Hybrid tree-finite difference methods for the Heston and Bates model with stochastic interest rate.

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Introduction

We propose a mixed tree-finite difference method in order to approximate the Heston model.

We prove the convergence by embedding the procedure in a bivariate Markov chain.

Numerical results show reliability and the efficiency of the algorithm.

We show how to generalize the procedure to the Heston-Hull-White model, the Heston-Hull-White2D model, and the Bates model.
Previous tree and finite difference literature

Tree methods for the Heston model:


Finite differences for the 2D Heston PDE:

Heston Hull-White model

Finite differences for the Heston Hull-White model:


Fourier cosine methods:


References


The hybrid tree-finite difference approach: Main idea

Backward induction algorithm that works following a finite difference PDE method in the direction of the share process and following a tree method in the direction of the other random sources (volatility, interest rate and possibly dividend rate).
The Heston model

Under the risk neutral measure, the pair \((S, V)\) of the share price and the volatility process solves the SDE

\[
\frac{dS_t}{S_t} = (r - \delta)dt + \sqrt{V_t} dZ_S(t), \quad S(0) = S_0 > 0
\]

\[
dV_t = \kappa(\theta - V(t))dt + \sigma \sqrt{V_t} dZ_V(t), \quad V(0) = V_0 > 0
\]

where:

- \(r\) and \(\delta\) are the risk free interest rate and the continuous dividend rate respectively,
- \(\sigma\) is the volatility of the volatility,
- \(\kappa\) is the reversion speed,
- \(\theta\) is the long run variance,
- \(Z_S\) and \(Z_r\) are correlated Brownian motions:

\[
d\langle Z_S, Z_V \rangle_t = \rho dt, \quad |\rho| < 1.
\]
Hybrid Tree-Finite Difference for the Heston model

The hybrid procedure:

- a binomial tree for the volatility $V$;
- a transformation which keeps the diffusion processes $S$ and $V$ uncorrelated;
- a finite difference approach in the $S$-direction.
Take $N$ “large” and $h = T/N$. For $n = 0, 1, \ldots, N$, consider the lattice used in Appolloni, Caramellino, Zanette (2014)

$$\mathcal{V}_n^h = \{v_{n,k}\}_{k=0,1,\ldots,n} \text{ with } v_{n,k} = \left(\sqrt{V_0} + \frac{\sigma}{2} (2k - n)\sqrt{h}\right)^2 1_{\sqrt{V_0} + \frac{\sigma}{2} (2k-n)\sqrt{h} > 0}$$

We define the multiple jumps

$$k_d^h(n, k) = \max\{k^* : 0 \leq k^* \leq k \text{ and } v_{n,k} + \mu_V (v_{n,k}) h \geq v_{n+1,k^*}\},$$
$$k_u^h(n, k) = \min\{k^* : k + 1 \leq k^* \leq n + 1 \text{ and } v_{n,k} + \mu_V (v_{n,k}) h \leq v_{n+1,k^*}\}$$

in which $\mu_V$ denotes the drift coefficient of $V$. 
The robust tree method for $V$

Figure: Standard jumps and multiple jumps for the discrete approximation of the process $V$. 
The robust tree method for $V$

Starting from the node $(n, k)$, the discrete process can reach the up-jump node $(n + 1, k_u^h(n, k))$ or the down-jump node $(n + 1, k_d^h(n, k))$ with transition probability:

up-jump: $p_{k_u^h(n, k)}^h = 0 \lor \frac{\mu V(v_n, k)h + v_{n, k} - v_{n+1, k_d^h(n, k)}}{v_{n+1, k_u^h(n, k)} - v_{n+1, k_d^h(n, k)}} \wedge 1,$

down-jump: $p_{k_d^h(n, k)}^h = 1 - p_{k_u^h(n, k)}^h.$

Multiple jumps & jump probabilities are set in order to match the first local moment of the tree and of the process $V$ up to order 1 w.r.t. $h$. As a consequence, as $h \to 0$ one gets weak convergence on the path space.

Remark. In order to obtain the convergence, we do not need to require the Feller condition $2\kappa \theta \geq \sigma^2$. 
The transformed process $Y$

We consider the diffusion pair $(Y, V)$, where

$$Y_t = \log S_t - \frac{\rho}{\sigma} V_t.$$ 

Set $\bar{\rho} = \sqrt{1 - \rho^2}$ and $(W, Z)$ standard Brownian motion in $\mathbb{R}^2$. Then,

$$dY_t = \left( r - \delta - \frac{1}{2} V_t - \frac{\rho}{\sigma} \kappa(\theta - V_t) \right) dt + \bar{\rho} \sqrt{V_t} dZ_t, \quad (1)$$

$$dV_t = \kappa(\theta - V_t) dt + \sigma \sqrt{V_t} dW_t, \quad (2)$$

with $Y_0 = \log S_0 - \frac{\rho}{\sigma} V_0$. We set

$$\mu_Y(\nu) = r - \delta - \frac{1}{2} \nu - \frac{\rho}{\sigma_v} \kappa(\theta - \nu) \quad \text{and} \quad \mu_V(\nu) = \kappa(\theta - \nu).$$
The approximation of $Y$

Let $\bar{V}^h = (\bar{V}^h_n)_{n=0,...,N}$ denote the tree process approximating $V$ and set $V^h_t = \bar{V}^h_{[t/h]}$, $t \in [0, T]$, the associated piecewise constant and càdlàg approximating path.

In order to approximate $Y$, we construct a Markov chain from the finite difference method.

We start from the Euler scheme: $Y_0^h = Y_0$ and for $t \in (nh, (n + 1)h]$, $n = 0, \ldots, N$, set

$$Y_t^h = Y_{nh}^h + \mu Y(V_{nh}^h)(t - nh) + \bar{\rho} \sqrt{V_{nh}^h} (Z_t - Z_{nh}),$$

$Z$ being independent of the noise driving $\bar{V}^h$. 
The approximation of the pair \((Y, V)\)

Let \(f\) be a suitable function depending on both variables \((y, v)\). Then,

\[
\mathbb{E}(f(Y_{(n+1)h}, V_{(n+1)h}) \mid Y_{nh} = y, V_{nh} = v) \\
\approx \mathbb{E}(f(Y_{(n+1)h}^h, V_{(n+1)h}^h) \mid Y_{nh}^h = y, V_{nh}^h = v) \\
= \mathbb{E}(u^h(nh, y; v, V_{(n+1)h}^h) \mid V_{nh}^h = v)
\]

where

\[
u^h(nh, y; v, z) = \mathbb{E}(f(Y_{(n+1)h}^h, z) \mid Y_{nh}^h = y, V_{nh}^h = v).
\]

and

\[
u^h(nh, y; v, z) = u^h(s, y; v, z) |_{s=nh}
\]
The approximation of the pair \((Y, V)\)

Main fact: \((s, y) \mapsto u^h(s, y; v, z)\) solves the PDE

\[
\frac{\partial_s u^h}{\partial_s} + \mu_Y(v) \frac{\partial_y u^h}{\partial_y} + \frac{1}{2} \rho^2 v \frac{\partial^2 u^h}{\partial y^2} = 0, \quad y \in \mathbb{R}, \; nh < s < (n + 1)h,
\]

\[
u^h((n + 1)h, y; v, z) = f(y, z), \quad y \in \mathbb{R}. \tag{3}
\]

\[\Rightarrow\] simple problem: one dimensional, constant coefficients.
The solution $u^h(s, y; v, z)$ of the PDE problem (3) can be easily numerically found by using a finite difference method.

