Portfolio Optimization using the Multivariate MixedTS distribution

Asmerilda Hitaj $^1$, Lorenzo Mercuri $^3$ and Edit Rroji $^1$

$^1$University of Milano-Bicocca

$^3$University of Milano

CMS 2017 (Bergamo), May 30 2017
1. Mixed Tempered Stable distribution

2. Multivariate Mixed Tempered Stable

3. Portfolio Optimization

4. Conclusions
Tempered Stable and stdCTS

**Definition**

A random variable \( X \) follows a Tempered Stable distribution if its Lévy measure is given by:

\[
\nu(dx) = \left( \frac{C_+ e^{-\lambda_+ x}}{x^{1+\alpha_+}} 1_{x>0} + \frac{C_- e^{-\lambda_- |x|}}{|x|^{1+\alpha_-}} 1_{x<0} \right) dx
\]

with \( \alpha_+, \alpha_- \in (0, 2) \) and \( C_+, C_-, \lambda_+, \lambda_- \in (0, +\infty) \).

The **standardized Classical Tempered Stable distribution**, i.e. \( X \sim \text{stdCTS}(\alpha, \lambda_+, \lambda_-) \), is obtained adding the conditions:

- \( \alpha_+ = \alpha_- = \alpha \)
- \( C = C_+ = C_- = \frac{1}{\Gamma(2-\alpha)(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2})} \)
Mixed Tempered Stable

**Definition**

The continuous random variable $Y$ follows a Mixed Tempered Stable distribution if:

$$Y \overset{d}{=} \mu + \beta V + \sqrt{V} \tilde{X}$$

where $\tilde{X} | V \sim stdCTS(\alpha, \lambda_+ \sqrt{V}, \lambda_- \sqrt{V})$

Let $\Phi_Y(u) = \log E[e^{uY}]$, $\Phi_V(u) = \log E[e^{uV}]$ and

$$\Phi_H(u) = \frac{(\lambda_+ - u)^\alpha - \lambda_+^\alpha + (\lambda_- + u)^\alpha - \lambda_-^\alpha}{\alpha(\alpha - 1)(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2})} + \frac{(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1})u}{(\alpha - 1)(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2})}.$$  

Then we have

$$\Phi_Y(u) = \mu u + \Phi_V(\beta u + \Phi_H(u)).$$
Special cases

Remark

*It is more flexible than NVMM that have the form:*

\[ Y = \mu + \beta V + \sigma \sqrt{V} X \]

*where* \( X \sim N(0, 1) \). *In the NVMM the mixing r.v* \( V \) *should be heavy-tailed in order to have heavy tails for* \( Y \).

In the MixedTS with Gamma mixing r.v. we can have the following special cases:

- VG
- Standardized CTS
- Geo-Stable
Tail Behaviour for $V \sim \Gamma(a, b)$

Suppose $E(e^{uY}) < +\infty$ for some $\epsilon > 0 : u \in [-\epsilon, +\epsilon]$. The behaviour of left tail is $\log[F(y)] \sim -q^* y$ while the right tail $\log[1 - F(y)] \sim -r^* y$

$q^*$ and $r^*$ are respectively:

$$-q^* := \inf \{ u : E(e^{uY}) < \infty \} \quad \text{and} \quad r^* := \sup \{ u : E(e^{uY}) < \infty \}.$$ 

1. If $\max \{-\beta \lambda_- + \Phi_H(-\lambda_-), \beta \lambda_+ + \Phi_H(\lambda_+)\} < b$ then $q^* = \lambda_-$ and $r^* = \lambda_+$. 
2. If $-\beta \lambda_- + \Phi_H(-\lambda_-) < b < \beta \lambda_+ + \Phi_H(\lambda_+)$ then $q^* = \lambda_-$ and $r^* = u_+$. $u_+$ is the unique real solution to $\beta u + \Phi_H(u) = b$. 
3. If $-\beta \lambda_- + \Phi_H(-\lambda_-) > b > \beta \lambda_+ + \Phi_H(\lambda_+)$ then $q^* = u_-$ and $r^* = \lambda_+$. $u_-$ is the unique real solution to $\beta u + \Phi_H(u) = b$. 
4. If $b < \min \{-\beta \lambda_- + \Phi_H(-\lambda_-), \beta \lambda_+ + \Phi_H(\lambda_+)\}$ then $q^* = u_-$ and $r^* = u_+$. $u_- < u_+$ are the two real solutions of $\beta u + \Phi_H(u) = b$. 

