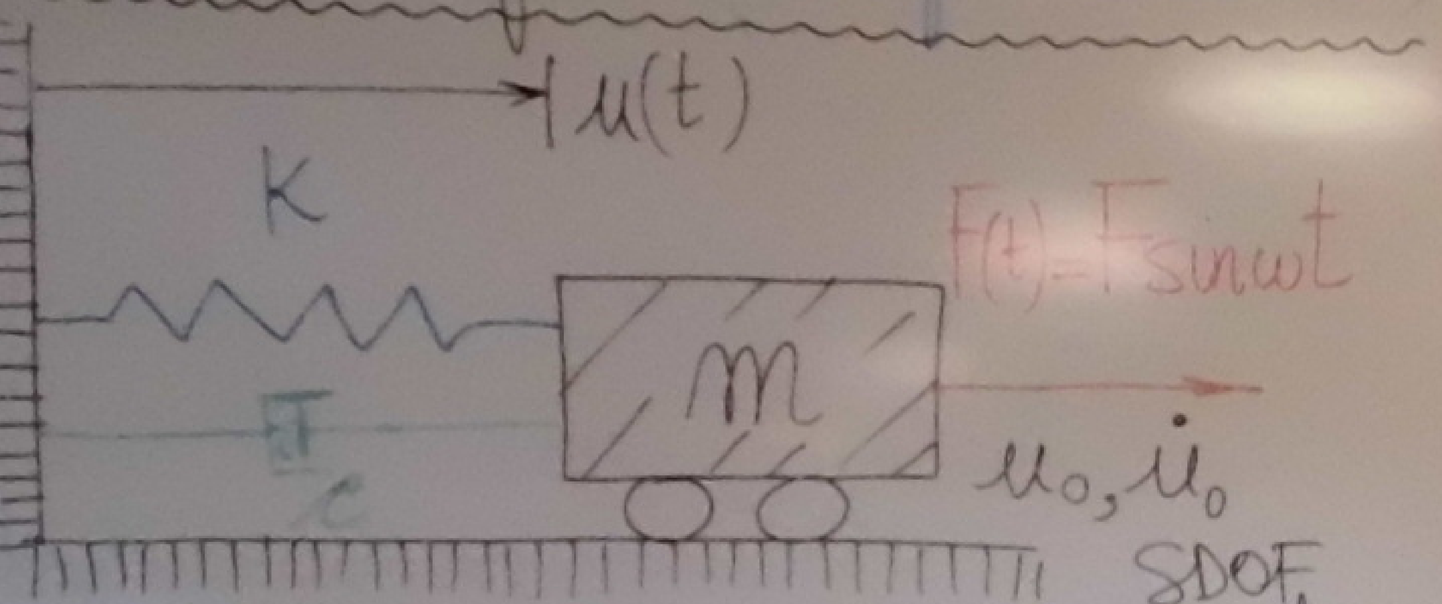
Harmonic force (damped vibration)



SDOF system

$$\ddot{u} + \frac{c}{m}\dot{u} + \frac{Ku}{m} = \frac{K}{K}\frac{F}{m}\sin\omega t \quad \text{eq. of motion}$$

$$\zeta = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{Km}}$$

$$\ddot{u} + 2\zeta\omega_1\dot{u} + \omega_1^2 u = \omega_1^2 u_{st}\sin\omega t \qquad \beta = \frac{\omega}{\omega_1} \text{ frequency ratio}$$

