental devices* (*i.e.* by the « preparation » apparatus and by the detector). In so doing, we automatically choose as *reference frame the one in which source and detector are at rest*, *i.e.* the laboratory frame, and all our discussions will be done with reference to it. In particular, notice that for simplicity we are assuming source and detector at rest *one with respect to the other*.

If we want to pass from the laboratory to another frame, we then ought not to forget that in the new frame the detector (and source) will *no longer* be at rest.

Modifying ref. (6), let us further define, in a natural way,

$$\langle t(\bar{x})\rangle_x \equiv \frac{\int_{-\infty}^{\infty}\varrho_x(t,\bar{x})\,t\,\mathrm{d}t}{\int_{-\infty}^{\infty}\varrho_x(t,\bar{x})\,\mathrm{d}t} \qquad \left(\varrho \equiv |F(t,\bar{x})|^2\right), \tag{7}$$

where the index x reminds us that in the multidimensional cases we should choose values of $\boldsymbol{x}$ only along a particular ray (*e.g.* along the average direction of the wave packet motion). Thus we can write

$$\langle t(\bar{x})\rangle_x \equiv \langle F(t,\bar{x})|\hat{t}|F(t,\bar{x})\rangle \equiv \frac{1}{N}\int_{-\infty}^{\infty}|F(t,\bar{x})|^2\,t\,\mathrm{d}t = \frac{1}{N}\int_{-\infty}^{\infty}F^*\,tF\,\mathrm{d}t\,. \tag{7 bis}$$

We have thus proven that our definition (7) is equivalent to defining $\hat{t}$ (in a very *immediate* way) as the operator that, in the space t, is just multiplication by t. *This is our starting point.*

Afterwards, by direct calculations, one finds easily

$$(8) \qquad \langle t(\bar{x})\rangle_x = \frac{1}{N}\int_0^\infty \mathrm{d}E \left\{ f^*(E,\bar{x})\cdot\left[-i\frac{\partial}{\partial E}\right]\cdot f(E,\bar{x}) + \left[|f(E,\bar{x})|^2\right]_0^\infty \right\}.$$

Since we are using weight functions, $f(E,\bar{x})$, square-integrable over the interval $0 \leqslant E < \infty$, we must first have $f \to 0$ for $E \to \infty$. Secondly, *if we assume the subsidiary condition* $f(0,\bar{x}) = 0$, the second addendum in the r.h.s. of the last equation vanishes:

$$(8') \qquad \langle t(\bar{x})\rangle_x = -\frac{i}{N}\int_0^\infty \mathrm{d}E\, f^* \frac{\partial}{\partial E} f\,,$$

and we may choose

$$(9) \qquad \hat{t}_1 = -i\frac{\partial}{\partial E}$$

as time operator.

By the way, the continuous, differentiable functions $f(E,\bar{x})$, satisfying eqs. (6) and the subsidiary condition $f(0,\bar{x}) = 0$, constitute a space dense [8] in the Hilbert space of L^2-functions defined over $0 \leqslant E < \infty$; therefore $\hat{t}_1$ may be shown to be not only Hermitian [*] but also symmetric in the sense of ref. [9].

Nevertheless, we deem the subsidiary condition $f(0,\bar{x}) = 0$ as physically distasteful, since it does not appear to be necessary from a physical point of view. Actually, it is possible to avoid such a restriction.

In fact, our calculations themselves show that such a program can be performed when we require substitution of the derivative $\partial/\partial E$ with a « radiance derivative » $\overleftrightarrow{\partial}/\partial E$. Namely, by means of our operator (3) we may write

$$(8\ bis) \qquad \frac{1}{N}\int_0^\infty \mathrm{d}E \left[f^*\left(-\frac{i}{2}\frac{\overleftrightarrow{\partial}}{\partial E}\right) f\right] = \langle t(\bar{x})\rangle_x\,;$$

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(8) J. VON NEUMANN: *Matematische Grundlagen der Quantum Mechanik* (Berlin, 1932).

