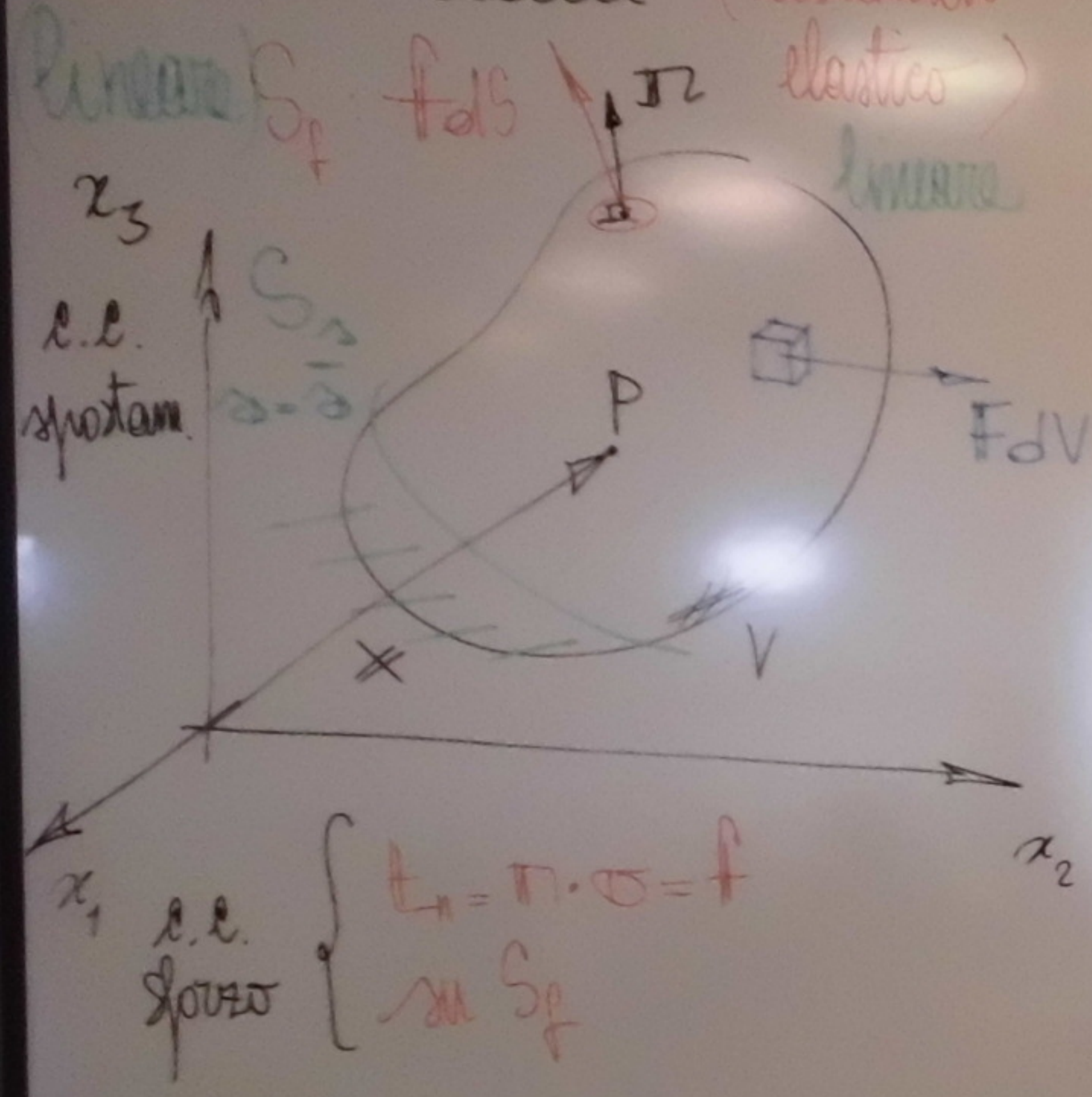


Problema elastico

legame costitutivo elastico



Risposta tenso-deformativa:

- sforzo: tensore sforzo di Cauchy $\sigma(x)$ di comp. σ_{ij} (6 comp.)
- deformazione:
 - campo di spostamento $s(x)$ di comp. s_i (3 comp.)
 - campo di deformazione $E(x)$ di comp. E_{ij} (6 comp.)