Seek particular integral in the form:

$$u_p(t) = N u_{st}\sin(\omega t - \xi) \qquad N: \text{dynamic ampl. factor}$$

$$\frac{d}{dt}: \quad \dot{u}_p(t) = \omega N u_{st}\cos(\omega t - \xi) \qquad \xi: \text{phase shift}$$

$$\ddot{u}_p(t) = -\omega^2 N u_{st}\sin(\omega t - \xi) = -\omega^2 u_p(t)$$

By substituting

$$\forall t \quad \left(\frac{\omega_1^2}{\omega_1^2} - \frac{\omega^2}{\omega_1^2}\right) N u_{st}\underbrace{\sin(\omega t - \xi)}_{\sin\omega t\cos\xi - \cos\omega t\sin\xi} + 2\zeta\frac{\omega_1}{\omega_1}\underbrace{\frac{\omega}{\omega_1}}_{\beta} N u_{st}\underbrace{\cos(\omega t - \xi)}_{\cos\omega t\cos\xi + \sin\omega t\sin\xi} = \frac{\omega_1^2}{\omega_1^2}u_{st}\sin\omega t$$

$$\begin{cases} (1-\beta^2)\cos\xi + 2\zeta\beta\sin\xi = \dfrac{1}{N} & (*) \\ -(1-\beta^2)\sin\xi + 2\zeta\beta\cos\xi = 0 \Rightarrow \tan\xi = \dfrac{\sin\xi}{\cos\xi} = \dfrac{2\zeta\beta}{1-\beta^2} \end{cases}$$

$$\boxed{\underset{(\beta,\zeta)}{\xi} = \operatorname{atan}\frac{2\zeta\beta}{1-\beta^2}}$$

From $(*)$:

$$(1-\beta^2)\frac{1-\beta^2}{\sqrt{D}} + 2\zeta\beta\frac{2\zeta\beta}{\sqrt{D}} = \frac{1}{N} = \frac{D}{\sqrt{D}} = \sqrt{D}$$

$$\underset{(\beta,\zeta)}{N} = \frac{1}{\sqrt{D}} = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\zeta\beta)^2}}$$

- $\zeta = 0 \to N = \dfrac{1}{|1-\beta^2|}$
- $\beta = 1 \to \boxed{N = \dfrac{1}{2\zeta}}$

$$\cos\xi = \frac{1}{\sqrt{1+\tan^2\xi}} = \frac{1}{\sqrt{1+\dfrac{(2\zeta\beta)^2}{(1-\beta^2)^2}}} = \frac{1-\beta^2}{\underbrace{\sqrt{(1-\beta^2)^2+(2\zeta\beta)^2}}_{D(\beta,\zeta)}} = \boxed{\frac{1-\beta^2}{\sqrt{D}} = \cos\xi = (1-\beta^2)N}$$

$$\sin\xi = \tan\xi\cos\xi = \frac{2\zeta\beta}{1-\beta^2}\,\frac{1-\beta^2}{\sqrt{D}} = \boxed{\frac{2\zeta\beta}{\sqrt{D}} = \sin\xi} = 2\zeta\beta N \qquad \xi = \frac{\pi}{2},\ \beta = 1\ (\zeta \neq 0)$$

Altern.
- $2\zeta\beta = \dfrac{1}{N}\sin\xi \Rightarrow \sin\xi = 2\zeta\beta N$
- $1-\beta^2 = \dfrac{1}{N}\cos\xi \Rightarrow \cos\xi = (1-\beta^2)N$

ζ	0	0.01	0.05
N	∞	50	10

$$N = N(\beta,\zeta) = \frac{1}{\sqrt{D}}$$

Stationary points (peaks)

$$\frac{dN}{d\beta} = N'(\beta) = \frac{-\frac{1}{2\sqrt{D}}}{D} \qquad \boxed{D' = 0}$$