In practice we consider the finite grid $\mathcal{Y}^h = \{y_j\}_{j \in \mathcal{I}_{M_h}}$ with equally spaced points

$$y_j = Y_0 + j \Delta y_h, \ j \in \mathcal{I}_{M_h} = \{-M_h, \ldots, M_h\}.$$  

The approximation of $u^h(nh, y; v, z)$ is done by adding to (3) suitable boundary conditions - we use Neumann type conditions (but others can be chosen).
The finite difference scheme

The behavior of the solution of problem (3) changes with respect to the magnitude of the rate between the diffusion coefficient ($\rho^2 v/2$) and the advection term ($\mu \gamma(v)$).

We fix a small real threshold $\epsilon_h > 0$. Then:

- case $v > \epsilon_h$: we use an implicit scheme

$$ \frac{u_{j}^{n+1} - u_{j}^{n}}{h} + \mu \gamma(v) \frac{u_{j+1}^{n} - u_{j-1}^{n}}{2\Delta y} + \frac{1}{2} \rho^2 v \frac{u_{j+1}^{n} - 2u_{j}^{n} + u_{j-1}^{n}}{\Delta y^2} = 0, $$

with boundary conditions:

$$ u_{-M-1}^{n} = u_{-M+1}^{n}, \quad u_{M+1}^{n} = u_{M-1}^{n}; $$
The finite difference scheme

- case $v < \varepsilon_h$: we use an **explicit scheme**:
  - if $\mu_Y(v) \geq 0$:
    $$\frac{u_j^{n+1} - u_j^n}{h} + \mu_Y(v) \frac{u_{j+1}^{n+1} - u_j^{n+1}}{\Delta y} + \frac{1}{2} \rho^2 v \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta y^2} = 0;$$
  - if $\mu_Y(v) < 0$:
    $$\frac{u_j^{n+1} - u_j^n}{h} + \mu_Y(v) \frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{\Delta y} + \frac{1}{2} \rho^2 v \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta y^2} = 0;$$

here, the boundary conditions are

$$u_{-M-1}^{n+1} = u_{-M+1}^{n+1}, \quad u_{M+1}^{n+1} = u_{M-1}^{n+1}.$$
We obtain a finite-dimensional stochastic matrix $\Pi^h(\nu)$ that gives the discrete solution $\{u^n_i\}_{i \in J_M}$ of (3) at time $nh$ in terms of the solution $\{u^{n+1}_i\}_{i \in J_M}$ at time $(n + 1)h$:

$$u^n = \Pi^h(\nu)u^{n+1}.$$  

By resuming, we get

$$\mathbb{E}(f(Y_{(n+1)h}, V_{(n+1)h}) \mid Y_{nh} = y_i, V_{nh} = \nu) 
\approx \sum_{j \in J_{Mh}} \Pi^h(\nu)_{i,j} \mathbb{E}(f(y_j, V_{(n+1)h}^h) \mid V_{nh}^h = \nu), \quad i \in J_{Mh}.$$
We take $h$ small and we call $\tilde{X}^h = (\tilde{X}^h_n)_{n=0,1,...,N}$ the 2-dimensional Markov chain with transition probability law

$$\mu^h(y_j, \nu_{n+1,k^*} \mid y_i, \nu_{n,k}) = \begin{cases} 
\Pi^h(\nu_{n,k})_{ij} p^h_{k_u(n,k)} & \text{if } k^* = k^h_u(n,k) \\
\Pi^h(\nu_{n,k})_{ij} p^h_{k_d(n,k)} & \text{if } k^* = k^h_d(n,k) \\
0 & \text{otherwise,}
\end{cases}$$

for every $(y_i, \nu_{n,k}) \in Y^h \times V^h_n$ and $(y_j, \nu_{n+1,k^*}) \in Y^h \times V^h_{n+1}$. 
2-dimensional Markov chain

Then, for $n = 0, 1, \ldots, N - 1$, $y_i \in \mathcal{Y}^h$ and $v_{n,k} \in \mathcal{V}_n^h$ we get

$$
\mathbb{E}(f(Y_{(n+1)h}, V_{(n+1)h}) \mid Y_{nh} = y_i, V_{nh} = v_{n,k}) 
\approx \sum_{k^*} \prod_{i,j}^h (v_{n,k})_{i,j} p_{k^*}^h f(y_j, v_{n+1,k^*}),
$$

the above sum running on $k^* \in \{k_u^h(n, k), k_d^h(n, k)\}$ and $j \in J_{Mh}$.

$\Rightarrow$ we have constructed an approximation of $(Y, V)$ at times $t_n = nh$, $n = 0, 1, \ldots, N$, through the Markov chain $(\bar{X}_n^h)_{n=0,1,\ldots,N}$.

Before studying the convergence, we briefly discuss how to use the above procedure to numerically price American options.
American option pricing

Consider an American option with maturity $T$ and payoff function

$$\Psi(Y_t, V_t) = \Phi(e^{Y_t - \frac{\rho}{\sigma}V_t}), \quad t \in [0, T].$$

By considering the discrete dynamic programming principle and by using the approximation of $(Y, V)$ through the Markov chain $\tilde{X}_n^h$, $n = 0, 1, \ldots, N$, we approximate the price as follows:

for $n = 0, 1, \ldots, N$, we define $\tilde{P}_h(nh, y, v)$ for $(y, v) \in \mathcal{Y}^h \times \mathcal{V}_n^h$ by

$$\begin{cases} 
\tilde{P}_h(T, y_i, v_{N,k}) = \Psi(y_i, v_{N,k}) & i \in \mathcal{I}_{M_h} \text{ and } v_{N,k} \in \mathcal{V}_n^h \\
\tilde{P}_h(nh, y_i, v_{n,k}) = \max \left\{ \Psi(y_i, v_{n,k}), e^{-rh} \times \right. \\
\left. \sum_{k^*\cdot j} \prod^n(v_{n,k})_{i,j} \tilde{P}_h\left((n+1)h, y_j, v_{n+1,k^*}\right)p_h^{k^*} \right\}, \\
i \in \mathcal{I}_{M_h} \quad \text{and} \quad v_{n,k} \in \mathcal{V}_n^h.
\end{cases}$$
\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial y} + \frac{1}{\rho \nu} \frac{\partial}{\partial y} \left( \frac{1}{2} \left( \frac{\partial u}{\partial y} \right)^2 \right) = 0
\]
Convergence conditions