Eq. governanti

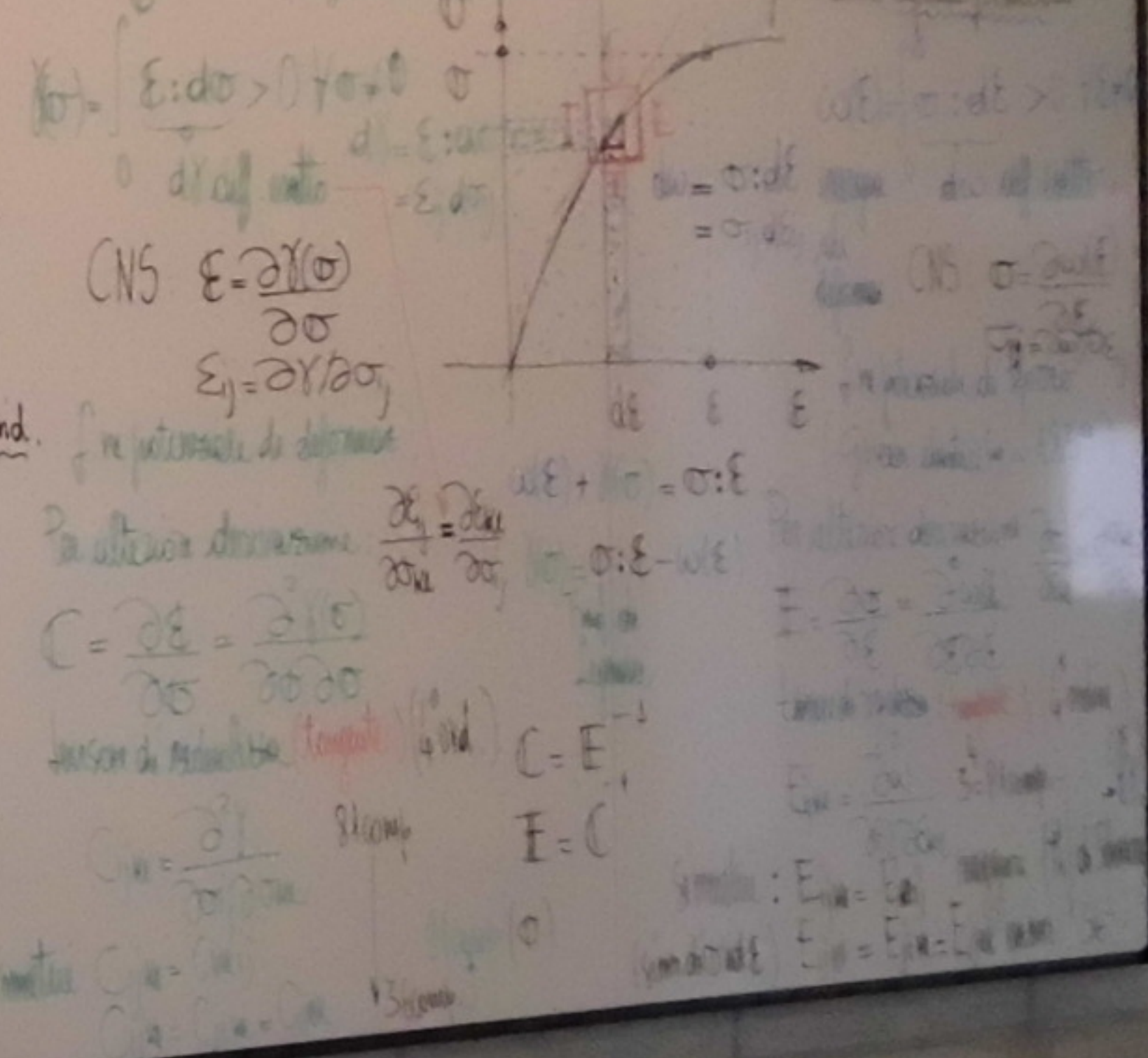
$\text{div } \sigma + F = 0 \text{ in } V$
 $\sigma_{ij,i} + F_j = 0; j=1,2,3 \quad 3 \text{ eq.}$
 congruenza di
 $\text{eq. in di DSV} \quad E_{ij,k} + E_{kj,i} = E_{ik,j} + E_{ji,k} \quad 8 \text{ eq.}$
 $E = \frac{1}{2}(\nabla u + \nabla u^T) \text{ in } V$
 $E_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad i,j=1,2,3 \quad 6 \text{ eq.}$
 3 Identities di Biot
 3 eq. ind.

Legame costitutivo (comp. meccanico dei materiali)