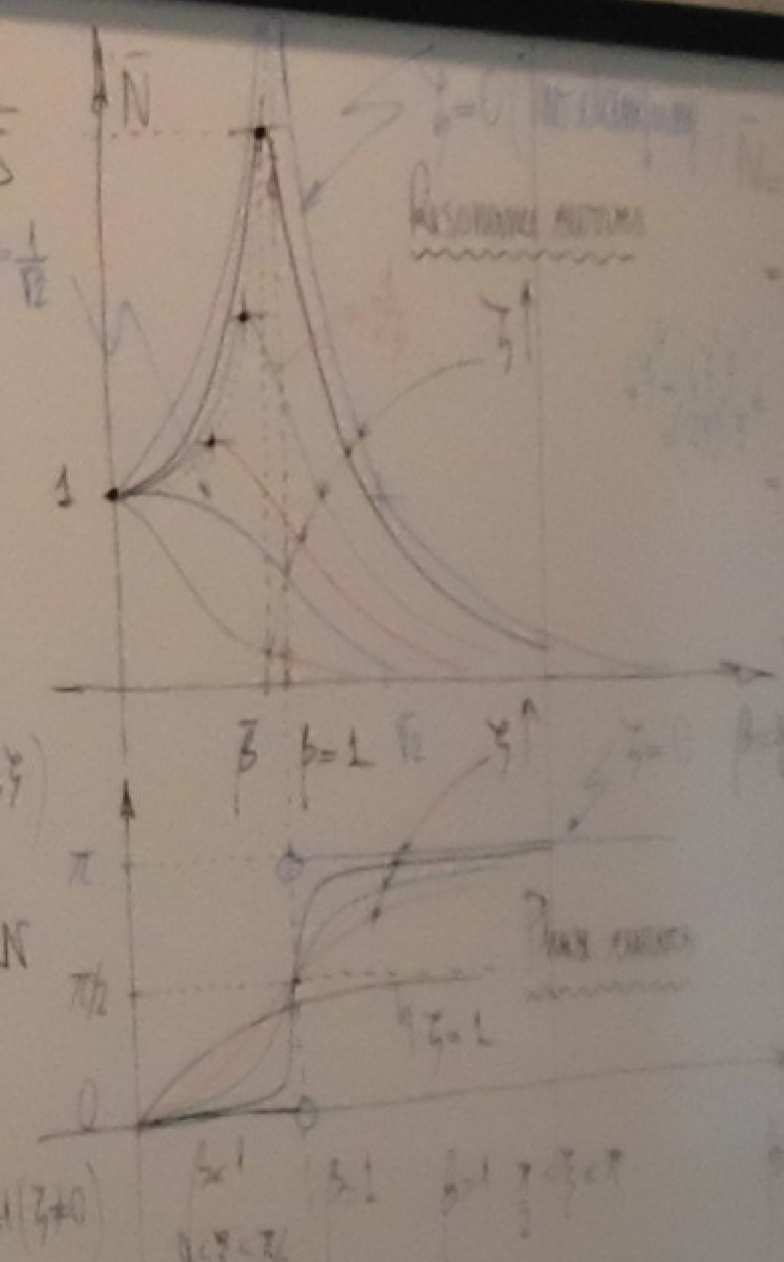
stationary cond. $D' = 2(1-\beta^2)(-2\beta) + 2(2\zeta\beta)2\zeta = 0$

$$\beta(-1+\beta^2+2\zeta^2) = 0 < \begin{matrix}\beta = 0 \\ \bar{\beta}^2 = 1-2\zeta^2\end{matrix}$$

$$\bar{\beta} = \sqrt{1-2\zeta^2} \quad (\zeta \ll 1,\ \bar{\beta} \simeq 1) \qquad \xi = \xi(\beta,\zeta)$$

$$1-2\zeta^2 \geq 0;\ \zeta^2 < \frac{1}{2};\ \zeta < \frac{1}{\sqrt{2}}$$

$$\zeta = \frac{1}{\sqrt{2}}$$



Harmonic force damped vibration
SDOF system
eq. of motion
By substituting
stationary cond.
From (*)
Stationary points (peaks)
Resonance curves
Phase curves
Envelope of maxima