- We set up the dependence on the time step $h$ for the space-step $\Delta y$, the number $M$ giving the points of the grid $\mathcal{Y}_M$ and the threshold $\epsilon$ that allows us to use the explicit or the implicit finite differences method.

\[ \Delta y \equiv \Delta y_h = c_y h^p, \quad M \equiv M_h = c_M h^{-q}, \quad \epsilon \equiv \epsilon_h = c_\epsilon h^p \quad (4) \]

where $c_M > 0$ and the constants $c_y, c_\epsilon, p, q > 0$ are chosen as follows

\[ p < 1, \quad q > p, \quad \frac{2c_y}{\rho^2} |r - \delta - \frac{\rho}{\sigma_v} \kappa \theta| < c_\epsilon, \quad \text{or} \]
\[ p = 1, \quad q > p, \quad \frac{2c_y}{\rho^2} |r - \delta - \frac{\rho}{\sigma_v} \kappa \theta| < c_\epsilon \left( \frac{1}{2} - \frac{1}{c_y} |r - \delta - \frac{\rho}{\sigma_v} \kappa \theta| \right) \frac{c_y^2}{\rho^2}, \quad (5) \]

- The constraint in (5) can be really satisfied, for example by choosing $c_y > 4 |r - \delta - \frac{\rho}{\sigma_v} \kappa \theta|$.
Convergence theorem

- The algorithm is actually given by approximating in law the diffusion pair \( X = (Y, V) \) with the Markov chain \( \tilde{X}^h = (\tilde{Y}^h, \tilde{V}^h) \).
- We set \( X^h = (Y^h, V^h) \) as the piecewise constant and càdlàg interpolation in time of \( \tilde{X}^h \), that is
  \[
  X^h_t = \tilde{X}^h_n, \quad t \in [nh, (n + 1)h), \quad n = 0, 1, \ldots, N - 1. \tag{6}
  \]
- We set \( D([0, T]; \mathbb{R}^2) \) the space of the \( \mathbb{R}^2 \)-valued and càdlàg functions on the interval \([0, T]\), that we assume to be endowed with the Skorohod topology. Our main result is the following:

**Theorem**

Suppose that (4) and (5) hold. Then as \( h \to 0 \), the sequence \( \{X^h\}_h = \{(Y^h, V^h)\}_h \) weakly converges in the space \( D([0, T]; \mathbb{R}^2) \) to the diffusion process \( X = (Y, V) \) solution to (1)-(2).
Let us consider pricing European options with payoff function $f : \mathcal{D}([0, T]; \mathbb{R}) \to \mathbb{R}_+$. The transformed payoff function $g(y, v) = f(e^{y + \frac{\rho}{\sigma}v})$, $(y, v) \in \mathcal{D}([0, T]; \mathbb{R}^2)$.

The associated option prices on the continuous and the discrete model as seen at time 0 are given by

\[ P_{Eu} = \mathbb{E}(\tilde{g}(Y, V)) \quad \text{and} \quad P_{Eu}^h = \mathbb{E}(\tilde{g}(Y^h, V^h)), \]

respectively, $\tilde{g}$ denoting the discounted payoff, i.e. $\tilde{g} = e^{-rT}g$.
The weak convergence in Theorem 24 ensures the convergence $P_{Eu}^h \to P_{Eu}$ of the European price when the discounted payoff-function fulfills the following requests: $(y, \nu) \mapsto \tilde{g}(y, \nu)$ is continuous and there exists $a > 0$ and $h_* > 0$ such that

$$\sup_{h < h_*} \mathbb{E}(|\tilde{g}(Y^h, V^h)|^{1+a}) < \infty.$$ 

As for American style options, even for simple payoffs things are more difficult because of the presence of optimal stopping times. However, due to the results in Amin and Khanna 94, we can deduce the convergence of the prices for suitable payoffs.

European and American put options can be numerically evaluated by means of the approximating algorithm.
Numerical results: vanilla options

- We compare the performance of the hybrid tree-finite difference algorithm with the tree method of Vellekoop and Nieuwenhuis. We consider as benchmark the Carr Madan FFT method in the European case and the Monte Carlo Longstaff-Schwartz method in the American case.
- Parameters: $S_0 = 100$, $K = 100$, $T = 1$, $r = \log(1.1)$, $\delta = 0$, $V_0 = 0.1$, $\theta = 0.1$, $\kappa = 2$ and $\rho = -0.5$.
- In order to study the numerical robustness of the algorithms we choose different values for $\sigma$: 0.04, 0.5, 1. For $\sigma = 1$ the Feller condition $2\kappa\theta \geq \sigma^2$ is not satisfied.
- HTFD1: fixed number of time steps $N_t = 100$ and varying number of space steps $N_S = 50, 100, 200, 400$;
- HTFD2: number of time steps equal to the number of space steps: $N_t = N_S = 50, 100, 200, 400$. 

### Numerical results: European options

<table>
<thead>
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**Table:** Prices of European put options. $\sigma = 0.04, 0.5, 1$. $S_0 = 100$, $K = 100$, $T = 1$, $r = \log(1.1)$, $\delta = 0$, $V_0 = 0.1$, $\theta = 0.1$, $\kappa = 2$, $\rho = -0.5$.  

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**Introduction**

The hybrid tree-finite difference approach

**Convergence results**

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Generalization to other models

## Numerical results: American options

<table>
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**Table**: Prices of American put options. $\sigma = 0.04, 0.5, 1$. $S_0 = 100$, $K = 100$, $T = 1$, $r = \log(1.1)$, $\delta = 0$, $V_0 = 0.1$, $\theta = 0.1$, $\kappa = 2$, $\rho = -0.5$. 

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Numerical results: computational time

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Table: Computational times (in seconds) for European put options for $\sigma = 0.5$. 

American vanilla options: comparison with finite difference methods

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<tr>
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<th>$N_S$</th>
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Table: Prices of American put options. $S_0$ = 8, 9, 10, 11, 12, $K = 10$, $T = 0.25$, $r = 0.1$, $V_0 = 0.25$, $\sigma = 0.16$, $\Delta = 5$, $\gamma = 0.1$. 

