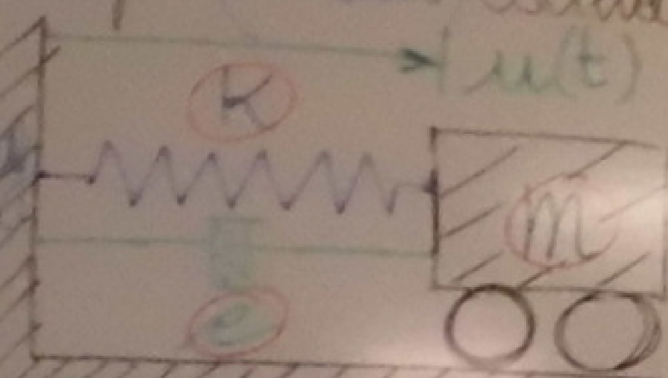
Dynamics – simple (linear) oscillator

Single **D**egree **O**f **F**reedom (dynamic) systems



elastic spring

dash-pot

time-invariant

$F(t)$ → $[F]$ external action

$u(t)$ displacement

$\dot{u}(t)$ velocity

$\ddot{u}(t)$ acceleration

m mass $[M]$

• Hooke's spring

k

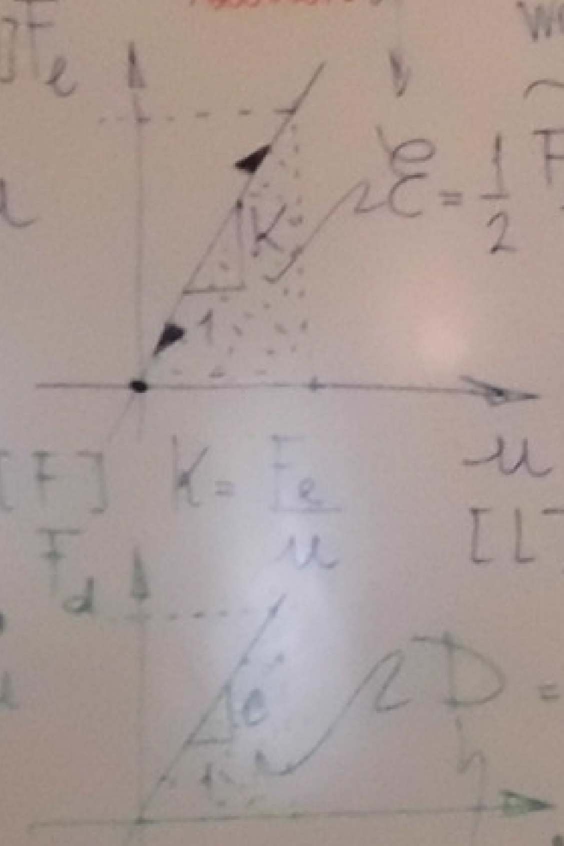
$F_e = K u$

$[K] = \frac{[F]}{[L]}$

u F_e

elastic energy (quadratic function)

work



$\mathcal{E} = \frac{1}{2} \underbrace{F_e u}_{\text{work}} < \begin{matrix} \frac{1}{2} K u^2 \\ \frac{1}{2}\frac{1}{K} F_e^2 \end{matrix}$

$F_e = \frac{d\mathcal{E}}{du} = Ku$

$[F]$ $K = \frac{F_e}{u}$ u $[L]$

They represent physical properties of a structural system

- stiffness (example)
- damping (measured)

e.g.

• Viscous damper

$F_d = c\dot{u}$

c damping coefficient

$[c] = \frac{[F]}{[L]}[t]$ $\dot{u} = \frac{1}{c} F_d$

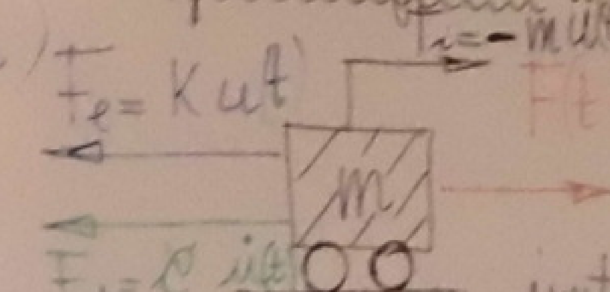
$\mathcal{D} = \frac{1}{2} \underbrace{F_d \dot{u}}_{\text{power}} < \begin{matrix} \frac{1}{2} c \dot{u}^2 \\ \frac{1}{2}\frac{1}{c} F_d^2 \end{matrix}$

$c = \frac{F_d}{\dot{u}}$

$\dot{u}$ $[L]/[t]$

Rayleigh dissipation function $F_d = \frac{d\mathcal{D}}{d\dot{u}} = c\dot{u}$

• Equation of motion (d'Alembert principle)



$F = ma$

$\underbrace{F - ma}_{} = 0$

inertia force F_I

dynamic equilibrium

$Ku(t) + c\dot{u}(t) = F(t) - m\ddot{u}$

$$\boxed{m\ddot{u}(t) + c\dot{u}(t) + Ku(t) = F(t)}$$

2^{nd}-order linear differential equation in $u(t)$

$t \geq t_0 = 0$

Initial conditions

$$\begin{cases} u(t = t_0 = 0) = u_0 \\ \dot{u}(t = t_0 = 0) = \dot{u}_0 \end{cases}$$