By substituting

$$(1-\beta^2)\cos\xi + 2\zeta\beta\sin\xi = \frac{1}{N} \quad (*)$$

From (*)

$$\xi$$	0	0.01	0.05
N	∞	50	10

$$\cos\xi = \frac{1}{\sqrt{1+\tan^2\xi}} = \frac{1}{\sqrt{1+\frac{(2\zeta\beta)^2}{(1-\beta^2)^2}}} = \frac{1-\beta^2}{\underbrace{\sqrt{(1-\beta^2)^2+(2\zeta\beta)^2}}_{D(\beta,\zeta)}} = \boxed{\frac{1-\beta^2}{\sqrt{D}} = \cos\xi} = (1-\beta^2)N$$

$$= 2\zeta\beta N \qquad \xi = \frac{\pi}{2},\ \beta = 1\ (\zeta \neq 0)$$

$$\forall t$$

$$N = N(\beta,\zeta) = \frac{1}{\sqrt{D}} \qquad \zeta = \frac{1}{\sqrt{2}}$$

Stationary points (peaks)

$$\frac{dN}{d\beta} = N'(\beta) = \frac{-\frac{1}{2\sqrt{D}}D'}{D} \qquad \boxed{D' = 0}$$

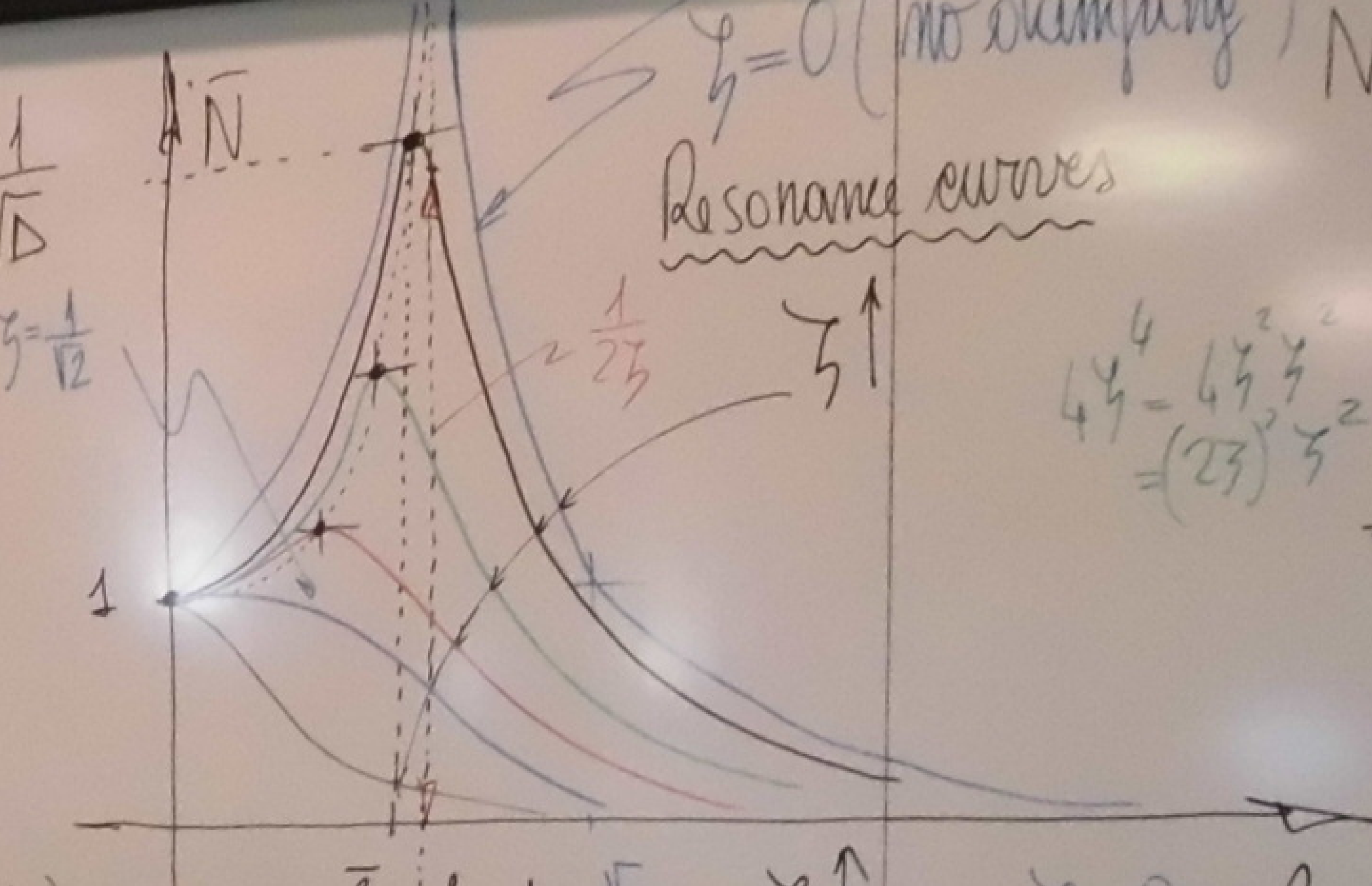
stationary cond.