[$\gamma$ is a volatility parameter, $\Delta$ is the time step size, $\gamma$ is the interest rate, $V_0$ is the initial variance.]
Numerical results: Convergence behaviour

In order to study the convergence behaviour of HTFD2 we consider the following convergence ratio proposed in D’Halluin et al. :

\[
\text{ratio} = \frac{P_{N}^\frac{1}{2} - P_{N}^\frac{1}{4}}{P_{N} - P_{N}^\frac{1}{2}},
\]

(7)

where \(P_{N}\) denotes here the approximated price obtained with \(N = N_t = N_S\) number of time steps.

<table>
<thead>
<tr>
<th>(N)</th>
<th>(S_0 = 8)</th>
<th>(S_0 = 9)</th>
<th>(S_0 = 10)</th>
<th>(S_0 = 11)</th>
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The table suggests that the convergence ratio for HTDF2 is linear, as it is expected to be because of the tree contribution.
We study continuously monitored barrier options and we compare our hybrid tree-finite difference algorithm with the numerical results of the method of lines MOL of Chiarella et al.

We consider European and American up-and-out call options with the following set of parameters: $K = 100$, $T = 0.5$, $r = 0.03$, $\delta = 0.05$, $V_0 = 0.1$, $\theta = 0.1$, $\kappa = 2$, $\rho = -0.5$. The up barrier is $H = 130$. We choose different values for $S_0$: $S_0 = 80, 100, 120$. 
### Numerical results: European barrier options

<table>
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<tr>
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<th>HTFD2</th>
<th>MOL</th>
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**Table:** Prices of European call up-and-out options. *Up barrier is $H = 130$. $K = 100$, $T = 0.5$, $r = 0.03$, $\delta = 0.05$, $V_0 = 0.1$, $\theta = 0.1$, $\kappa = 2$, $\rho = -0.5$.**
### Numerical results: American barrier options

<table>
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<th>HTFD2</th>
<th>MOL</th>
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**Table:** Prices of American call up-and-out options. Up barrier is $H = 130$. $K = 100$, $T = 0.5$, $r = 0.03$, $\delta = 0.05$, $V_0 = 0.1$, $\theta = 0.1$, $\kappa = 2$, $\rho = -0.5$. 
Numerical results: computational time

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**Table:** *Computational times (in seconds) for European Barrier options.*
Conclusion

- The numerical results show that our method is very stable and robust, on the contrary tree methods fail when $\sigma_r$ increases.
- The proposed method is efficient and reliable.
Heston-Hull-White model

The model is a Heston model with a stochastic interest rate $r$: under the risk neutral measure, the model is

$$
\frac{dS_t}{S_t} = (r_t - \delta)dt + \sqrt{V_t} dZ_t, \quad S_0 > 0,
$$

$$
dV_t = \kappa_V (\theta_V - V_t) dt + \sigma_V \sqrt{V_t} dW^1_t, \quad V_0 > 0
$$

$$
dr_t = \kappa_r (\theta_r(t) - r_t) dt + \sigma_r dW^2_t, \quad r_0 > 0.
$$

where $Z$, $W^1$ and $W^2$ are Brownian motions with correlations

$$
d\langle Z, W^1 \rangle_t = \rho_1 dt, \quad d\langle Z, W^2 \rangle_t = \rho_2 dt, \quad d\langle W^1, W^2 \rangle_t = 0.
$$

Here, $r$ follows a generalized OU process: $\theta_r$ is a (deterministic) function determined by the market values of the zero-coupon bonds.
Heston-Hull-White model

The hybrid procedure:

- a 2-dimensional binomial tree for the pair $(V, r)$;
- a finite difference approach in the $S$-direction.
By passing to the logarithm $Y = \ln S$ in the first component and taking into account the above mentioned correlations, we reduce to the dynamics

$$dY_t = (r_t - \eta - \frac{1}{2} V_t)dt + \sqrt{V_t} \left( \rho_1 dW_t^1 + \rho_2 dW_t^2 + \rho_3 dW_t^3 \right), \quad Y_0 = \ln S_0 \in \mathbb{R}$$

$$dV_t = \kappa_V (\theta_V - V_t)dt + \sigma_V \sqrt{V_t} dW_t^1, \quad V_0 > 0,$$

$$dr_t = \kappa_r (\theta_r(t) - r_t)dt + \sigma_r dW_t^2, \quad r_0 > 0,$$

where $W = (W^1, W^2, W^3)$ is a standard Brownian motion in $\mathbb{R}^3$ and the correlation parameter $\rho_3$ is given by

$$\rho_3 = \sqrt{1 - \rho_1^2 - \rho_2^2}, \quad (\rho_1, \rho_2) \in B_1(0),$$

$B_1(0)$ denoting the open ball in $\mathbb{R}^2$ centered in 0 and with radius 1.
The process \( r \) can be written in the following way:

\[
    r_t = \sigma_r X_t + \varphi_t
\]

where

\[
    X_t = -\kappa_r \int_0^t X_s \, ds + W_t^2 \quad \text{and} \quad \varphi_t = r_0 e^{-\kappa_r t} + \kappa_r \int_0^t \theta_r(s) e^{-\kappa_r (t-s)} \, ds.
\]

So, we can consider the triple \((Y, V, X)\), whose dynamics is given by

\[
    dY_t = \mu_Y(V_t, X_t, t) \, dt + \sqrt{V_t} \left( \rho_1 dW_t^1 + \rho_2 dW_t^2 + \rho_3 dW_t^3 \right), \quad Y_0 = \ln S_0 \in \mathbb{R}
\]

\[
    dV_t = \mu_V(V_t) \, dt + \sigma_V \sqrt{V_t} \, dW_t^1, \quad V_0 > 0,
\]

\[
    dX_t = \mu_X(X_t) \, dt + dW_t^2, \quad X_0 = 0,
\]

where

\[
    \mu_Y(v, x, t) = \sigma_r x + \varphi_t - \eta - \frac{1}{2} v,
\]

\[
    \mu_V(v) = \kappa_V(\theta_V - v),
\]

\[
    \mu_X(x) = -\kappa_r x.
\]
The tree for $X$

For $n = 0, 1, \ldots, N$, consider the lattice for the process $X$

$$X_n^h = \{x_{n,j}\}_{j=0,1,\ldots,n} \text{ with } x_{n,j} = (2j - n)\sqrt{h}$$

(14)

We define the multiple jumps

$$k_d^h(n, k) = \max \{k^* : 0 \leq k^* \leq k \text{ and } x_{n,k} + \mu x(x_{n,k})h \geq x_{n+1,k^*}\},$$

$$k_u^h(n, k) = \min \{k^* : k + 1 \leq k^* \leq n + 1 \text{ and } x_{n,k} + \mu x(x_{n,k})h \leq x_{n+1,k^*}\}$$