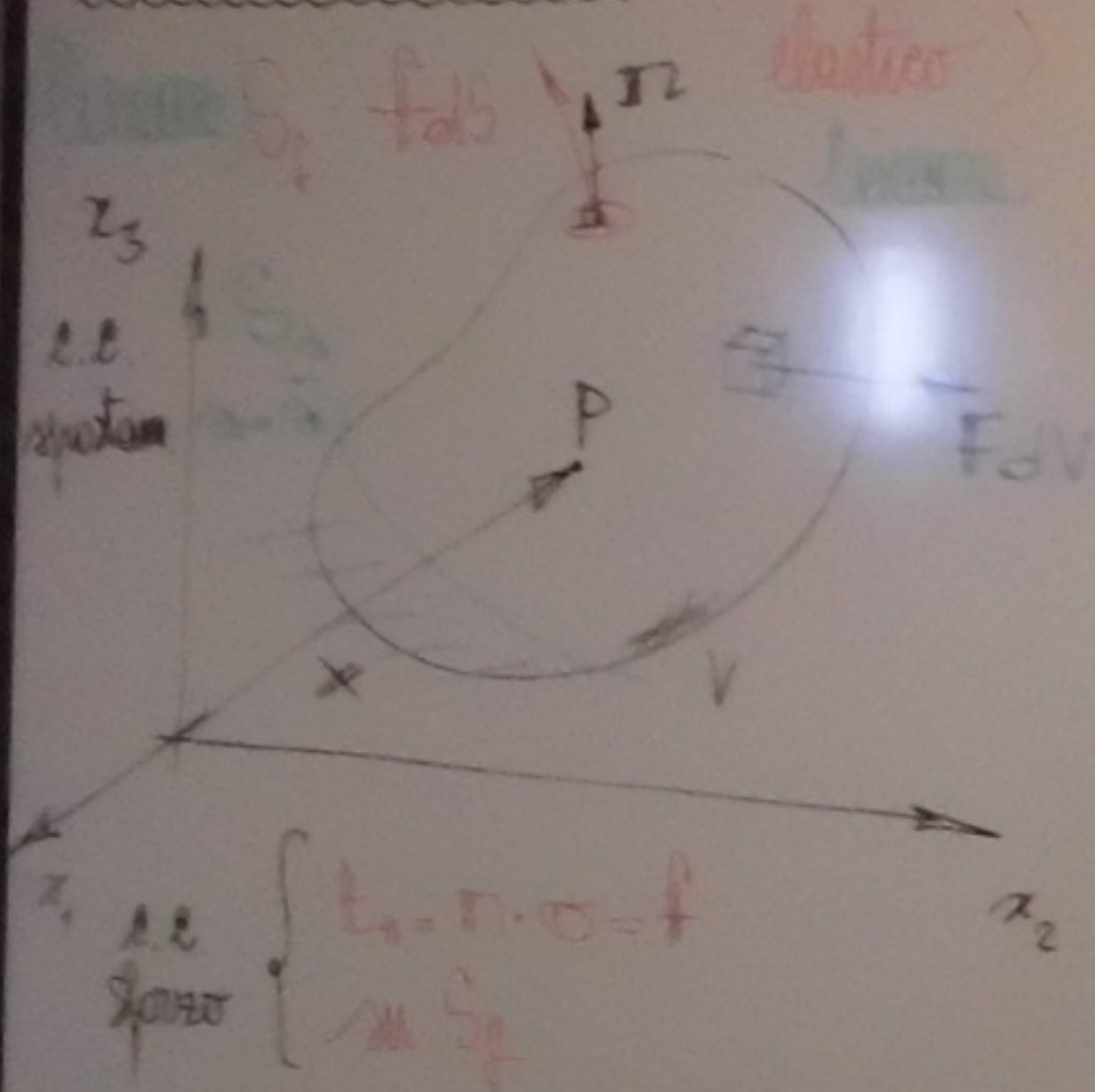
$\sigma = \sigma(E) \text{ o } E = E(\sigma) \quad i,j,k,l=1,2,3 \quad 6 \text{ eq.}$
 $\sigma_{ij} = \sigma_{ij}(E_{kl}) \text{ o } E_{ij} = E_{ij}(\sigma_{kl})$
 (equil. in scalo in inf. + relas. deform. spost.) 9 eq. in
 + comp. del materiale (elastico) 15 eq. in
 (3 eq. equil. + 3 di comp. + 6 eq. cost.) 12 eq. in

n. di componenti incognite 15 comp.
 (senza det. del campo di spostamento) 12 inc.

Energia deformativa (Energia, lavoro)



Problema elastico



Risposta tenso-deformativa:

- sforzo: tensore sforzo di Cauchy $\sigma(x)$ di comp. σ_{ij} (6 comp.)
- deformazione: campo di spostamento $s(x)$ di comp. s_i (3 comp.)
- campo di deformazione $E(x)$ di comp. E_{ij} (6 comp.)

Eqm governante

$$\text{div } \sigma + F = 0 \text{ in } V$$

$$\sigma_{ij,i} + F_j = 0, \quad j=1,2,3 \quad 3 \text{ eq.}$$

congruenza di

$$E_{ij} = \frac{1}{2} (\nabla_i u_j + \nabla_j u_i) \quad 6 \text{ eq.}$$

$$E_{ij} = \frac{1}{2} (\partial_i u_j + \partial_j u_i) \quad 6 \text{ eq.}$$

3 Identieta di Bianchi

Legame costitutivo (comp. meccanico del materiale)

$\sigma = \sigma(E)$ o $E = E(\sigma)$ $i,j,k,l=1,2,3$ 6 eq.

(equil. in sede inel. + elast. deform. spst) 9 eq. in

+ comp. del materiale (elastico) 15 eq. in

(3 eq. equil. + 3 di comp. + 6 eq. cost.) 12 eq. in

Energia complementare (Engesser, Castigliano)

$$W(\sigma) = \int_V E : d\sigma > 0 \quad \forall \sigma \neq 0$$

$$dW = E : d\sigma = \sigma_{ij} d\epsilon_{ij}$$

$$CNS \quad E = \frac{\partial W(\sigma)}{\partial \sigma}$$

$$\epsilon_{ij} = \frac{\partial W(\sigma)}{\partial \sigma_{ij}}$$

funzione potenziale di deformazione

Per ulteriori derivazioni: $\frac{\partial \epsilon_{ij}}{\partial \sigma_{kl}} = \frac{\partial^2 W(\sigma)}{\partial \sigma_{kl} \partial \sigma_{ij}}$

Legge di elasticita (tangente) (4° ord.)

$$C_{ijkl} = \frac{\partial^2 W(\sigma)}{\partial \sigma_{ij} \partial \sigma_{kl}} \quad 81 \text{ comp.}$$

$$C_{ijkl} = C_{jikl} = C_{klij} = C_{lki j}$$

simmetria $C_{ijkl} = C_{jikl} = C_{klij} = C_{lki j}$ 36 comp.

Legame iperelastico

Postuliamo l'esistenza di una densita di deformazione definita positivamente

$$W(E) = \int_V \sigma : dE > 0 \quad \forall E \neq 0$$

$$dW = \sigma : dE = \sigma_{ij} d\epsilon_{ij}$$

$$CNS \quad \sigma = \frac{\partial W(E)}{\partial E}$$

$$\sigma_{ij} = \frac{\partial W(E)}{\partial \epsilon_{ij}}$$

funzione potenziale di sforzo

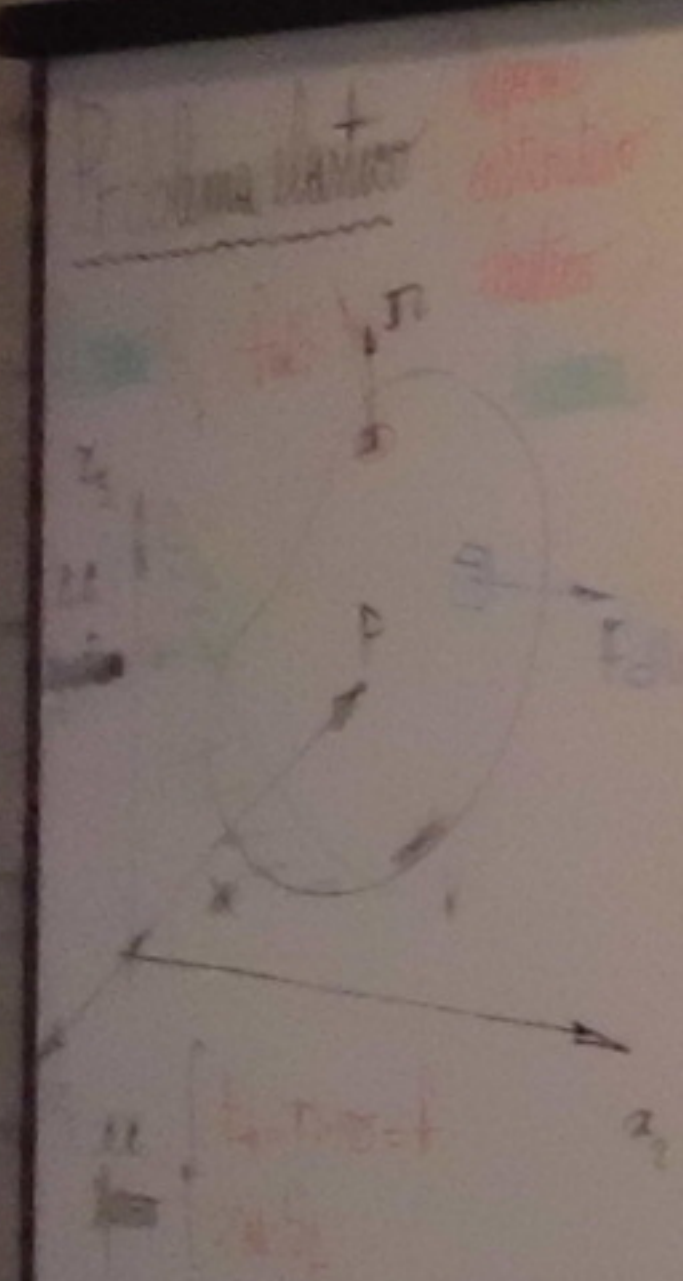
Per ulteriori derivazioni: $\frac{\partial \sigma_{ij}}{\partial \epsilon_{kl}} = \frac{\partial^2 W(E)}{\partial \epsilon_{kl} \partial \epsilon_{ij}}$

Legge di elasticita (tangente) (4° ord.)

$$E_{ijkl} = \frac{\partial^2 W(E)}{\partial \epsilon_{ij} \partial \epsilon_{kl}} \quad 81 \text{ comp.}$$

$$E_{ijkl} = E_{jikl} = E_{klij} = E_{lki j}$$

simmetria: $E_{ijkl} = E_{jikl} = E_{klij} = E_{lki j}$ 36 comp.



Resposta lineare-elastica

- Spesso tensor spaziale di Cauchy $\sigma(x)$
- campo σ_{ij} (6 comp.)
- campo di spostamento $u(x)$
- campo u_i (3 comp.)
- campo di deformazione $E(x)$
- campo E_{ij} (6 comp.)

Equi governante

$\text{div } \sigma + F = 0 \text{ in } V$

$\sigma_{j,i} + F_i = 0, j=1,2,3 \quad 3 \text{ eq.}$

conservazione di eq. m di B.S.V. $\epsilon_{ij,k} + \epsilon_{kij} = \epsilon_{ik,j} + \epsilon_{ijk} \quad 81 \text{ eq.}$

$\epsilon = \frac{1}{2}(\nabla u + \nabla u^T) \quad 3 \text{ Identities di Biot}$

$\epsilon_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j) \quad 3 \text{ eq. ind.}$

Legame costitutivo (comp. meccanico del materiale)

$\sigma = \sigma(E) \text{ o } E = E(\sigma) \quad 12 \text{ eq.}$

$\sigma_i = \sigma_i(\epsilon_{ij}) \text{ o } \epsilon_{ij} = \epsilon_{ij}(\sigma_{ij})$

Legge di Hooke + elast. deform. spaziale 9 eq. m

+ campo del materiale (elastico) 15 eq. m

(3 eq. eq. + 3 di campo + 6 eq. rest.) 12 eq. m

Energia complementare (Engesser, Castigliano)

$\gamma(\sigma) = \int_0^\sigma \epsilon : d\sigma > 0 \quad \forall \sigma \neq 0$

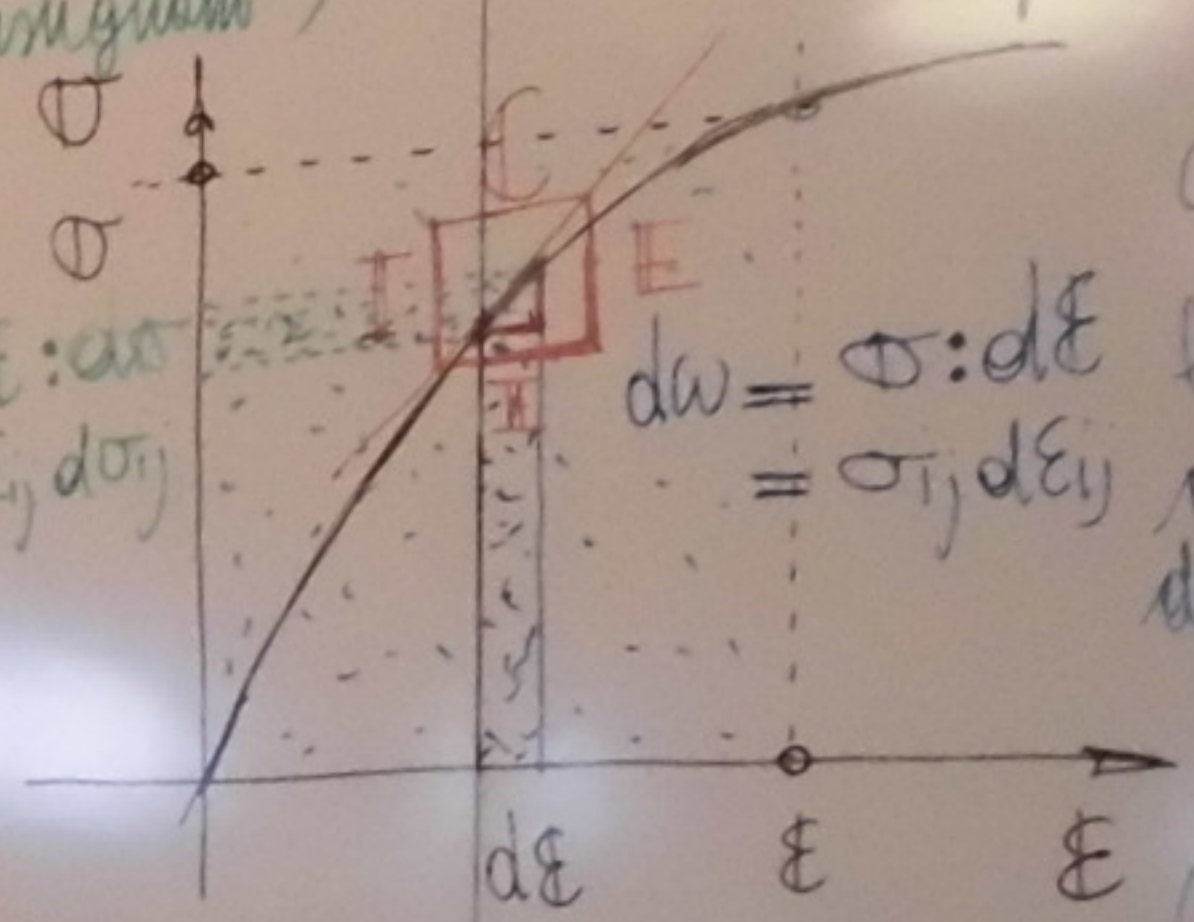
$d\gamma = \epsilon : d\sigma = \epsilon_{ij} d\sigma_{ij}$

$d\gamma \text{ diff. esatto}$

CNS: $E = \frac{\partial \gamma(\sigma)}{\partial \sigma}$

$\epsilon_{ij} = \frac{\partial \gamma(\sigma)}{\partial \sigma_{ij}}$

γ m potenziale di deformazione



Per ulteriore derivazione: $\frac{\partial \epsilon_{ij}}{\partial \sigma_{kl}} = \frac{\partial^2 \gamma}{\partial \sigma_{kl} \partial \sigma_{ij}}$

$C = \frac{\partial^2 \gamma}{\partial \sigma \partial \sigma}$

Tensor di elasticità (tangent) (4° ord.)

$C_{ijkl} = \frac{\partial^2 \gamma}{\partial \sigma_{ij} \partial \sigma_{kl}}$ 81 comp.

Simmetrie: $C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}$ 24 eq. m

$C = E^{-1}$

$E = C^{-1}$

Postuliamo l'Enza di energia di deformazione definita positiva

$$\omega(E) = \int_0^E \sigma : dE > 0 \quad \forall E \neq 0$$

$d\omega \text{ diff. esatto}$

energia di deformazione

CNS: $\sigma = \frac{\partial \omega(E)}{\partial E}$

$\sigma_{ij} = \frac{\partial \omega}{\partial \epsilon_{ij}}$

γ m potenziale di sforzo (Green elasticity ~ 1839)

Per ulteriore derivazione $\frac{\partial \sigma_{ij}}{\partial \epsilon_{kl}} = \frac{\partial^2 \omega}{\partial \epsilon_{kl} \partial \epsilon_{ij}}$

$E = \frac{\partial^2 \omega}{\partial E \partial E}$

Tensor di rigidità (tangent) (4° ordine)

$E_{ijkl} = \frac{\partial^2 \omega}{\partial \epsilon_{ij} \partial \epsilon_{kl}}$ 3 = 81 comp.

Simmetrie: $E_{ijkl} = E_{jikl} = E_{ijlk} = E_{klij}$ maggiore (th) di Schwarz

(Simmetria di σ ed E) $E_{ijkl} = E_{jike} = E_{ijek}$ minori 36

Legame iperelastico lineare (legge di Hooke generalizzata)

$$\gamma = \frac{1}{2} \sigma : C : \sigma = \frac{1}{2} \sigma_{ij} C_{ijkl} \sigma_{kl}$$

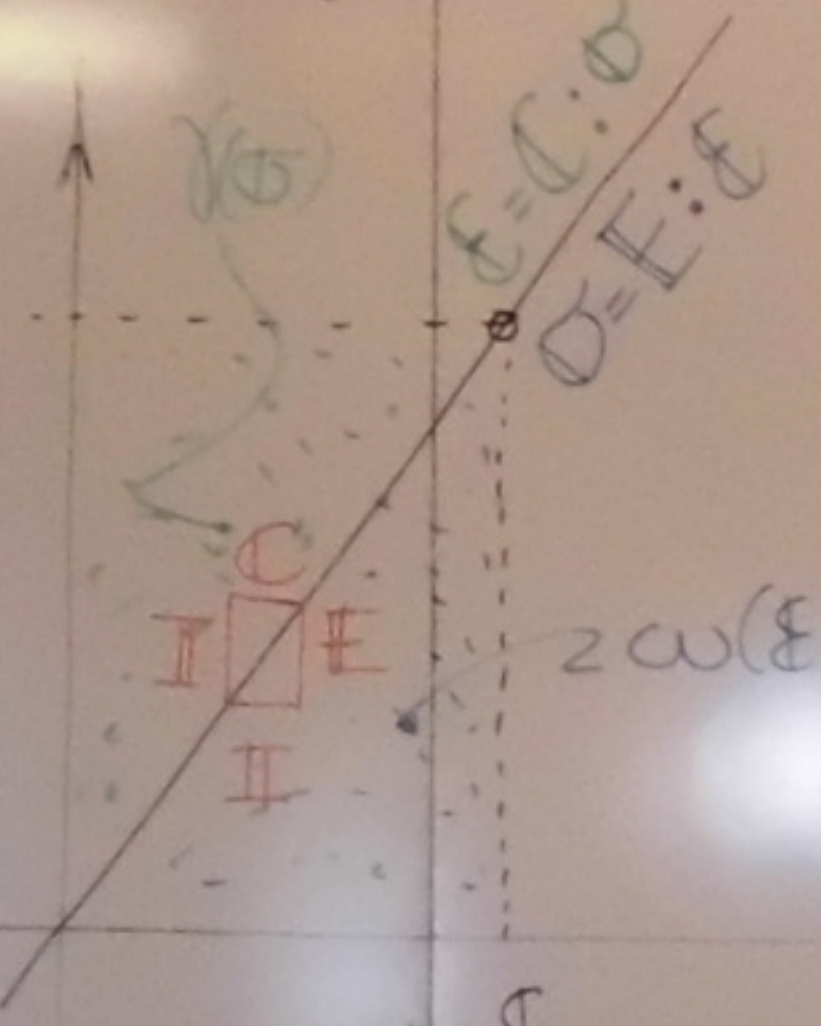
$$\downarrow$$

$$\epsilon = \frac{\partial \gamma}{\partial \sigma} = C : \sigma \Rightarrow \epsilon_{ij} = C_{ijkl} \sigma_{kl}$$

$$\downarrow$$

$$C = \frac{\partial^2 \gamma}{\partial \sigma \partial \sigma} = \text{cost. tensore di rigidità}$$

C_{ijkl} 21



$$\omega = \frac{1}{2} \epsilon : E : \epsilon = \frac{1}{2} \epsilon_{ij} E_{ijkl} \epsilon_{kl} > 0 \forall \epsilon \neq 0$$

$$\downarrow$$

$$\sigma = \frac{\partial \omega}{\partial \epsilon} = E : \epsilon \Rightarrow \sigma_{ij} = E_{ijkl} \epsilon_{kl}$$

$$\downarrow$$

$$E = \frac{\partial^2 \omega}{\partial \epsilon \partial \epsilon} = \text{cost. tensore di rigidità}$$

E_{ijkl} 21

Comportamento isotropo

Comp. meccanico, indep. della dizez. - Comp. è simmetrico
 - 21 param. indep. $\rightarrow$ 2 parametri indep. ($\nu \rightarrow -1, G \rightarrow \infty$)
 $G = \text{shear modulus}$
 $\nu = \text{coefficient of Poisson}$
 $E = \text{Young's modulus}$
 $K = \text{bulk modulus}$
 $\epsilon = -\frac{\nu}{1+\nu} \text{tr}(\sigma) I + \frac{1+\nu}{E} \sigma \Leftrightarrow \{\epsilon\} = [C] \cdot \{\sigma\}$

$$C = E^{-1} \quad \epsilon_{nm} \quad \epsilon$$

$$\omega \neq \gamma = \frac{1}{2} \sigma : \epsilon = \frac{1}{2} \epsilon : \sigma$$

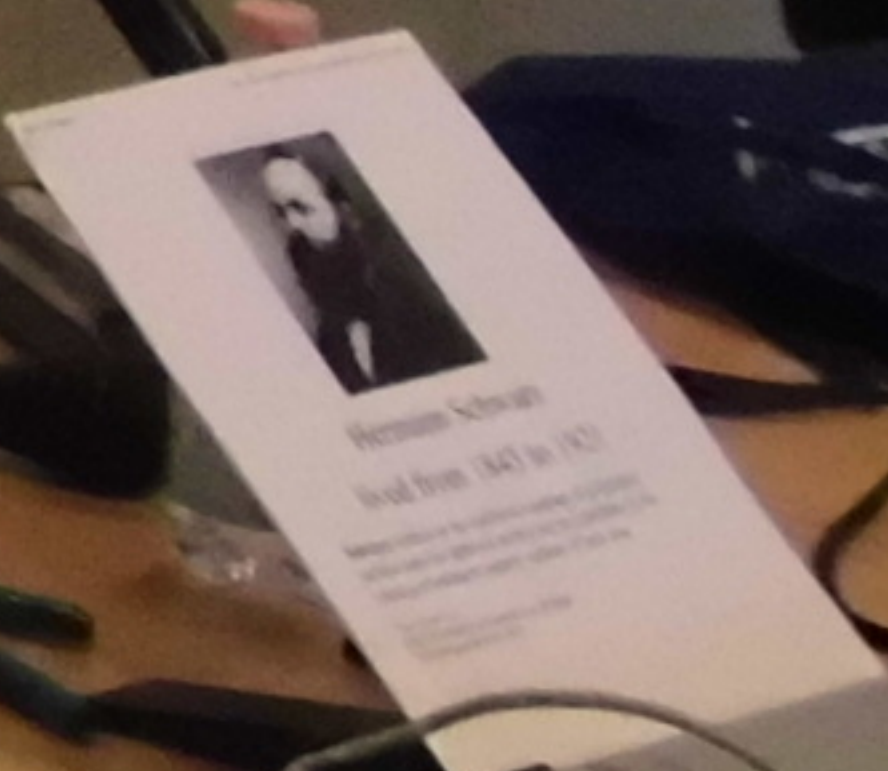
$$E = C^{-1}$$

$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{23} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{bmatrix}$$

$\frac{1}{G} \quad \frac{1}{G} \quad \frac{1}{G}$

Materiali anisotropi

Materiali anisotropi: sim. rispetto a piani materiali or.
 3 moduli elastici E_1, E_2, E_3
 3 coeff. di Poisson ν_1, ν_2, ν_3
 3 moduli elast. tang. G_1, G_2, G_3



Legge di Hooke generalizzata

$$\omega = \frac{1}{2} \epsilon : E : \epsilon = \frac{1}{2} \epsilon_{ij} E_{ijkl} \epsilon_{kl} > 0 \quad \forall \epsilon \neq 0$$

$$\sigma = \frac{\partial \omega}{\partial \epsilon} = E : \epsilon \Leftrightarrow \sigma_{ij} = E_{ijkl} \epsilon_{kl}$$

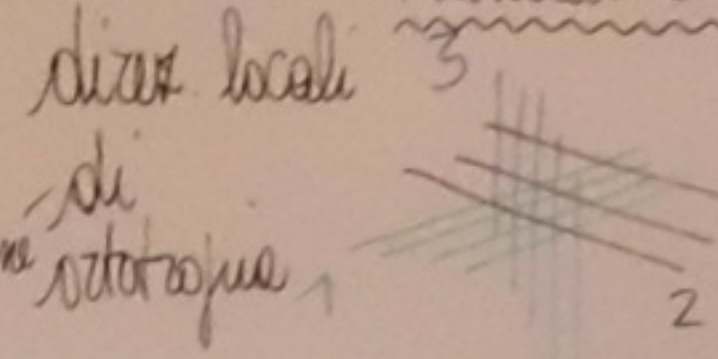
$$\sigma = \lambda \text{tr} \epsilon \mathbb{I} + \mu \epsilon$$

cost. di Lamé isotropia

$$E = \frac{\partial \omega}{\partial \epsilon \partial \epsilon} = \text{cost. tensore di rigidità}$$

$$C = \text{matrice di rigidità (DP.)}$$

Materiale ortotropo (sim. rispetto a tre piani mutuamente ort.)



$$C = \begin{bmatrix} E_1 & -\nu_{12} E_1 & -\nu_{13} E_1 & 0 & 0 & 0 \\ -\nu_{21} E_2 & E_2 & -\nu_{23} E_2 & 0 & 0 & 0 \\ -\nu_{31} E_3 & -\nu_{32} E_3 & E_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{12} \end{bmatrix}$$

3 moduli elast. lineari E_1, E_2, E_3
 3 coeff. di Poisson $\nu_{12}, \nu_{13}, \nu_{23}$
 3 moduli elast. tang. G_{12}, G_{13}, G_{23}

Mat. trasversalmente isotropo
 (asse preferenziale)

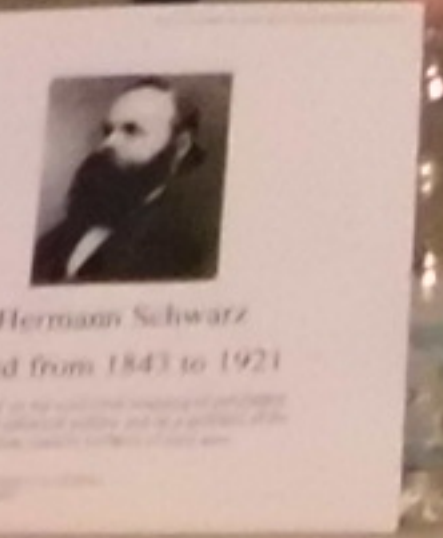
comp. isotropo nel piano ortogonale all'asse trasversale

$$C = \begin{bmatrix} E_L & 0 & 0 & 0 & 0 & 0 \\ 0 & E_T & 0 & 0 & 0 & 0 \\ 0 & 0 & E_T & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{LT} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{TT} & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{TT} \end{bmatrix}$$

Costanti elastiche

21 param. indep. $\rightarrow$ 2 param. indep. $\nu \rightarrow 1/\mu = G$
 $E = \frac{G(1+\nu)}{1-2\nu}$
 $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$
 $\mu = G = \frac{E}{2(1+\nu)}$
 $\nu = \frac{E - 2G}{E + G}$
 $K = \frac{E}{3(1-2\nu)}$

$$C = \begin{bmatrix} E_{11} & E_{12} & E_{13} & 0 & 0 & 0 \\ E_{22} & E_{22} & E_{23} & 0 & 0 & 0 \\ E_{33} & E_{23} & E_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{12} \end{bmatrix}$$



Hermann Schwarz
 lived from 1843 to 1921

$$\sigma = \frac{1}{2} \epsilon : E : \epsilon = \frac{1}{2} \epsilon_{ij} E_{ijkl} \epsilon_{kl} \quad \epsilon_{ij} > 0 \quad \forall \epsilon \neq 0$$

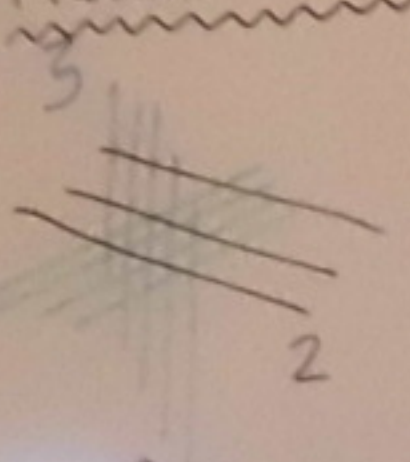
$$\sigma = \frac{1}{2} \epsilon : E : \epsilon = \frac{1}{2} \epsilon_{ij} E_{ijkl} \epsilon_{kl}$$

$$\sigma = \lambda \text{tr} \epsilon \mathbf{I} + \mu \epsilon$$

$$E_{ijkl} = \frac{\lambda}{3} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + \frac{2\mu}{3} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

$$C = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{13}}{E_1} \\ -\frac{\nu_{21}}{E_2} & \frac{1}{E_2} & -\frac{\nu_{23}}{E_2} \\ -\frac{\nu_{31}}{E_3} & -\frac{\nu_{32}}{E_3} & \frac{1}{E_3} \end{bmatrix}$$

direz. locali
di
ortotropia



• Materiale ortotropo (sim. rispetto a tre piani mutuam. ort.)

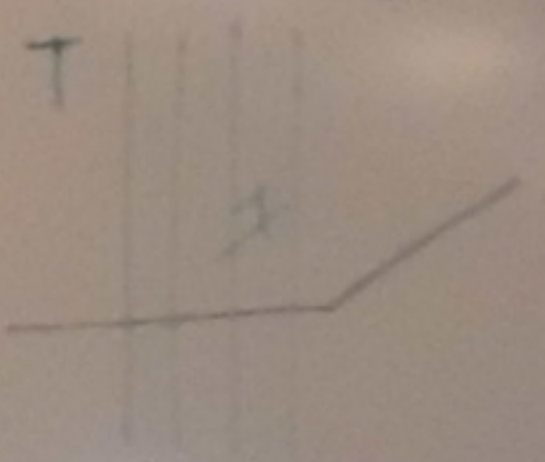
costante di Lamé
 $\mu = G$
 $\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$ (so mat. fibroso: legno, mat. compositi)
 $E > 0$
 $-1 < \nu < 1/2$

$$C = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{13}}{E_1} & 0 & 0 & 0 \\ -\frac{\nu_{21}}{E_2} & \frac{1}{E_2} & -\frac{\nu_{23}}{E_2} & 0 & 0 & 0 \\ -\frac{\nu_{31}}{E_3} & -\frac{\nu_{32}}{E_3} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{12}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{23}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{31}} \end{bmatrix}$$

5 parametri indipendenti

- 3 moduli elast. long. E_1, E_2, E_3
- 3 coeff. di Poisson $\nu_{12}, \nu_{13}, \nu_{23}$
- 3 moduli elast. tang. G_{12}, G_{23}, G_{31}

• Mat. trasversalmente isotropo
(caso particolare)



un solo
array di
fibre
comp. isotropo nel piano
ortogonale all'asse trasverso

5 parametri indip.

- E, E_T
- ν, ν_T
- G_T