$$D' = 2(1-\beta^2)(-2\beta) + 2(2\zeta\beta)2\zeta = 0$$

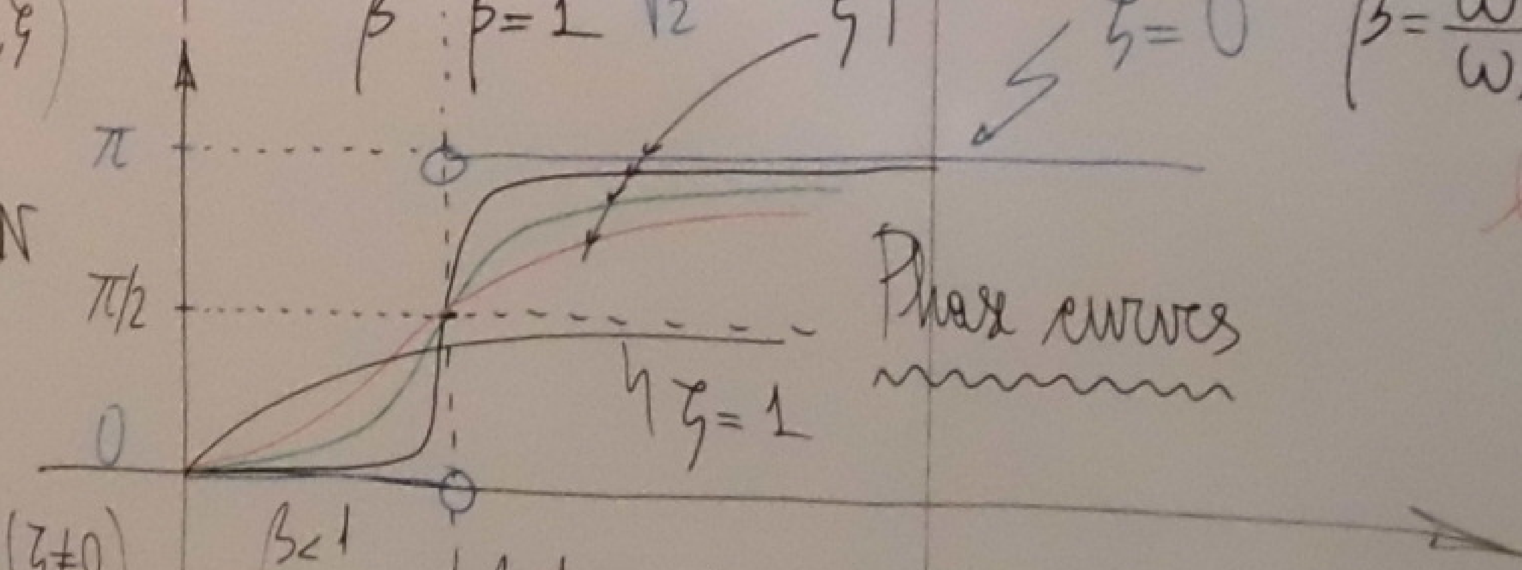
$$\beta(-1+\beta^2+2\zeta^2) = 0 < \begin{matrix}\beta = 0 \\ \bar\beta^2 = 1-2\zeta^2\end{matrix}$$