The transition probabilities are defined as follows

$$p_{u}^{X,h}(n, j) = 0 \vee \frac{\mu x(x_{n,j})h + x_{n,j} - x_{n+1,j_d^h(n,j)}}{x_{n+1,j_u^h(n,j)} - x_{n+1,j_d^h(n,j)}} \wedge 1 \quad \text{and} \quad p_{d}^{X,h}(n, j) = 1 - p_{u}^{X,h}(n, j)$$

(15)

respectively.
The tree for the pair \((V, X)\)

For \(n = 0, 1, \ldots, N\), consider the lattice

\[
V_n^h \times X_n^h = \{(v_{n,k}, x_{n,j})\}_{k,j=0,1,\ldots,n}. \tag{16}
\]

Starting from the node \((n, k, j)\), which corresponds to the position \((v_{n,k}, x_{n,j}) \in V_n^h \times X_n^h\), we define the four possible jump by setting the four nodes at time \(n + 1\)

\[
\begin{align*}
(n + 1, k_u^h(n, k), j_u^h(n, j)) & \quad \text{with probability} \quad p_{uu}(n, k, j) = p_u^V(n, k)p_u^X(n, j), \\
(n + 1, k_u^h(n, k), j_d^h(n, j)) & \quad \text{with probability} \quad p_{ud}(n, k, j) = p_u^V(n, k)p_d^X(n, j), \\
(n + 1, k_d^h(n, k), j_u^h(n, j)) & \quad \text{with probability} \quad p_{du}(n, k, j) = p_d^V(n, k)p_u^X(n, j), \\
(n + 1, k_d^h(n, k), j_d^h(n, j)) & \quad \text{with probability} \quad p_{dd}(n, k, j) = p_d^V(n, k)p_d^X(n, j),
\end{align*}
\]

\(\text{(17)}\)

One might include correlations between any two of the Brownian motions driving the processes \(V, X\). The jump probabilities are no more of a product-type but they solve a linear system of equations that must include the matching of the local cross-moments up to order one in \(h\).
We go now back to (10), that is

\[ dY_t = \mu_Y(V_t, X_t, t)dt + \sqrt{V_t} (\rho_1 dW^1_t + \rho_2 dW^2_t + \rho_3 dW^3_t), \quad Y_0 = \ln S_0, \]
\[ dV_t = \mu_V(V_t)dt + \sigma_V \sqrt{V_t} dW^1_t, \quad V_0 > 0, \]
\[ dX_t = \mu_X(X_t)dt + dW^2_t, \quad X_0 = 0, \]

By isolating \( \sqrt{V_t} dW^1_t \) in the second line and \( dW^2_t \) in the third one, we obtain

\[ dY_t = \frac{\rho_1}{\sigma_V} dV_t + \rho_2 \sqrt{V_t} dX_t + \mu(V_t, X_t, t)dt + \rho_3 \sqrt{V_t} dW^3_t \quad (18) \]

with

\[ \mu(v, x, t) = \mu_Y(v, x, t) - \frac{\rho_1}{\sigma_V} \mu_V(v) - \rho_2 \sqrt{v} \mu_X(x) \]
\[ = \sigma_r x + \varphi_t - \eta - \frac{1}{2} v - \frac{\rho_1}{\sigma_V} \kappa_V(\theta_V - v) + \rho_2 \kappa_r x \sqrt{v}. \quad (19) \]

The main point is that the noise \( W^3 \) is independent of the processes \( V \) and \( X \).
The hybrid Monte Carlo algorithm

- The approximation we have set-up for the Heston-Hull-White processes can be used to construct a **Monte Carlo algorithm**;
- we simulate a continuous process in space (the component $Y$) starting from a discrete process in space (the 2-dimensional tree for $(V, X)$).

**Euler Scheme** We set $\hat{Y}_0^h = Y_0$ and for $t \in [nh, (n + 1)h]$ with $n = 0, 1, \ldots, N - 1$ then

$$
\hat{Y}_{n+1}^h = \hat{Y}_n^h + \frac{\rho_1}{\sigma_V} (\hat{V}_{n+1}^h - \hat{V}_n^h) + \rho_2 \sqrt{\hat{V}_n^h} (\hat{X}_{n+1}^h - \hat{X}_n^h) + \mu h + \rho_3 \sqrt{h\hat{V}_n^h} \Delta_{n+1},
$$

(20)

where $\mu$ is defined in (19) and $\Delta_1, \ldots, \Delta_N$ denote i.i.d. standard normal r.v.’s, independent of the noise driving the chain $(\hat{V}, \hat{X})$.

One let the pair $(V, X)$ evolve on the tree and simulate the process $Y$ at time $nh$ by using (7.1)
The approximating scheme for the triple \((Y, V, X)\)

\[
\tilde{Y}_t^h = Y_{nh} + \frac{\rho_1}{\sigma_V} (\tilde{V}_t^h - \tilde{V}_{nt}^h) + \rho_2 \sqrt{\tilde{V}_{nh}^h} (\tilde{X}_t^h - \tilde{X}_{nh}^h) + \mu(\tilde{X}_{nh}^h, \tilde{V}_{nh}^h, nh)(t-nh) + \rho_3 \sqrt{\tilde{V}_{nh}^h} (W_t^3 - W_{nh}^3).
\]

(21)

If we set

\[
\tilde{Z}_t^h = \tilde{Y}_t^h - \frac{\rho_1}{\sigma_V} (\tilde{V}_t^h - \tilde{V}_{nt}^h) - \rho_2 \sqrt{\tilde{V}_{nh}^h} (\tilde{X}_t^h - \tilde{X}_{nt}^h), \quad t \in [nh, (n+1)h]
\]

(22)

then we have

\[
d\tilde{Z}_t^h = \mu(\tilde{X}_{nh}^h, \tilde{V}_{nh}^h, nh) dt + \rho_3 \sqrt{\tilde{V}_{nh}^h} dW_t^3, \quad t \in (nh, (n+1)h],
\]

(23)

\[
\tilde{Z}_{nh}^h = \tilde{Y}_{nh}^h
\]
Take now a function $f$: we are interested in approximating

$$\mathbb{E}(f(Y_{(n+1)}h) \mid Y_{nh} = y, V_{nh} = v, X_{nh} = x).$$

By using our scheme and the process $\bar{Z}^h$ in (22), we approximate it with

$$\mathbb{E}(f(\bar{Y}_{(n+1)}h) \mid \bar{Y}_{nh} = y, \bar{V}_{nh} = v, \bar{X}_{nh} = x)$$

$$= \mathbb{E}(f(\bar{Z}_{(n+1)}h + \frac{\rho_1}{\sigma_V} (\bar{V}_{(n+1)}h - \bar{V}_{nh}) + \rho_2 \sqrt{\bar{V}_{nh}} (\bar{X}_{(n+1)}h - \bar{X}_{nh})) \mid \bar{Z}_{nh} = y, \bar{V}_{nh} = v, \bar{X}_{nh} = x).$$
Since \((\vec{V}^h, \vec{X}^h)\) is independent of the Brownian noise \(W^3\) driving \(\vec{Z}^h\) in (22), we can write

\[
\mathbb{E}(f(\vec{Y}^h_{(n+1)h}) \mid \vec{Y}^h_{nh} = y, \vec{V}^h_{nh} = v, \vec{X}^h_{nh} = x) = \mathbb{E}\left(\psi_f\left(\frac{\rho_1}{\sigma_v}(\vec{V}^h_{(n+1)h} - v) + \rho_2 \sqrt{v}(\vec{X}^h_{(n+1)h} - x); y, v, x\right) \mid \vec{V}^h_{nh} = v, \vec{X}^h_{nh} = x\right),
\]

in which

\[
\psi_f(\xi; y, v, x) = \mathbb{E}(f(\vec{Z}^h_{(n+1)h} + \xi) \mid \vec{Z}^h_{nh} = y, \vec{V}^h_{nh} = v, \vec{X}^h_{nh} = x).
\]
Now, in order to compute the above quantity $\Psi_f(\xi)$, consider a generic function $g$ and set

$$u(s, z; v, x) = \mathbb{E}(g(\bar{Z}^h_{(n+1)h}) \mid \bar{Z}^h_{s} = z, \bar{V}^h_{s} = v, \bar{X}^h_{s} = x), \quad s \in [nh, (n+1)h].$$

By (23) and the Feynman-Kac representation formula we can state that, for every fixed $x \in \mathbb{R}$ and $v \geq 0$, the function $(s, z) \mapsto u(s, z; v, x)$ is the solution to

$$\begin{aligned}
\partial_s u + \mu(v, x, s)\partial_z u + \frac{1}{2} \rho^2 v \partial_z^2 u &= 0, \quad s \in [nh, (n + 1)h), \ z \in \mathbb{R},
\partial_z u((n + 1)h, z; v, x) &= g(z),
\end{aligned}$$

(26)

$\mu$ being given in (19). In order to solve the above PDE problem, we use a finite difference approach.
The scheme on the $Y$-component

We get the approximation

$$
\Psi_f (\xi; y_i, v_{n,k}, x_{n,j}) \simeq \sum_{\ell \in \mathcal{J}_M} \Pi_{i\ell}^h(v_{n,k}, x_{n,j}) f(y_\ell + \xi), \quad i \in \mathcal{J}_M.
$$

Therefore, the expectation is computed on the approximating tree for $(V, X)$ by means of the above approximation:

$$
E(f(\tilde{Y}^h_{(n+1)h}) | \tilde{Y}^h_{nh} = y_i, \tilde{V}^h_{nh} = v_{n,k}, \tilde{X}^h_{nh} = x_{n,j}) \simeq \sum_{a,b \in \{d,u\}} \sum_{\ell \in \mathcal{J}_M} \Pi_{i\ell}^h(v_{n,k}, x_{n,j}) T_{n,k,j} f(\ell, a, b) p^h_{ab}(n, k, j)
$$

where

$$
T_{n,k,j} f(\ell, a, b) = f(y_\ell + \frac{\rho_1}{\sigma_V}(v_{n+1,k_a(n,k)} - v) + \rho_2 \sqrt{v}(x_{n+1,j_b(n,j)} - x))
$$
The algorithm for the pricing of American options

Finally, we can summarize the backward induction giving our approximating algorithm as follows. For \( n = 0, 1, \ldots, N \), we define \( \tilde{P}_h(n h, y, v, x) \) for \((y, v, x) \in \mathcal{D}_{n,M}^h\) as

\[
\tilde{P}_h(T, y_i, v_{N,k}, x_{N,j}) = \psi(y_i) \quad \text{for } (y_i, v_{N,k}, x_{N,j}) \in \mathcal{D}_{N,M} \text{ and as } n = N - 1, \ldots, 0:
\]

\[
\tilde{P}_h(n h, y_i, v_{n,k}, x_{n,j}) = \max \left\{ \psi(y_i), e^{-(\sigma r x_{n,j} + \varphi_{n} h) h} \times \sum_{a, b \in \{d, u\}} \sum_{\ell \in \mathcal{J}_M} \Pi^h_{i,\ell}(v_{n,k}, x_{n,j}) p^h_{ab}(n, k, j) T_{n,k,j} P_h(\ell, a, b) \right\}.
\]

for \((y_i, v_{n,k}, x_{n,j}) \in \mathcal{D}_{n,M}^h\).

\[(28)\]
Heston-Hull-White2D model

\[
\frac{dS_t}{S_t} = (r_t - \eta_t) dt + \sqrt{V_t} dZ_t,
\]
\[
dV_t = \kappa_V (\theta_V - V_t) dt + \sigma_V \sqrt{V_t} dW^1_t,
\]
\[
dr_t = \kappa_r (\theta_r(t) - r_t) dt + \sigma_r dW^2_t,
\]
\[
d\eta_t = \kappa_\eta (\theta_\eta(t) - \eta_t) dt + \sigma_\eta dW^3_t,
\]

with initial data \(S_0, V_0, r_0, \eta_0 > 0\), where \(Z, W^1, W^2\) and \(W^3\) are suitable and possibly correlated Brownian motions. **Note that the process \(\eta\) evolves as a generalized OU process:** \(\theta_\eta\) is a deterministic function of the time. We consider non null correlations between the Brownian motions driving the pairs \((S, V), (S, r)\) and \((S, \eta)\), that is

\[
d\langle Z, W^1 \rangle_t = \rho_1 dt, \quad d\langle Z, W^2 \rangle_t = \rho_2 dt, \quad d\langle Z, W^3 \rangle_t = \rho_3 dt.
\]
Heston-Hull-White model

The hybrid procedure:

- a 3-dimensional binomial tree for the triple \((V, r, \eta)\);
- a finite difference approach in the \(S\)-direction.
Numerical results

We consider

- **HTFD1** refers to the (fixed) number of time steps $N_t = 50$ and varying number of space steps $N_S = 50, 100, 150, 200$;

- **HTFD2** refers to $N_t = N_S = 50, 100, 150, 200$.

- **HMC1** and **AMC1** refer to 50 000 iterations,

- **HMC2** and **AMC2** refer to 200 000 iterations.

The benchmark value **B-AMC** is obtained using the Alfonsi Monte Carlo method **AMC** with a huge number of Monte Carlo simulations (1 million iterations) and $N_t = 300$ discretization time steps.

In the American case, in absence of reliable numerical methods, we consider the Longstaff-Schwartz algorithm **MC-LS** with 20 exercise dates.
### Numerical results: European Options

<table>
<thead>
<tr>
<th>$\rho_{SV} = -0.5$</th>
<th>$N_S$</th>
<th>HTFD1</th>
<th>HTFD2</th>
<th>$B$-AMC</th>
<th>HMC1</th>
<th>HMC2</th>
<th>AMC1</th>
<th>AI</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_{Sr} = -0.5$</td>
<td>50</td>
<td>11.202744</td>
<td>11.202744</td>
<td>11.34±0.04</td>
<td>11.30±0.16</td>
<td>11.32±0.08</td>
<td>11.34±0.16</td>
<td>11.37</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>11.319814</td>
<td>11.331040</td>
<td></td>
<td>11.41±0.16</td>
<td>11.38±0.08</td>
<td>11.31±0.16</td>
<td>11.36</td>
</tr>
<tr>
<td></td>
<td>150</td>
<td>11.340665</td>
<td>11.34902</td>
<td></td>
<td>11.36±0.16</td>
<td>11.36±0.08</td>
<td>11.35±0.16</td>
<td>11.38</td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>11.346972</td>
<td>11.35772</td>
<td></td>
<td>11.34±0.16</td>
<td>11.37±0.08</td>
<td>11.44±0.16</td>
<td>11.38</td>
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<tr>
<td>$\rho_{Sr} = 0$</td>
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<td>12.526779</td>
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<td>12.66±0.18</td>
<td>12.69±0.09</td>
<td>12.68±0.18</td>
<td>12.71</td>
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<tr>
<td></td>
<td>100</td>
<td>12.720651</td>
<td>12.70577</td>
<td></td>
<td>12.74±0.18</td>
<td>12.79±0.09</td>
<td>12.63±0.18</td>
<td>12.78</td>
</tr>
<tr>
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<td>12.754610</td>
<td>12.74952</td>
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<td>12.74±0.18</td>
<td>12.79±0.09</td>
<td>12.68±0.18</td>
<td>12.81</td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>12.760365</td>
<td>12.76683</td>
<td></td>
<td>12.74±0.18</td>
<td>12.80±0.09</td>
<td>12.75±0.18</td>
<td>12.78</td>
</tr>
<tr>
<td>$\rho_{Sr} = 0.5$</td>
<td>50</td>
<td>13.853193</td>
<td>13.853193</td>
<td>14.04±0.04</td>
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<td>13.97±0.20</td>
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<tr>
<td></td>
<td>100</td>
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<td></td>
<td>150</td>
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<td>13.94±0.19</td>
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<td>200</td>
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<td>13.99±0.19</td>
<td>14.07±0.10</td>
<td>13.90±0.19</td>
<td>14.06</td>
</tr>
</tbody>
</table>

**Table:** Prices of European call options. $S_0 = 100$, $K = 100$, $T = 1$, $r_0 = 0.04$, $\kappa_r = 1$, $\sigma_r = 0.2$, $\eta = 0.03$, $V_0 = 0.1$, $\theta_V = 0.1$, $\kappa_V = 2$, $\sigma_V = 0.3$, $\rho_{Sr} = -0.5, 0, 0.5$, $\rho_{SV} = -0.5$. 
### Numerical results: American Options

<table>
<thead>
<tr>
<th>$\rho_{SV} = -0.5$</th>
<th>$N_S$</th>
<th>HTFD1</th>
<th>HTFD2</th>
<th>MC-LS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_{Sr} = -0.5$</td>
<td>50</td>
<td>12.090433</td>
<td>12.090433</td>
<td>12.18±0.01</td>
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<tr>
<td></td>
<td>100</td>
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</tr>
<tr>
<td></td>
<td>150</td>
<td>12.224432</td>
<td>12.231392</td>
<td></td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>12.230288</td>
<td>12.237054</td>
<td></td>
</tr>
<tr>
<td>$\rho_{Sr} = 0$</td>
<td>50</td>
<td>12.912708</td>
<td>12.912708</td>
<td>13.14±0.01</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>13.119121</td>
<td>13.101073</td>
<td></td>
</tr>
<tr>
<td></td>
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<td>13.156492</td>
<td>13.149182</td>
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</tr>
<tr>
<td></td>
<td>200</td>
<td>13.162893</td>
<td>13.168602</td>
<td></td>
</tr>
<tr>
<td>$\rho_{Sr} = 0.5$</td>
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<td>13.944266</td>
<td>13.944266</td>
<td>14.15±0.01</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>14.125059</td>
<td>14.122918</td>
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<tr>
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<td>14.146240</td>
<td>14.152060</td>
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<td></td>
<td>200</td>
<td>14.153288</td>
<td>14.160288</td>
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</table>

**Table:** Prices of American call options. $S_0 = 100$, $K = 100$, $T = 1$, $r_0 = 0.04$, $\kappa_r = 1$, $\sigma_r = 0.2$, $\eta = 0.03$, $V_0 = 0.1$, $\theta_V = 0.1$, $\kappa_V = 2$, $\sigma_V = 0.3$, $\rho_{Sr} = -0.5, 0, 0.5$, $\rho_{SV} = -0.5$. 
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Numerical results
Generalization to other models

<table>
<thead>
<tr>
<th>$N_S$</th>
<th>HTFD1</th>
<th>HTDF2</th>
<th>B-AMC</th>
<th>HMC1</th>
<th>HMC2</th>
<th>AMC1</th>
<th>AMC</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>0.41</td>
<td>0.41</td>
<td>223.67</td>
<td>0.77</td>
<td>3.05</td>
<td>2.16</td>
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<td>100</td>
<td>0.84</td>
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<td>1.59</td>
<td>6.11</td>
<td>4.00</td>
<td>14.61</td>
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</tr>
<tr>
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<tr>
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<td>1.87</td>
<td>213.06</td>
<td>3.11</td>
<td>12.73</td>
<td>7.61</td>
<td>28.85</td>
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</table>

**Table:** Computational times (in seconds) for European call options.
Numerical results: European Options

<table>
<thead>
<tr>
<th>( \rho_{SV} = -0.5, ) ( \rho_{S\eta} = -0.5 )</th>
<th>( N_S )</th>
<th>HTFD1</th>
<th>HTFD2</th>
<th>B-AMC</th>
<th>HMC1</th>
<th>HMC2</th>
<th>AMC1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho_{Sr} = -0.5 )</td>
<td>30</td>
<td>13.470572</td>
<td>13.470572</td>
<td>13.79 ± 0.04</td>
<td>13.82±0.20</td>
<td>13.74±0.10</td>
<td>13.83±0.20</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>13.688842</td>
<td>13.671173</td>
<td>13.96±0.20</td>
<td>13.81±0.10</td>
<td>13.88±0.20</td>
<td>13.68±0.20</td>
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<tr>
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<td>100</td>
<td>13.790205</td>
<td>13.781519</td>
<td>14.00±0.20</td>
<td>13.82±0.10</td>
<td>13.68±0.20</td>
<td>13.71±0.20</td>
</tr>
<tr>
<td>( \rho_{Sr} = 0 )</td>
<td>30</td>
<td>14.736242</td>
<td>14.736242</td>
<td>15.04 ± 0.05</td>
<td>15.10±0.22</td>
<td>14.99±0.11</td>
<td>14.95±0.22</td>
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<tr>
<td></td>
<td>50</td>
<td>14.958094</td>
<td>14.946029</td>
<td>15.23±0.22</td>
<td>15.04±0.11</td>
<td>14.98±0.22</td>
<td>14.80±0.21</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>15.019204</td>
<td>15.032709</td>
<td>15.23±0.22</td>
<td>15.04±0.11</td>
<td>14.80±0.21</td>
<td>15.01±0.21</td>
</tr>
<tr>
<td>( \rho_{Sr} = 0.5 )</td>
<td>30</td>
<td>15.805046</td>
<td>15.805046</td>
<td>16.19 ± 0.03</td>
<td>15.21±0.22</td>
<td>15.04±0.11</td>
<td>16.04±0.23</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>16.052315</td>
<td>16.032043</td>
<td>16.13±0.23</td>
<td>16.06±0.11</td>
<td>16.09±0.23</td>
<td>15.93±0.23</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>16.155354</td>
<td>16.145308</td>
<td>16.33±0.23</td>
<td>16.10±0.11</td>
<td>15.93±0.23</td>
<td>16.04±0.23</td>
</tr>
</tbody>
</table>

**Table:** Prices of European call options. \( S_0 = 100, K = 100, T = 1, \)
\( r_0 = 0.04, \kappa_r = 1, \sigma_r = 0.2, \eta_0 = 0.03, \kappa_\eta = 1, \sigma_\eta = 0.2, V_0 = 0.1, \)
\( \theta_V = 0.1, \kappa_V = 2, \sigma_V = 0.3, \rho_{Sr} = -0.5, 0, 0.5, \rho_{SV} = -0.5, \)
\( \rho_{S\eta} = -0.5. \)
Numerical results : American Options

<table>
<thead>
<tr>
<th>$\rho_{SV} = -0.5$, $\rho_{SN} = -0.5$</th>
<th>$N_S$</th>
<th>HTFD1</th>
<th>HTFD2</th>
<th>MC-LS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_{Sr} = -0.5$</td>
<td>30</td>
<td>14.057963</td>
<td>14.057963</td>
<td>14.37 ± 0.01</td>
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<tr>
<td></td>
<td>50</td>
<td>14.290597</td>
<td>14.263254</td>
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<td>100</td>
<td>14.400377</td>
<td>14.381552</td>
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</tr>
<tr>
<td>$\rho_{Sr} = 0$</td>
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<td>14.989844</td>
<td>14.989844</td>
<td>15.32 ± 0.01</td>
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<td>15.253011</td>
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<td>100</td>
<td>15.320569</td>
<td>15.331744</td>
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<td>$\rho_{Sr} = 0.5$</td>
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<td>16.146080</td>
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<td>100</td>
<td>16.270439</td>
<td>16.248656</td>
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Table: Prices of American call options. $S_0 = 100$, $K = 100$, $T = 1$, $r_0 = 0.04$, $\kappa_r = 1$, $\sigma_r = 0.2$, $\eta_0 = 0.03$, $\kappa_\eta = 1$, $\sigma_\eta = 0.2$, $V_0 = 0.1$, $\theta_V = 0.1$, $\kappa_V = 2$, $\sigma_V = 0.3$, $\rho_{Sr} = -0.5, 0, 0.5$, $\rho_{SV} = -0.5$, $\rho_{SN} = -0.5$. 
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### Numerical results

<table>
<thead>
<tr>
<th>$N_s$</th>
<th>HTFD1</th>
<th>HTDF2</th>
<th>B-AMC</th>
<th>HMC1</th>
<th>HMC2</th>
<th>AMC1</th>
<th>AMC2</th>
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</thead>
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<td>2.22</td>
<td>284.84</td>
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<td>4.19</td>
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<td>2.02</td>
<td>8.06</td>
<td>5.05</td>
<td>18.70</td>
</tr>
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</table>

**Table:** Computational times (in seconds) for European call options.
Bates model

The Bates model differs from the Heston model in the presence of jumps in the equation for the share price $S$:

$$\frac{dS_t}{S_t} = (r - \delta)dt + \sqrt{V_t} dZ_S(t) + dN_t, \quad S(0) = S_0 > 0$$
$$dV_t = \kappa(\theta - V_t)dt + \sigma \sqrt{V_t} dZ_V(t), \quad V(0) = V_0 > 0.$$

Here, $N_t$ is a compound Poisson process independent of the correlated Brownian motion $(Z_S, Z_V)$ with intensity $\lambda$ and independent jumps $J_1, J_2, \ldots$ whose common law is

$$\log(1 + J) \sim N\left( \log(1 + \gamma) - \frac{1}{2} \alpha^2, \alpha^2 \right).$$

So, the PDE problem becomes a PIDE problem.
\[
\begin{align*}
\partial_s \hat{u}^h(s, y; v, g_{t+h}) + L^{(v)} \hat{u}^h(s, y; v, g_{t+h}) &= 0 \quad y \in \mathbb{R}, s \in (t, t + h), \\
\hat{u}^h(t + h, y; v, g_{t+h}) &= f(y, g_{t+h}) \quad y \in \mathbb{R},
\end{align*}
\] (29)

with

\[
L^{(v)} u = \tilde{\mu} \gamma(v) \partial_y u + \frac{1}{2} \rho^2 v \partial_{yy} u + \int_{-\infty}^{+\infty} [u(y + x, s) - u(y)] p(x) dx.
\]
To solve the PIDE perform the following steps:

- **Localisation.** We choose a spatial bounded computational domain $\Omega = [-l, l]$, which implies that we must choose some artificial boundary conditions.

- **Truncation of large jumps.** This step corresponds to truncating the integration domain in the integral part.

- **Discretisation.** The derivatives of the solution are replaced by finite differences, and the integral terms are approximated using the trapezoidal rule. Then the problem is solved by using an explicit-implicit scheme (see Cont and Voltchkova, Briani et al).
We have introduced a new hybrid tree-finite difference method and a new Monte Carlo method for numerically pricing options in a stochastic volatility with jumps framework with stochastic interest rates.

The numerical comparisons show that both methods provide good approximation of the option prices with efficient time computations.