(*) The definitions adopted in this paper for linear operators mapping the Hilbert space $\mathscr{H}$ into itself are the following: 1) the operator A is *Hermitian* if $(x|Ay) = (Ax|y)$, $\forall\, x, y \in \mathscr{D}(A)$; $\overline{\mathscr{D}(A)} \subset \mathscr{H}$; 2) A is *symmetric* if $(x|Ay) = (Ax|y)$, $\forall\, x, y \in \mathscr{D}(A)$; $\overline{\mathscr{D}(A)} \equiv \mathscr{H}$; 3) A is *self-adjoint* if it is symmetric and $A = A^\dagger$, $\mathscr{D}(A) = \mathscr{D}(A^\dagger)$, so that $(x|Ay) = (A^\dagger y|x)$.

(9) N. I. AKHIESER and I. M. GLADSMAN: *Theorie der Linearen Operatoren in Hilbert Raum* (Berlin, 1954).

and this equation (8 *bis*) generalizes eq. (7′) for *all* weight-functions satisfying conditions (6), *without any supplementary condition.* Of course, eq. (8 *bis*) coincides with eq. (8′) on the more restricted domain in which eq. (8′) holds.

We may well choose as time operator in the « space E » also the operator

$$\hat{t} \equiv \hat{t}_2 = -\frac{i}{2}\frac{\overleftrightarrow{\partial}}{\partial E}, \tag{10}$$

since

$$\langle F|t|F\rangle \equiv \langle f|-\frac{i}{2}\frac{\overleftrightarrow{\partial}}{\partial E}|f\rangle \tag{11}$$

holds, and therefore also $\hat{t} \equiv \hat{t}_2$ *reduces just to the multiplication by t in the « space t ».*

We prefer the operator $\hat{t}_2$ rather than the operator $\hat{t}_1$ not only for the reason that the domain of $\hat{t}_2$ is larger than the domain of $\hat{t}_1$, but also because the operator $\hat{t}_2$ seems to hold even in the relativistic case [6,7].

Later on, we shall come back to the properties of operator (10). For the moment, let us only observe that the weight-function derivatives must be finite at $E = \infty$ in order that integrals (8), (8 *bis*) exist.

One could proceed analogously in the three-dimensional case [6].

The physical meaning of relations such as (8 *bis*) becomes clear if we write [6]

$$\langle t(\bar{x})\rangle_x = \frac{1}{N}\int_0^\infty \mathrm{d}E|f|^2\left(\frac{\bar{x}}{v} + \frac{\partial \arg f}{\partial E}\right). \tag{12}$$

Therefore, from eq. (12) we have, according to the Ehrenfest principle,

$$\langle \hat{t}\rangle = t_0 + \bar{x}\langle v^{-1}\rangle. \tag{12 bis}$$

Let us repeat that, when doing calculations in another reference frame, source and detector will no longer be at rest in the new frame; it is easy to recognize that only the packet characteristics *with reference to the detector* (*and source*) are still essential. This is enough for the Galilei invariance of times calculated by means of our operator, *i.e.* by eq. (12).

Let us explicitly repeat also that, even in the relativistic case [6], we can find an analogous result [7] for the time operator, *i.e.* ($c = 1$)

$$\hat{t} = -\frac{i}{2}\frac{\overleftrightarrow{\partial}}{\partial p_0}. \tag{13}$$

Needless to say, in the nonrelativistic limit one has $p_0 \to E +$ const and $\partial/\partial p_0 \to \partial/\partial E$, the quantitity E being the kinetic energy. Therefore, the form (10) of the time operator is *a priori* to be used both in the nonrelativistic case and in the relativistic one [7]. But we want to mention that for the relativistic case « nonpunctual » operators [4] have been proposed too (see *e.g.* ref. [7]), which are not Hermitian. (Moreover, KALNAY and co-workers [10] *seem* to have shown that no position operator exists that is both Hermitian and Lorentz covariant; and the same *might* be valid also for time operator.)