$$\bar\beta = \sqrt{1-2\zeta^2} \quad (\zeta \ll 1,\ \bar\beta \simeq 1)$$

$$1-2\zeta^2 \geq 0;\ \zeta^2 \leq \frac{1}{2};\ \zeta < \frac{1}{\sqrt{2}} \qquad \xi = \xi(\beta,\zeta)$$



$$\beta = \frac{\omega}{\omega_1}$$



$$\beta = \frac{\omega}{\omega_1}$$

$$\bar N = N(\bar\beta) = \frac{1}{\sqrt{(1-(1-2\zeta^2))^2+(2\zeta)^2(1-2\zeta^2)}}$$

$$(1-1+2\zeta^2)^2+(2\zeta)^2(1-2\zeta^2)$$

$$4\zeta^4 - 4\zeta^2\zeta^2 = (2\zeta)^2\zeta^2$$

$$= \frac{1}{2\zeta}\frac{1}{\sqrt{\zeta^2+(1-2\zeta^2)}}$$

$$\boxed{= \frac{1}{2\zeta}\frac{1}{\sqrt{1-\zeta^2}} = \bar N}$$

$$\zeta \ll 1, \quad \boxed{\bar N \simeq \frac{1}{2\zeta}} = \frac{1/\zeta}{2}$$

light damping

Envelope of maxima:

$$\bar N(\bar\beta) = \frac{1}{\sqrt{1-\bar\beta^4}}$$

$$2\zeta^2 = 1-\bar\beta^2$$

- $u_p(t) = N u_{st} \sin(\omega t - \xi)$

$$= N u_{st} (\sin\omega t \cos\xi - \cos\omega t \sin\xi)$$

$$= \underbrace{N u_{st} \cos\xi}_{Z_1} \sin\omega t - \underbrace{N u_{st} \sin\xi}_{Z_2} \cos\omega t$$

$$= Z_1 \sin\omega t - Z_2 \cos\omega t$$

$$\begin{cases} Z_1 = N u_{st} \cos\xi = \frac{1}{\sqrt{D}} u_{st} \frac{1-\beta^2}{\sqrt{D}} = \frac{1-\beta^2}{D} u_{st} \\ Z_2 = N u_{st} \sin\xi = \frac{1}{\sqrt{D}} u_{st} \frac{2\zeta\beta}{\sqrt{D}} = \frac{2\zeta\beta}{D} u_{st} \end{cases}$$

$$\sqrt{Z_1^2 + Z_2^2} = N u_{st}$$

- General integral: $\omega_d = \omega_1 \sqrt{1-\zeta^2}$

$$u(t) = e^{-\zeta\omega_1 t} (A \sin\omega_d t + B \cos\omega_d t) + Z_1 \sin\omega t - Z_2 \cos\omega t$$

- By imposing the i.c.s.:

$$\begin{cases} u_0 = B - Z_2 \Rightarrow \boxed{B = u_0 + Z_2} \\ \dot{u}_0 = -\zeta\omega_1 B + \omega_d A + \omega Z_1 \end{cases}$$

$$\rightarrow A = \frac{1}{\omega_d}\left(\dot{u}_0 + \zeta\omega_1(u_0 + Z_2) - \omega Z_1\right)$$

$$= \frac{\dot{u}_0 + \zeta\omega_1 u_0}{\omega_d} + \frac{\zeta\omega_1 Z_2 - \omega Z_1}{\omega_d} \quad \frac{\omega_1}{\omega_d}(\zeta Z_2 - \beta Z_1) = \boxed{\frac{\dot{u}_0 + \zeta\omega_1 u_0}{\omega_d} + \frac{\zeta Z_2 - \beta Z_1}{\sqrt{1-\zeta^2}} = A}$$

- Final solution:

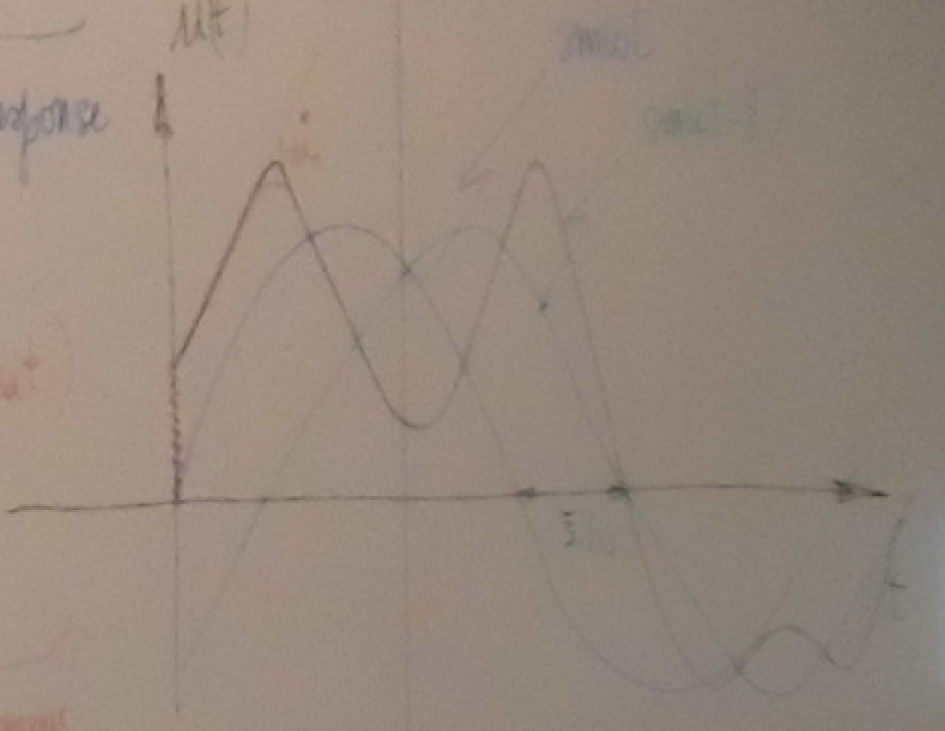
$$u(t) = \underbrace{e^{-\zeta\omega_1 t}\left(\left(\frac{\dot{u}_0 + \zeta\omega_1 u_0}{\omega_d} + \frac{\zeta Z_2 - \beta Z_1}{\sqrt{1-\zeta^2}}\right)\sin\omega_d t + (u_0 + Z_2)\cos\omega_d t\right)}_{\text{"transient response" } (\to 0,\ t\to\infty)} \underbrace{+ Z_1 \sin\omega t - Z_2 \cos\omega t}_{\text{"steady state" response}}$$

$$= \underbrace{e^{-\zeta\omega_1 t}\left(\frac{\dot{u}_0 + \zeta\omega_1 u_0}{\omega_d} \sin\omega_d t + u_0 \cos\omega_d t\right)}_{\text{linked to non-homogeneous i.c.}} + e^{-\zeta\omega_1 t}\left(\frac{\zeta Z_2 - \beta Z_1}{\sqrt{1-\zeta^2}} \sin\omega_d t + Z_2 \cos\omega_d t\right) + Z_1 \sin\omega t - Z_2 \cos\omega t$$

$(0,$ if $u_0 = 0, \dot{u}_0 = 0)$

$e^{-\zeta\omega_1 t} \to 0, t \to \infty$

response to harmonic force for homogeneous initial conditions



- $u_p(t) = N u_{st} \sin(\omega t - \xi)$

  $= N u_{st} (\sin\omega t \cos\xi - \cos\omega t \sin\xi)$

  $= \underbrace{N u_{st} \cos\xi}_{Z_1} \sin\omega t - \underbrace{N u_{st} \sin\xi}_{Z_2} \cos\omega t$

  $= Z_1 \sin\omega t - Z_2 \cos\omega t$

where $\begin{cases} