Analysis of long term natural gas contracts with Vine copula in optimization problems.

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Outline

1. Introduction and Objectives
2. Methodology
3. Data and Dependence Structure
4. Pair Copula Construction
5. Optimal Composition of LTNG contracts
6. Conclusions
The current situation of the European natural gas market can be described as an hybrid pricing system composed by:

- Long-Term Natural Gas (LTNG) contracts, traditionally concluded over long periods (20 years or more)

- Spot prices at the market hub
The LTNG contracts are characterized by

- A *quantity clause*, (Take or Pay), obligates the buyer to take a certain quantity of natural gas or to pay for it

- A *price indexation clause*, that relates the price at which the gas is bought to some index, traditionally the price of crude and oil-base products

- Have been historically introduced to allow for risk sharing between gas producers and mid-streamers that respectively face price and volume risks

- The price clause of these contracts provides producers with some *price stability and reduction of revenues’ volatility*

- The quantity and price indexation also allows for *hedging mid-streamers’ volume risk* (see Abada et al., 2014)
The Hub pricing approach developed in the nineties in the US and UK and is now developing in Europe. In this system, natural gas is traded, every day, on a spot market that determines prices and volume on the short term.

In continental Europe, Zeebrugge (ZEE) (Belgium) and the Title Transfer Facility (TTF) (Netherlands), are the two dominant spot market-places and many others are emerging (see Melling, 2010).

**Figure 1:** European Gas Hubs and Gas Exchange, source "The Oxford Institute for Energy Studies" - Continental European Gas Hubs: Are they fit for purpose?
The fall of the European gas demand combined with the increase of the US shale gas exports and the Liquified Natural Gas availability have led to a reduction of gas spot prices in Europe. Oil-indexed LTNG contracts have failed to adjust their position promptly ⇒ losses for mid-streamers who have asked for re-negotiations ⇒ decline of oil-indexation while hub-linked pricing has rapidly become the basis for an increasing number of transactions.
Considering these arrangements our main contribution concerns:

- Evaluating via Pair Copula Construction (PCC) the portfolio dependence risk structure across the constituents of LTNG contracts
- Applying Vine copula models to classical portfolio optimization problems
- Defining the optimal composition of LTNG contracts
Copula Function

A Copula is a distribution function $C : [0, 1]^d \to [0, 1]$ of random variables $(X_1, \ldots, X_d)$ with standard uniform marginals. In the bivariate case:

$$C_\theta(u_1, u_2) = Pr(U_1 < u_1, U_2 < u_2)$$

- It can be defined as a function linking univariate marginal distributions to multivariate distribution functions.
- The use of copulas allows us to split the distribution of a random vector into its individual marginal components, and model the dependence structure without losing any information.
The Theorem

Given a vector of random variables \( X = (X_1, \ldots, X_d) \) with \( d \)-dimensional joint c.d.f \( F(x_1, \ldots, x_d) \), and marginal cumulative distributions \( F_i(x_i) \) with \( i = 1, \ldots, d \) there exist a \( d \)-dimensional copula \( C \) such that:

\[
F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d); \theta),
\]

where \( \theta \) denotes the set of parameters of the copula.

If all \( F_1, \ldots, F_d \) are continuous, then \( C \) is unique. Conversely, if \( C \) is a copula and \( F_1, F_2, \ldots, F_d \) are distribution functions, then the function \( F \) is a joint distribution function with margins \( F_1, F_2, \ldots, F_d \).

This ensures that the joint c.d.f. can be written as a function of the cumulative marginal functions and vice versa.
Pros and Cons

Copula are suitable when dealing with:

- Non-normal distributions, typical of financial asset returns distribution
- Dependency among assets, different from a perfect one

But when we are dealing with a large number $n$ of marginal we are facing a tradeoff:

- Elliptical copulas (Normal, T-Student) are useful for large $n$, but they are symmetric in the tails
- Archimedean copulas can account for asymmetry, but is hard to extend to a multivariate setting
If the dependency structure of different couple of variables in a multivariate problem is very different, the copula approach has drawbacks. Joe (1996) proposed the **Pair Copula Construction** (PCC) approach and then Bedford and Cooke (2001, 2002) introduced its graphical representation via *R*-vines.

- PCCs are a collection of potentially different bivariate copulas used to construct the joint distribution, allowing us to represent different types and strengths of dependency.
- The dependence structure is simply determined by the bivariate copulas using a nested set of trees.
R-vines are a particular type of graphical models, using a nested set of trees to represent the decomposition of the joint distribution into its bivariate components, incorporating the dependence structure of our variables.

A C-vine is an R-vine where each tree is a star with one unique node that connects it to all the others.

A D-vine is an R-vine represented by line trees.
An Example: Three dimensions

$$f(x_1, x_2, x_3) = f_3(x_3)f_2(x_2)f_1(x_1) \cdot c_{12}[F_1(x_1), F_2(x_2)] \cdot c_{23}[F_2(x_2), F_3(x_3)] \cdot c_{13|2}[F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)]$$
An Example: Three dimensions

\[
f(x_1, x_2, x_3) = f_3(x_3)f_2(x_2)f_1(x_1) \cdot \\
\cdot c_{12}[F_1(x_1), F_2(x_2)] \cdot c_{23}[F_2(x_2), F_3(x_3)] \cdot \\
\cdot c_{13|2}[F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)]
\]

Marginals

Unconditional pairs

Conditional pairs
An Example: Three dimensions

\[ f(x_1, x_2, x_3) = f_3(x_3)f_2(x_2)f_1(x_1) \cdot \\
\cdot c_{12}[F_1(x_1), F_2(x_2)] \cdot c_{23}[F_2(x_2), F_3(x_3)] \cdot \\
\cdot c_{13|2}[F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)] \]

Marginals

Unconditional pairs

Conditional pairs
An Example: Three dimensions

\[ f(x_1, x_2, x_3) = f_3(x_3)f_2(x_2)f_1(x_1) \cdot \]
\[ \cdot c_{12}[F_1(x_1), F_2(x_2)] \cdot c_{23}[F_2(x_2), F_3(x_3)] \cdot \]
\[ \cdot c_{13|2}[F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)] \]

Marginals
Unconditional pairs
Conditional pairs

1  12  2  23  3
12  13|2  23
The historical daily prices (log-returns) from 01/04/2012 to 07/24/2014 referred to oil-based commodities and spot gas traded at the hub.

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Max</th>
<th>Min</th>
<th>Std. Dev</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brent</td>
<td>-0.01%</td>
<td>6.57%</td>
<td>-5.67%</td>
<td>1.26%</td>
<td>-0.068</td>
<td>2.063</td>
</tr>
<tr>
<td>Gasoil</td>
<td>-0.01%</td>
<td>5.08%</td>
<td>-3.11%</td>
<td>1.06%</td>
<td>0.049</td>
<td>1.077</td>
</tr>
<tr>
<td>JetF</td>
<td>-0.01%</td>
<td>5.05%</td>
<td>-3.98%</td>
<td>1.03%</td>
<td>0.075</td>
<td>1.599</td>
</tr>
<tr>
<td>Naphta</td>
<td>0.00%</td>
<td>5.27%</td>
<td>-7.72%</td>
<td>1.30%</td>
<td>-0.440</td>
<td>3.654</td>
</tr>
<tr>
<td>Lsfo</td>
<td>-0.02%</td>
<td>3.54%</td>
<td>-6.80%</td>
<td>1.13%</td>
<td>-0.758</td>
<td>4.551</td>
</tr>
<tr>
<td>Gas NBP</td>
<td>-0.04%</td>
<td>9.10%</td>
<td>-7.69%</td>
<td>1.59%</td>
<td>0.294</td>
<td>3.620</td>
</tr>
<tr>
<td>Gas HenryHub</td>
<td>0.04%</td>
<td>13.27%</td>
<td>-11.93%</td>
<td>2.78%</td>
<td>0.290</td>
<td>2.342</td>
</tr>
</tbody>
</table>

Table 1: Basic statistics of log return time series referred to the period January 4, 2012 - July 24, 2014
Stationarity of the series

Figure 2: Log returns (a); 30-day horizon rolling standard deviation on log returns (b); and volatility (c) associated to the Gas NBP time series.

Table 2: Results of the unit root tests

<table>
<thead>
<tr>
<th></th>
<th>Brent</th>
<th>Gasoil</th>
<th>JetF</th>
<th>Naphta</th>
<th>Lsfo</th>
<th>Gas NBP</th>
<th>Gas HenryHub</th>
</tr>
</thead>
<tbody>
<tr>
<td>ADF</td>
<td>&lt; 0.01</td>
<td>&lt; 0.01</td>
<td>&lt; 0.01</td>
<td>&lt; 0.01</td>
<td>&lt; 0.01</td>
<td>&lt; 0.01</td>
<td>&lt; 0.01</td>
</tr>
<tr>
<td>PP</td>
<td>&lt; 0.01</td>
<td>&lt; 0.01</td>
<td>&lt; 0.01</td>
<td>&lt; 0.01</td>
<td>&lt; 0.01</td>
<td>&lt; 0.01</td>
<td>&lt; 0.01</td>
</tr>
<tr>
<td>KPSS</td>
<td>&gt; 0.1</td>
<td>&gt; 0.1</td>
<td>&gt; 0.1</td>
<td>&gt; 0.1</td>
<td>&gt; 0.1</td>
<td>0.013</td>
<td>&gt; 0.1</td>
</tr>
</tbody>
</table>
Filter for volatility

To model the volatility of the series, we have selected TGARCH models with a skewed-t distribution for innovations to capture the asymmetry in volatility (i.e. the leverage effect). ARMA models have been used to compensate for autocorrelation modelling the conditional mean when needed.

Table 3: P-value of the ARMA GARCH models. The relative statistics are reported in parentheses.

The results show that the selected models are appropriate and their adequacy is also confirmed by Sign Bias and the Adjusted Pearson Goodness-of-Fit Tests.
Filter for volatility

Figure 3: ACF and the Partial ACF (PACF) of the Gas NBP log return series and of the corresponding residuals. This proves that there is no evidence of autocorrelation at any lag.
**Figure 4:** The left-hand side of Figure 6 reports the corresponding ACF of the squared mean adjusted log return series, while the right-hand side of Figure 6 illustrates the ACF of the squared mean adjusted residuals. The latter shows the volatility reduction after the ARMA GARCH modelling.
Correlation search

From the standardized residuals we can obtain the so-called u-data (between 0 and 1) using the "Probability integral transformation" and then we analyse Kendall’s $\tau$ correlation matrix:

<table>
<thead>
<tr>
<th></th>
<th>Brent.u</th>
<th>Gasoil.u</th>
<th>JetF.u</th>
<th>Naphta.u</th>
<th>Lsfo.u</th>
<th>Gas NBP.u</th>
<th>Gas HenryHub.u</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brent.u</td>
<td>1.0000</td>
<td>0.6037</td>
<td>0.6002</td>
<td>0.3143</td>
<td>0.2750</td>
<td>0.0851</td>
<td>0.0213</td>
</tr>
<tr>
<td>Gasoil.u</td>
<td>0.6037</td>
<td>1.0000</td>
<td>0.7805</td>
<td>0.2887</td>
<td>0.2608</td>
<td>0.0847</td>
<td>0.0341</td>
</tr>
<tr>
<td>JetF.u</td>
<td>0.6002</td>
<td>0.7805</td>
<td>1.0000</td>
<td>0.2797</td>
<td>0.2524</td>
<td>0.0863</td>
<td>0.0408</td>
</tr>
<tr>
<td>Naphta.u</td>
<td>0.3143</td>
<td>0.2887</td>
<td>0.2797</td>
<td>1.0000</td>
<td>0.2976</td>
<td>0.1004</td>
<td>-0.0428</td>
</tr>
<tr>
<td>Lsfo.u</td>
<td>0.2750</td>
<td>0.2608</td>
<td>0.2524</td>
<td>0.2976</td>
<td>1.0000</td>
<td>0.0990</td>
<td>-0.0025</td>
</tr>
<tr>
<td>Gas NBP.u</td>
<td>0.0851</td>
<td>0.0847</td>
<td>0.0863</td>
<td>0.1004</td>
<td>0.0990</td>
<td>1.0000</td>
<td>0.0111</td>
</tr>
<tr>
<td>Gas HenryHub.u</td>
<td>0.0213</td>
<td>0.0341</td>
<td>0.0408</td>
<td>-0.0428</td>
<td>-0.0025</td>
<td>0.0111</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Table 4: Kendall’s $\tau$ correlation between u-data
**Figure 5:** In Figure 7a, we can observe that the data related to JetF and Gasoil lie on the smooth curve describing perfect positive dependence. This is also confirmed in Figure 7b where our data are not in the interval bounded by the dashed line that defines data independence.
Building the dependence structure (Czado et al., 2012)

- Let’s choose the first root node/variable
- Provide an independence test for each couple of marginals
- Fit a copula family for each of them (Vuong and Clark test, AIC and BIC based test, goodness-of-fit test)
- Estimate the parameter(s) of the bivariate copula
- Use h-function to compute the input of the pair copula of the second tree.
- Restart from point 1
- After having built all the trees, provide an estimation with sequential MLE and Joint MLE
**Figure 6**: The first tree of the $C$-vine with a 5% confidence level. The letters reported between the root assets indicate the type of the bivariate copulas used to model the dependence, while the numbers refer to the corresponding Kendall’s $\tau$ correlation.

**Figure 7**: The first tree of the $D$-vine with a 5% confidence level. The letters reported between the root assets indicate the type of the bivariate copulas used to model the dependence, while the numbers refer to the corresponding Kendall’s $\tau$ correlation.
The whole $C$- or $D$-vine structure can be represented by a $T$ matrix. Each column identifies a specific tree and each number denotes the type of pair-copula family.

$$T(C) = \begin{pmatrix} 2 & 2 \\ 2 & 2 \\ 14 & 2 & 1 \\ 1 & 13 & 1 & 7 \\ 3 & 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 30 & 0 & 0 \end{pmatrix} \quad T(D) = \begin{pmatrix} 2 & 1 \\ 2 & 2 \\ 1 & 5 & 0 \\ 7 & 5 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

**Table 5:** Copula family type: 0 = independence copula; 1 = Gaussian copula; 2 = Student-t copula (t-copula); 3 = Clayton copula; 5 = Frank copula; 7 = BB1 copula; 13 = rotated Clayton copula (180 degrees); 14 = rotated Gumbel copula (180 degrees); 30 = rotated BB8 copula (90 degrees)
Optimization portfolio problems

We compute the optimal weights of each asset defining the long-term gas contract under the minimum portfolio risk. We simulate the portfolio returns based on the dependence structures specified in the $C$-vine and $D$-vine models described above and estimate the risk of the seven-dimensional long-term natural gas contract.

We consider the five well-known risk measures:

- Mean Variance (EV)
- Mean Absolute Deviation (MAD)
- MiniMax
- Conditional Value-at-Risk (CVaR)
- Conditional Drawdown at Risk (CDaR)
## Results

### Table 6: Optimal weights for long-term natural gas portfolio (C-vine)

<table>
<thead>
<tr>
<th></th>
<th>EV</th>
<th>MAD</th>
<th>MiniMax</th>
<th>CVaR</th>
<th>CDaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brent</td>
<td>0.1040</td>
<td>0.1011</td>
<td>0.1064</td>
<td>0.1142</td>
<td>0.2868</td>
</tr>
<tr>
<td>Gasoil</td>
<td></td>
<td></td>
<td>0.1204</td>
<td></td>
<td>0.0300</td>
</tr>
<tr>
<td>JetF</td>
<td>0.1920</td>
<td>0.1928</td>
<td></td>
<td>0.1593</td>
<td>0.1323</td>
</tr>
<tr>
<td>Naptha</td>
<td>0.1205</td>
<td>0.1315</td>
<td></td>
<td>0.0317</td>
<td>0.1667</td>
</tr>
<tr>
<td>Lsfo</td>
<td>0.3373</td>
<td>0.3414</td>
<td>0.5696</td>
<td>0.3517</td>
<td></td>
</tr>
<tr>
<td>Gas NBP</td>
<td>0.1564</td>
<td>0.1500</td>
<td>0.1222</td>
<td>0.1726</td>
<td>0.4142</td>
</tr>
<tr>
<td>Gas HenryHub</td>
<td>0.0897</td>
<td>0.0832</td>
<td>0.0814</td>
<td>0.1405</td>
<td></td>
</tr>
<tr>
<td><strong>Min Risk</strong></td>
<td><strong>0.0006</strong></td>
<td><strong>0.0058</strong></td>
<td><strong>0.0271</strong></td>
<td><strong>0.0171</strong></td>
<td><strong>0.7927</strong></td>
</tr>
</tbody>
</table>

### Table 7: Optimal weights for long-term natural gas portfolio (D-vine)

<table>
<thead>
<tr>
<th></th>
<th>EV</th>
<th>MAD</th>
<th>MiniMax</th>
<th>CVaR</th>
<th>CDaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brent</td>
<td>0.1292</td>
<td>0.1086</td>
<td>0.2035</td>
<td>0.1355</td>
<td>0.3179</td>
</tr>
<tr>
<td>Gasoil</td>
<td>0.0027</td>
<td></td>
<td>0.1145</td>
<td>0.0826</td>
<td>0.1778</td>
</tr>
<tr>
<td>JetF</td>
<td>0.1775</td>
<td>0.1864</td>
<td></td>
<td>0.0734</td>
<td></td>
</tr>
<tr>
<td>Naptha</td>
<td>0.1302</td>
<td>0.1549</td>
<td></td>
<td>0.0912</td>
<td>0.1414</td>
</tr>
<tr>
<td>Lsfo</td>
<td>0.3026</td>
<td>0.3155</td>
<td>0.3180</td>
<td>0.2658</td>
<td></td>
</tr>
<tr>
<td>Gas NBP</td>
<td>0.1784</td>
<td>0.1579</td>
<td>0.2780</td>
<td>0.2460</td>
<td>0.3628</td>
</tr>
<tr>
<td>Gas HenryHub</td>
<td>0.0794</td>
<td>0.0767</td>
<td>0.0860</td>
<td>0.1055</td>
<td></td>
</tr>
<tr>
<td><strong>Min Risk</strong></td>
<td><strong>0.0006</strong></td>
<td><strong>0.0058</strong></td>
<td><strong>0.0257</strong></td>
<td><strong>0.0164</strong></td>
<td><strong>0.7812</strong></td>
</tr>
</tbody>
</table>
Results and Conclusions

- Variance is the measure that leads to a minimum risk value of the long-term natural gas portfolio. All assets are considered with the exception of Gasoil.
- MAD leads to a similar optimal composition, but with a slightly higher risk.
- CDaR leads to the highest risk portfolio and the least diversified one.
- The CVaR risk measure has to be applied to take into account all assets (Lsfo is the asset with the highest weight).
- The $D$-vine structure allows for the construction of less risky portfolios, in all cases.
- Finally, the weight globally covered by spot gas assets (Gas NBP and Gas HenryHub) is between 20% and 41%. This significant proportion reflects the re-negotiation policy advocated by mid-streamers in Europe.
Pair-Copula LTNG Contracts

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Pair-copula LTNG Contracts

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NOTATION (Risk measures)

- $M$ assets ($m = 1, \ldots, M$)
- $T$ time periods ($t = 1, \ldots, T$)
- $w_m$ the weights associated to each asset $m$ of the portfolio
- $r_{t,m}$ the return of each asset $m$ in time period $t$
- $\mu_m$ the average return of asset $m$ that is $\mu_m = \frac{1}{T} \sum_{t=1}^{T} r_{t,m}$
- $\mu_p$ the portfolio target return.
The mean variance (EV) nonlinear optimization problem (Markowitz, 1952):

\[
\min_w \frac{1}{T} \sum_{t=1}^{T} \left( \sum_{m=1}^{M} w_m (r_{t,m} - \mu_m) \right)^2
\]

s.t.

\[
\sum_{m=1}^{M} w_m \mu_m = \mu_p
\]

\[
\sum_{m=1}^{M} w_m = 1
\]

\[
w_m \geq 0 \quad \forall j = 1, \ldots, M.
\]

This optimization problem aims at minimizing portfolio variance (1) under the portfolio target return (2), assuming that the sum of the asset weights has to be equal to one (3) and the non-negativity of weights \( w_m \) (4).
The MAD optimization problem (Konno and Yamazaki, 1991)

\[
\min_w \frac{1}{T} \sum_{t=1}^{T} \left| \sum_{m=1}^{M} (r_{t,m} - \mu_m) w_m \right|
\]  

(5)

\[
\sum_{m=1}^{M} w_m \mu_m = \mu_p
\]  

(6)

\[
\sum_{m=1}^{M} w_m = 1
\]  

(7)

\[
w_m \geq 0 \quad \forall m = 1, \ldots, M.
\]  

(8)
Problem (5)-(8) can be transformed in the following linear optimization problem:

\[
\min_{w, y} \frac{1}{T} \sum_{t=1}^{T} y_t \quad (9)
\]

s.t.

\[
\left| \sum_{m=1}^{M} (r_{t,m} - \mu_m) w_m \right| \leq y_t \quad (10)
\]

\[
\sum_{m=1}^{M} w_m \mu_m = \mu_p \quad (11)
\]

\[
\sum_{m=1}^{M} w_m = 1 \quad (12)
\]

\[
w_m \geq 0 \quad \forall m = 1, \ldots, M. \quad (13)
\]
The MiniMax optimization problem (Young, 1998).
Maximizing the minimum return:

\[ L_p = \min_t \left( \sum_{m=1}^{M} w_m r_{t,m} \right) \quad \forall t = 1, \ldots, T. \]

or minimizing the maximum loss:

\[
\max_{L_p, w} L_p \\
\text{s.t.} \\
\sum_{m=1}^{M} w_m r_{t,m} - L_p \geq 0 \quad \forall t = 1, \ldots, T 
\]

\[ \sum_{m=1}^{M} w_m \mu_m = \mu_p \] \hspace{1cm} (16)

\[ \sum_{m=1}^{M} w_m = 1 \] \hspace{1cm} (17)

\[ w_m \geq 0 \quad \forall m = 1, \ldots, M. \] \hspace{1cm} (18)
CVaR

The CVaR optimization problem (Rockafellar and Uryasev, 2000)

\[
\min_{w, d, v} \frac{1}{(1 - \alpha)T} \sum_{t=1}^{T} d_t + v
\]

\[
\text{s.t.}
\]

\[
\sum_{m=1}^{M} w_m r_{t,m} + v \geq -d_t \quad \forall t = 1, \ldots, T
\]

\[
\sum_{m=1}^{M} w_m \mu_m = \mu_p
\]

\[
\sum_{m=1}^{M} w_m = 1
\]

\[
w_m \geq 0 \quad \forall m = 1, \ldots, M
\]

\[
d_t \geq 0 \quad \forall t = 1, \ldots, T
\]

where \( v \) represents the VaR, \((1 - \alpha)\) is the coverage rate and \( d_t \) is the deviation value below the VaR.
the CDaR optimization problem (Chekhlov et al., 2005)

\[
\min_{w, u, v, z} \frac{1}{(1 - \alpha)T} \sum_{t=1}^{T} z_t + v
\]

s.t.

\[
\sum_{m=1}^{M} w_m r_{t,m} + u_t - u_{t-1} \geq 0, \quad u_0 = 0 \quad \forall t = 1, \ldots, T
\]  
\[
z_t - u_t + v \geq 0 \quad \forall t = 1, \ldots, T
\]  
\[
\sum_{m=1}^{M} w_m \mu_m = \mu_p
\]  
\[
\sum_{m=1}^{M} w_m = 1
\]  
\[
w_m \geq 0 \quad \forall m = 1, \ldots, M
\]  
\[
z_t \geq 0 \quad \forall t = 1, \ldots, T
\]  
\[
u_t \geq 0 \quad \forall t = 1, \ldots, T
\]

where \(z\) is an auxiliary vector of variables of the conditional drawdowns, \(u\) is the auxiliary vector of variables used to model the cumulative returns and \(v\) represents the Drawdown Risk at the quantile \((1 - \alpha)\).