We have $x \to -\infty$ and $x \to +\infty$, respectively:

$$\log [F(x)] = -q^* x + o(x) \quad \text{and} \quad \log [1 - F(x)] = -r^* x + o(x)$$

**Figure:** Tail behaviors of $\log F(x)$ for fixed parameters of the MixedTS($\mu = 0$, $\beta = 0$, $a = 1, \sigma = 1$, $\alpha = 1.25$, $\lambda_+ = 1.2$ and $\lambda_- = 1.9$).
Definition

A random vector $Y \in \mathbb{R}^N$ follows a multivariate MixedTS distribution if the $i^{th}$ component is defined as:

$$Y_i = \mu_i + \beta_i V_i + \sqrt{V_i} X_i,$$

where $V_i$ is the $i^{th}$ component of multivariate Gamma random vector $V$, defined as:

$$V_i = G_i + a_i Z,$$

where $G_i \sim \Gamma(l_i, m_i)$ and $Z \sim \Gamma(n, k)$, with $\{G_i\}_{i=1}^N$ and $Z$ independent.

$$X_i | V_i \sim stdCTS(\alpha_i, \lambda_{+,i} \sqrt{V_i}, \lambda_{-,i} \sqrt{V_i})$$
Main features

- The marginal component follows a MixedTS distribution and in order to ensure the marginal $Y_i$ to be MixedTS distributed with Gamma mixing density, we need to impose the restriction:

$$a_i = \frac{k}{m_i} \rightarrow a_i Z \sim \Gamma(n, m_i) \quad \forall i = 1, \ldots, N$$

- The characteristic function of the multivariate MixedTS is:

$$\varphi_Y(t) = e^{\sum_{h=1}^{N} t_h \mu_h} e^{\sum_{h=1}^{N} \left(i a_h t_h \beta_h + a_h L_{stdCTS}(t; \lambda_+, h, \lambda_-, h, \alpha_h)\right)} \times \prod_{h=1}^{N} e^{\Phi_{G_h}(i t_h \beta_h + L_{stdCTS}(t; \lambda_+, h, \lambda_-, h, \alpha_h))},$$

where $L_{stdCTS}(t; \lambda_+, h, \lambda_-, h, \alpha_h)$ is the characteristic exponent of a Standardized Classical Tempered Stable r.v.

- Analytical first four order moments.
Simulation Scheme

**Simulation:** The steps for the simulation of a multivariate MixedTS with $N$ components are:

1. Simulate independent Gamma r.v's $G_i$ and $Z$ for $i = 1 \ldots N$.
2. Compute $V_i = G_i + Z$ for $i = 1 \ldots N$.
3. Simulate $X_i | V_i \sim \text{stdCTS} \left( \alpha_i, \lambda_{+,i} \sqrt{V_i}, \lambda_{-,i} \sqrt{V_i} \right)$.
4. Compute $Y_i = \mu_i + \beta_i V_i + \sqrt{V_i} X_i$.
5. Repeat steps from 1 to 4.
Positive correlation

In Semeraro: \( \text{skew}(Y_i) \geq 0, \text{skew}(Y_j) \leq 0 \Rightarrow \text{Cov}(Y_i, Y_j) > 0 \). In the multivariate MixedTS the sign of the skewness of the marginals may be discordant and have positive covariance.
Negative correlation

In Semeraro: \( \text{skew}(Y_i) \geq 0, \text{skew}(Y_j) \geq 0 \Rightarrow \text{Cov}(Y_i, Y_j) < 0 \). In the multivariate MixedTS the same sign of the skewness can lead to negative covariance.
Problem formulation

Here we consider two different portfolio optimization problems based on CVaR.

- Minimal CVaR portfolio.

\[
\begin{align*}
\min_{\{w_i\}_{i=1}^N} & \quad \text{CVaR} \left( \sum_{i=1}^N w_i r_i \right) \\
\text{s.t.} & \\
& w_i \geq 0 \\
& \sum_{i=1}^N w_i = 1
\end{align*}
\]

- Optimal CVaR portfolio with minimum expected return $\bar{r}$ guaranteed.

\[
\begin{align*}
\min_{\{w_i\}_{i=1}^N} & \quad \text{CVaR} \left( \sum_{i=1}^N w_i r_i \right) \\
\text{s.t.} & \\
& w_i \geq 0 \\
& \sum_{i=1}^N w_i = 1 \\
& \sum_{i=1}^N w_i \mathbb{E}(r_i) \geq \bar{r}
\end{align*}
\]
We consider daily prices for the period 31/03/2015-31/03/2017 for the three assets: Deutsche Bank - DB, BNP Paris- BNP and Intesa San Paolo - ISP.
We fit an $ARMA(1,1)$-GARCH$(1,1)$ model

\[
\begin{align*}
  r_t &= \bar{\mu} + \theta_1 (r_{t-1} - \mu) + \theta_2 z_{t-1} + z_t \\
  z_t &= \sigma_t \epsilon_t \\
  \sigma^2_t &= \omega_0 + \alpha_1 z_{t-1}^2 + \beta_1 \sigma^2_{t-1}
\end{align*}
\]

to each log return time series and employ the multivariate MixedTS for the joint dynamics of the three sequences of residuals

\[
Y_1 = \epsilon_{t}^{db}, \quad Y_2 = \epsilon_{t}^{isp}, \quad Y_3 = \epsilon_{t}^{bs}.
\]

Results are obtained using Generalized Method of Moments (GMM).
### Fitting results

<table>
<thead>
<tr>
<th>Params</th>
<th>DB</th>
<th>BNP</th>
<th>ISP</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{\mu}$</td>
<td>-1.89E-003</td>
<td>-1.91E-005</td>
<td>-5.86E-004</td>
</tr>
<tr>
<td>$\theta_1$</td>
<td>3.70E-001</td>
<td>-6.45E-001</td>
<td>-5.15E-001</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>-3.03E-001</td>
<td>6.60E-001</td>
<td>4.32E-001</td>
</tr>
<tr>
<td>$\omega$</td>
<td>9.73E-005</td>
<td>4.35E-005</td>
<td>2.77E-005</td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>1.32E-001</td>
<td>1.42E-001</td>
<td>2.90E-001</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>7.48E-001</td>
<td>7.72E-001</td>
<td>7.09E-001</td>
</tr>
</tbody>
</table>

**Table:** Parameters of the ARMA(1,1)-GARCH(1,1)
## Fitting results 2

<table>
<thead>
<tr>
<th>Params</th>
<th>DB</th>
<th>BNP</th>
<th>ISP</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>-0.018</td>
<td>-0.002</td>
<td>-0.003</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.038</td>
<td>0.069</td>
<td>-0.004</td>
</tr>
<tr>
<td>$m$</td>
<td>0.798</td>
<td>0.970</td>
<td>0.940</td>
</tr>
<tr>
<td>$l$</td>
<td>1.146</td>
<td>0.918</td>
<td>1.14</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>1.231</td>
<td>1.250</td>
<td>1.232</td>
</tr>
<tr>
<td>$\lambda_+$</td>
<td>1.109</td>
<td>1.044</td>
<td>1.056</td>
</tr>
<tr>
<td>$\lambda_-$</td>
<td>1.130</td>
<td>1.047</td>
<td>1.049</td>
</tr>
</tbody>
</table>

**Table:** The common parameter $n$ is equal to 0.531.
Results of the optimal portfolio

Portfolio Optimization

Hitaj, Mercuri and Rroji (University of Milano-Bicocca, University of Milano)
Bergamo May 30, 2017
Towards worst case scenario

We can try to compute the worst case scenario CVaR. For the MixedTS $Y = \mu + \beta V + \sqrt{V} \tilde{X}$ with $\tilde{X}|V \sim stdCTS$ it is possible to prove that the coherent risk measure $\rho$ we have:

- $\mu \mapsto \rho(Y)$ is decreasing in $\Re$;
- $\beta \mapsto \rho(Y)$ is non-increasing in $\Re$;

The influence of parameters the scale parameter $b$ in the Gamma r.v. $V$ and of the tempering parameter $\lambda$ is still to be investigated.
Conclusions

- We considered a new infinitely divisible multivariate distribution that is a generalization of the Normal Variance Mean Mixtures.
- The flexibility of the MixedTS distribution is emphasized.
- We investigated its use in a portfolio optimization problem.

Thank You!