• By Lagrange's equations

Lagrangian function $\mathcal{L}(q_k, \dot{q}_k) = T_k - V_k$

Lagrangian coordinates

potential energy

$q_1 = u$

• $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) - \frac{\partial \mathcal{L}}{\partial q_k} = Q_k^*$

SDOF $T = \frac{1}{2} m \dot{u}^2$; $V_e = \frac{1}{2} K u^2$

$\frac{d}{dt}\left(\frac{1}{2} m 2\dot{u}\right) - \frac{1}{2} K 2u$

$= \frac{(F - F_d)\delta u}{\delta u}$

$m\ddot{u} + Ku = F(t) - F_d$

Dynamics
Single Degree Of Freedom
simple linear oscillator
F(t)
elastic energy quadratic function
external action
time-invariant
work
They represent physical properties of a structural system
Equation of motion (d'Alembert principle)
$F_e = k\,u(t)$
$F_i = -m\ddot{u}(t)$
F(t)
$F = ma$
$F - ma = 0$
inertia force F_i
dynamic equilibrium
$k\,u(t) + c\,\dot{u}(t) = F(t) - m\ddot{u}$
$m\,\ddot{u}(t) + c\,\dot{u}(t) + k\,u(t) = F(t)$
2nd-order linear differential equation in u(t)
$t \geq t_0 = 0$
Initial conditions
$u(t=t_0=0) = u_0$
$\dot{u}(t=t_0=0) = \dot{u}_0$
By Lagrange's equation(s)
Lagrangian function
Lagrangian coordinates
$q_1 = u$
potential energy
$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) - \frac{\partial \mathcal{L}}{\partial q_k} = Q_k^*$
SDOF
$\frac{d}{dt}\left(\frac{1}{2} m\, 2\dot{u}\right) + \frac{1}{2} k\, 2u$
T = Kinetic energy
$P = F \cdot v$
power


Dynamics
Single Degree Of Freedom
external action
work
Equation of motion (d'Alembert principle)
By Lagrange's equation(s)
F = m a
F - m a = 0
inertia force F_i
dynamic equilibrium
Lagrangian function L(q_k, q̇_k) = T(q̇_k) - V(q_k) = T + U
lagrangian coordinates
potential energy
potential
q_1 = u
V_f = -L_f
U = L_f
k u(t) + c u̇(t) = F(t) - m ü
m ü(t) + c u̇(t) + k u(t) = F(t)
2nd-order linear differential equation in u(t)
t ≥ t_0 = 0
Initial conditions
u(t=t_0=0) = u_0
u̇(t=t_0=0) = u̇_0
d/dt (∂L/∂q̇_k) - ∂L/∂q_k = Q*_k = δL*(δq_k)/δq_k
SDOF (u) T = 1/2 m u̇² ; V_e(u) = 1/2 K u²
d/dt (1/2 m 2 u̇) + 1/2 K 2 u = (F - F_d) δu/δu
m ü + K u = F(t) - F_d
c u̇
Q*_fk = -∂V_f/∂q_k = ∂U_f/∂q_k
Q*_dk = -∂D/∂q̇_k
-F_d
T = Kinetic energy
velocity
P = F · V
a = dv(t)/dt
power = m V̇ · V
= m 1/2 (V̇ · V + V · V̇)
time invariant
d/dt (V · V)
V²
= d/dt (1/2 m V²) = dT/dt
consrv. forces
Kinetic energy theorem
dissipation viscous forces


Examples of structural systems
point mass
m
u(t)
EJ
bending stiffness
F_e
$K u$
$u = \frac{F}{K}$
$= \frac{1}{K} F$
$K = 3 \frac{EJ_c}{l_c^3}$
EJ
$K = 48 \frac{EJ_b}{l_b^3}$
Frames
$K = 24 \frac{EJ_c}{l_c^3}$
$6 \frac{EJ_c}{l_c^2}$
$12 \frac{EJ_c}{l_c^3}$
$\frac{EA_c}{l_c} \to \infty$
$\frac{EJ_b}{l_b} \to \infty$
Shear-type frame
For the evaluation of K
F_e influence coefficients
$f_e = \frac{1}{3} \frac{l}{EJ}$
$= \frac{1}{\frac{3EJ}{l^3}} \to \frac{1}{K}$
Force method
Displacement method
$K_{11} = \frac{3EJ}{l^3}$
PLV → PVW
LE → EL


Examples of structural systems
point mass
Frames
m
EJ
u(t)
K = 48 EJ
rotations of the nodes (static dofs)
K = 24 EJ
K = 6 EJ
Shear-type frame
For the evaluation of K
influence coefficients
Force method
Displacement method
PLV => PVW
LE => EL
2 → see Mathematica file


Frames
m
EJ
u(t)
l
K = 48 EJ_b / l_b^3
K = 24 EJ_c / l_c^3
EJ_b
EJ_c
K
EA_c / l_c → ∞
EJ_b / l_b → ∞
Shear-type frame
rotions of the nodes (static dofs)
EJ_b / l_b → 0
K = 6 EJ_c / l_c^3
ρ = Σ (EJ_b / l_b) / Σ (EJ_c / l_c)
K = 24 EJ_c / l_c^3 · (12ρ + 1)/(12ρ + 4)
= 6 EJ_c / l_c^3 · (12ρ + 1)/(3ρ + 1)
K / (6 EJ_c / l_c^3)
4
1
ρ
see Mathematica file
beam to column stiffness ratio
Displacement method