At last, in the unidimensional case, it is easy to verify (by taking into account the proper boundary conditions for the weight-functions) that in the *impulse representation* we have the interesting correspondence ($\hbar = 1$)

$$\frac{i}{2}\frac{\overleftrightarrow{\partial}}{\partial E} \leftrightarrow \frac{m_0}{2}\left[\hat{x}\cdot\frac{1}{p} + \frac{1}{p}\cdot\hat{x}\right] + \frac{im_0}{2p^2}, \tag{14}$$

where the last addendum *vanishes* in the limit $\hbar \to 0$.

3. – Properties of the time operator. Pauli's objection.

We have seen that for a wave packet moving freely in the « space E » we may choose as time operator the quantity

$$\hat{t} = -\frac{i}{2}\frac{\overleftrightarrow{\partial}}{\partial E} \qquad (E \equiv E_{\text{tot}}), \tag{10'}$$

which is canonically conjugate to the (total) energy operator $\hat{E}$, and which in the « space t » is multiplication by t. We want now to investigate if it is also Hermitian.

The operator (10) has been defined onto the « space E ». And the vectors of this space are themselves defined over the energy range $[0, \infty)$. Because of our choice of the form (10) instead of the form (9), the operator $\hat{t}$ is *a priori* allowed to act on the Hilbert space of functions satisfying eq. (6*a*), *i.e.* on the weight-functions square-integrable over the interval $0 \leqslant E < \infty$, without further particular boundary conditions. But we must besides require that $\hat{t}$ transforms vectors of the Hilbert space into vectors of the Hilbert space, and we are brought to condition (6*b*). At last, in order that a good energy operator too exists, we are brought as well to condition (6*c*).

Now, VON NEUMANN has shown [8] that the continuous, differentiable functions satisfying conditions (6*a*), (6*b*), (6*c*) do constitute a space *dense* in the Hilbert space of L^2-functions defined over $0 \leqslant E < \infty$.

Moreover, under conditions (6) we have that

$$\int_0^\infty f_1^* \,\hat{t} f_2 \,\mathrm{d}E = \int_0^\infty (\hat{t} f_1)^* \cdot f_2 \,\mathrm{d}E\,. \tag{15}$$

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[10] J. C. GALLARDO, A. J. KÁLNAY and S. H. RISEMBERG: *Phys. Rev.*, **158**, 1484 (1967); A. J. KÁLNAY: *Phys. Rev. D*, **1**, 1092 (1970); **3**, 2357 (1971); A. J. KÁLNAY and P. L. TORRES: *Phys. Rev. D*, **3**, 2977 (1971).

In conclusion, our operator $\hat{t}$—satisfying eq. (15) onto a (space dense in a) Hilbert space—is also *Hermitian*, and even symmetric [9] (but *not* self-adjoint). It happens, therefore, that our time operator i) is canonically conjugate to the (total) energy, ii) is Hermitian and symmetric [9], iii) acts onto (a space dense in) the separable Hilbert space $L^2[0, \infty)$.

The occurrence of these three conditions might seem to contradict the famous Pauli's theorem [5], which may be summarized [11] as follows:

« If $\hat{T}$ is a Hermitian operator in Hilbert space and α is a real number, then $\exp[i\alpha\hat{T}]$ is a unitary operator in Hilbert space. Then, if $\hat{T}$ satisfies eq. (4) for some Hermitian operator $\hat{E}$ and ψ_E is an eigenfunction of $\hat{E}$, we shall have that $\exp[i\alpha\hat{T}]\cdot\psi_E$ is an eigenfunction of $\hat{E}$ with eigenvalue $E-\alpha\hbar$. Since α is an arbitrary number, if $\hat{T}$ is a Hermitian operator in Hilbert space and satisfies eq. (4), then $\hat{E}$ must have a continuum of eigenvalues from $-\infty$ to $+\infty$. Therefore, for any Hermitian operator in Hilbert space $\hat{E}$ which does not have a continuum of eigenvalues from $-\infty$ to ∞ no Hermitian operator $\hat{T}$ exists in Hilbert space which satisfies eq. (4). »

Pauli's arguments, however, refer to « time operators » that act on functions defined over the infinite interval $(-\infty, +\infty)$, and in this respect they are therefore *partly* tautological.

On the contrary, our operator (10) acts onto the space of continuous functions in $L^2[0, \infty)$ that satisfy eqs. (6). It is therefore simply maximal, but *not hypermaximal* [8], *i.e.* our operator $\hat{t}$ does *not* [8] admit spectral resolution [*].

The indirect meaning of Pauli's objection is just pointing out that fact.

But, notwithstanding that our $\hat{t}$ does not have true eigenfunctions, nevertheless we can calculate the *average values* of the operator (10) over the vectors of our space $\mathscr{P}$ (and, therefore, also over the corresponding wave-packet *physical states*), as we showed before and in ref. [6,7], and as we shall show when considering the nonfree case [12].

Our thesis is that even nonhypermaximal operators may be given a physical meaning and may represent observables, as in the present case. Confining ourselves only to hypermaximal operators in quantum mechanics [8] seems to be too restrictive a postulate, as already emphasized by ENGELMAN and FICK [13].

VON NEUMANN himself clarified the question by giving an example [8] that we do want to mention here. Let us consider the case of a semi-space,

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[11] D. M. ROSENBAUM: *Journ. Math. Phys.*, **10**, 1127 (1969).

[*] A Hermitian operator, when also hypermaximal, has been shown in ref. [8] to be actually self-adjoint. Therefore, usually, *self-adjoint* operators (and not the merely Hermitian ones) admit an identity resolution.

[12] V. S. OLKHOVSKY and E. RECAMI: *Lett. Nuovo Cimento*, **4**, 1165 (1970).

[13] F. ENGELMAN and E. FICK: *Zeits. Phys.*, **175**, 271 (1963); **178**, 551 (1964); *Suppl. Nuovo Cimento*, **12**, 63 (1959). See also M. RAZAVY: *Nuovo Cimento*, **63** B, 271 (1969).

limited by a rigid wall. In such a case the values of x will run only in the range $[0, \infty)$. No such limitation will be present for the impulse p_x, but the operator

$$\hat{p}_x \equiv -i\hbar \frac{\partial}{\partial x} \tag{16a}$$

will *not* be hypermaximal, even if it has a very clear physical sense and corresponds to an observable quantity. By the way, also in this case we should better choose the operator $\hat{p}_x$ in the form

$$\hat{p}_x \equiv -i\hbar \frac{\overleftrightarrow{\partial}}{\partial x} \tag{16b}$$

rather than in form (16*a*).

4. – The nonfree case.

Let us now consider the case of a (spinless) particle moving in a central potential $V(r)$. We want to show that we can still choose the time operator in the form (10) and that it will still be canonically conjugate to the total energy. Actually, if $V(r) \to 0$ when $r \to \infty$, the continuous spectra of the kinetic-energy operator $\hat{E} = \hat{E}_{\text{kin}} = \hat{H}_0$ and of the total-energy operator $\hat{H}$ will coincide. Therefore, for the present purpose we may indifferently take either $E = E_{\text{kin}}$, $\hat{E} = \hat{H}_0$ or $E = H$, $\hat{E} = \hat{H}$ in eqs. (4), (10) and so on, since we are considering only *scattering* processes.

Inside the potential region we shall have packets of partial l-waves distorted by the potential ($\hbar = 1$)

$$F_l(t, \bar{r}) = \int_0^\infty \mathrm{d}p \cdot f(p) \cdot \mu_l^{(+)}(p, \bar{r}) \cdot \exp[-iEt], \tag{17}$$

where—according to CALOGERO ([14])—the functions

$$\mu_l^{(+)}(p, \bar{r}) = \mu_l^{(\text{in})} - \mu_l^{(\text{out})} = \tag{18}$$
$$= \tfrac{1}{2} \cdot \alpha_l(p, \bar{r}) \cdot S_l^{-\frac{1}{2}}(p, \bar{r}) \cdot [p\bar{r} \cdot h_l^{(2)}(p\bar{r}) - S_l(p, \bar{r}) \cdot p\bar{r} \cdot h_l^{(1)}(p\bar{r})]$$

are the (outgoing wave) solutions (*) of the radial Schrödinger equation with potential, the functions $a_l(p, \bar{r})$ and $S_l(p, \bar{r}) = \exp[2i \cdot \delta_l(p, \bar{r})]$ being defined in ref. ([14]) for a wide class of potentials.

([14]) F. CALOGERO: *Variable Phase Approach to Potential Scattering* (New York, N. Y., and London, 1967).
(*) *I.e.*, the solutions that asymptotically are the sum of plane waves plus outgoing spherical wave. See ref. ([14]).

Expression (17) may be obtained by applying the time evolution operator ([12]) $U = \exp[-i\hat{H}t]$ to a plane-wave packet. In our calculations we neglect discrete spectra, since bound states (as we shall see) do not contribute to scattering durations or flight times.

Packet (17) may be represented as a sum of incoming and outgoing spherical-wave packets (by inserting eq. (18) into eq. (17)) as follows:

$$F_l(t, \bar{r}) = F_l^{(\text{in})}(t, \bar{r}) - F_l^{(\text{out})}(t, \bar{r}) , \tag{19}$$

where

$$F_l^{(\text{in})}(t, \bar{r}) = \int\limits_0^\infty \mathrm{d}p \cdot B_l^{(\text{in})}(p, \bar{r}) \cdot \exp[-iEt] , \tag{20}$$

$$F_l^{(\text{out})}(t, \bar{r}) = \int\limits_0^\infty \mathrm{d}p \cdot B_l^{(\text{out})}(p, \bar{r}) \cdot \exp[-iEt] , \tag{21}$$

and where

$$\begin{aligned} E &= p^2/2m_0 , \\ G_l &\equiv f(p) \cdot \alpha_l(p, \bar{r}) \cdot S_l^{-\frac{1}{2}}(p, \bar{r}) , \\ B_l^{(\text{in})} &\equiv G_l(p, \bar{r}) \cdot pr \cdot h_l^{(2)}(p\bar{r}) , \\ B_l^{(\text{out})} &\equiv G_l(p, \bar{r}) \cdot S_l(p, \bar{r}) \cdot p\bar{r} \cdot h_l^{(1)}(p\bar{r}) . \end{aligned}$$

The functions h_l are the Hankel spherical functions.

Let us now define, according to ref. ([6]), the *time duration* of the *l-partial scattering process* (*i.e.* the time spent by the l-wave packet inside a sphere with radius $\bar{r} < R$) as

$$\tau_l \equiv \langle \hat{t} \rangle_{l,\text{out}} - \langle \hat{t} \rangle_{l,\text{in}} \equiv \frac{\int_{-\infty}^{\infty} \mathrm{d}t \cdot t \cdot \varrho_{l,\text{out}}(t, \bar{r})}{\int_{-\infty}^{\infty} \mathrm{d}t \cdot \varrho_{l,\text{out}}(t, \bar{r})} - \frac{\int_{-\infty}^{\infty} \mathrm{d}t \cdot t \cdot \varrho_{l,\text{in}}(t, \bar{r})}{\int_{-\infty}^{\infty} \mathrm{d}t \cdot \varrho_{l,\text{in}}(t, \bar{r})} , \tag{22}$$

where

$$\varrho_{l,\text{out}} \equiv [F_l^{(\text{out})}]^2, \qquad \varrho_{l,\text{in}} \equiv [F_l^{(\text{in})}]^2 .$$

Through direct calculations we find the important result

$$\tau_l = \frac{1}{N_1} \int\limits_{\substack{0 \\ (+)}}^{\infty} \mathrm{d}E \cdot [B_l^{(\text{out})}]^* \left(-\frac{i}{2} \frac{\overleftrightarrow{\partial}}{\partial E} \right) [B_l^{(\text{out})}] - \frac{1}{N_2} \int\limits_{\substack{0 \\ (+)}}^{\infty} \mathrm{d}E \cdot [B_l^{(\text{in})}]^* \left(-\frac{i}{2} \frac{\overleftrightarrow{\partial}}{\partial E} \right) [B_l^{(\text{in})}] \tag{23}$$

with

$$N_1 \equiv \int\limits_{\substack{0\\(+)}}^{\infty} \mathrm{d}E\,[B_l^{(\mathrm{out})}]^2\,, \qquad N_2 \equiv \int\limits_{\substack{0\\(+)}}^{\infty} \mathrm{d}E\,[B_l^{(\mathrm{in})}]^2\,;$$

and therefore we verify that *time durations are still obtained by applying the time operator*

(10)
$$\hat{t} \equiv -\frac{i}{2}\,\frac{\overleftrightarrow{\partial}}{\partial E}$$

to the weight-functions $B(p, r)$ of packets (20) and (21).

For instance, we still have

(24)
$$\langle F|t|F\rangle_{\mathrm{in,out}} = \langle B|-\frac{i}{2}\,\frac{\overleftrightarrow{\partial}}{\partial E}\,|B\rangle_{\mathrm{in,out}}\,.$$

And again we will get

(4)
$$[\hat{t}, \hat{E}] = -i\hbar\,.$$

The physical meaning of eq. (23) may be got by writing it as follows ($p^2=2mE$):

(23 *bis*)
$$\tau_l = \frac{2}{N}\int\limits_{\substack{0\\(+)}}^{\infty} \mathrm{d}E\cdot|G_l(p, \bar{r})|^2\cdot\left\{\bar{r}v^{-1} + \left[\frac{\partial}{\partial p}\,\delta_l(p, \bar{r})\right]\cdot|p\bar{r}\cdot h_l^{(1)}|^2\right\}$$

with

$$N \equiv \int\limits_{\substack{0\\(+)}}^{\infty} \mathrm{d}E\cdot|G_l(p, \bar{r})|^2\cdot|p\bar{r}\cdot h_l^{(1)}|^2.$$

In the particular case of *bound states*, the probability densities are known not to change with time. In such a case, our time definition eq. (7) is no longer meaningful. We want only to *verify* that bound states bring no contribution to the scattering motion time ([15]) (which will no longer contain $\bar{x}$):

$$\langle t\rangle \equiv \frac{\int_{-\infty}^{\infty}\mathrm{d}t\cdot t\cdot\varrho(\bar{x})}{\int_{-\infty}^{\infty}\mathrm{d}t\cdot\varrho(\bar{x})} = \frac{\int_{-\infty}^{\infty}t\,\mathrm{d}t}{\int_{-\infty}^{\infty}\mathrm{d}t} = 0\,.$$

([15]) See, *e.g.*, E. RECAMI: *Acc. Naz. Lincei, Rendic. Sci.*, **49**, 77 (1970).

Let us pass to the *metastable states*, and consider formation and decays of a metastable system in the hypothesis that $V(r) = 0$ for $r > R$, the quantity R being the potential radius. We choose as $t = 0$ the initial formation instant of the unstable system [6]. Outside the interaction region, the emitted particle will be described by the wave packet (unidimensional case)

$$F(t, \bar{x}) = \int_0^\infty \mathrm{d}E \frac{\gamma}{E - E_0 + i\Gamma} \exp[i(p\bar{x} - Et)], \tag{25}$$

where γ and Γ are constants. If we choose γ as real, then the point $\bar{x} = 0$ will be defined as the final (exit) point of the interaction region. In the case of a narrow resonance, the integral may be extended from $-\infty$ to $+\infty$. Direct calculations give for the motion time the obvious result [16]

$$\langle t(\bar{x})\rangle \approx \bar{x}\langle v^{-1}\rangle + \frac{\hbar}{\Gamma}. \tag{26}$$

More rigorously, following the result of this Section, we may start writing the expression of τ_l, as given in eq. (23 *bis*), in the general case of scattering by finite-radius potential, *outside* the interaction region:

$$\tau_l = \frac{2}{N}\int_0^\infty \mathrm{d}E \cdot |G_l|^2 \cdot \left\{\bar{r}v^{-1} + \left[\frac{\partial}{\partial E}\delta_l(E)\right] \cdot |p\bar{r} \cdot h_l^{(1)}|^2\right\}, \tag{27}$$

where the Calogero *phase function* $\delta_l(p, r)$ reduced to the usual scattering phase shift.

Now we pass to consider the whole process of i) initial free flight, ii) metastable-state formation, iii) decay with final free flight. In the presence of *resonant* elastic scattering

$$S_l = \tilde{S}_l \frac{E - E_0 - i\Gamma}{E - E_0 + i\Gamma},$$

and therefore

$$\delta_l = \tilde{\delta}_l - \operatorname{arctg} \frac{\Gamma}{E - E_0}, \tag{28}$$

where $\tilde{S}$, $\tilde{\delta}$ will be functions slowly varying in the resonance region.

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(16) V. S. OLKHOVSKY: *Nuovo Cimento*, **48** B, 170 (1967); *Ukrainian Phys. Journ.*, **13**, 143 (1968).

In the narrow-resonance approximation we still obtain [16] *for sufficiently large values of* $\bar{r}$ that

$$(26') \qquad \tau_l \approx \frac{2}{\int\limits_{-\infty}^{+\infty} \mathrm{d}E\cdot|G_l|^2} \cdot \int\limits_{-\infty}^{\infty} \mathrm{d}E\cdot|G_l|^2\cdot\left[\bar{r}v^{-1} + \frac{\Gamma}{(E-E_0)^2+\Gamma^2}\right] \approx 2\bar{r}\langle v^{-1}\rangle + \frac{\hbar}{\Gamma},$$

where τ_l represents now the time spent by the l-wave packet inside the sphere with radius $\bar{r} > R$.

* * *

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● RIASSUNTO

Nell'ambito di una *descrizione spazio-temporale* di oggetti quantistici non relativistici mediante pacchetti d'onde, ci si può ridurre semplicemente a considerare (per ogni punto spaziale fissato $\vec{x}$: vedi eq. (5)) i «pacchetti d'onde» $F(t, \vec{x}) = \int \mathrm{d}E\, f(E, \vec{x})\cdot$ $\cdot\exp[-iEt]$, che noi assumeremo avere come funzioni-peso i vettori dello spazio funzionale $\mathscr{P}$ definito come segue. Lo spazio $\mathscr{P}$ è lo spazio delle funzioni L^2 continue i) definite sull'intervallo $0 \leqslant E < \infty$ dell'energia (totale), ii) aventi derivate prime a quadrato sommabile, e iii) per le quali esiste un operatore hermitiano per l'energia. Tale spazio $\mathscr{P}$ è *denso* nello spazio hilbertiano delle funzioni L^2. Si mostra, quindi, l'esistenza di un «buon» operatore tempo, $\hat{t} = -(i/2)(\overleftrightarrow{\partial}/\partial E)$, che agisce su $\mathscr{P}$ e che i) è «simmetrico» (ma non autoaggiunto), ii) è canonicamente coniugato all'energia (totale), e iii) soddisfa al principio di Ehrenfest e all'invarianza galileiana. Si riconosce come la nota, vecchia obiezione di Pauli sottolinei semplicemente che il nostro operatore $\hat{t}$ non può essere ipermassimale, come è stato chiarito da von Neumann. Ma anche gli operatori non ipermassimali possono avere significato fisico e rappresentare osservabili in meccanica quantistica. Come già rilevato da precedenti autori, il limitarsi ai soli operatori autoaggiunti in meccanica quantistica risulta troppo restrittivo. Anche se t non ammette vere autofunzioni, ciò nonostante risulta possibile calcolare il *valore medio* dell'operatore tempo $\hat{t}$ per i nostri « pacchetti d'onde » (e per gli stati fisici ad essi corrispondenti). Dapprima si esamina il caso di un pacchetto d'onde in moto libero, quindi si analizzano i casi di *scattering da un potenziale*.

Резюме не получено.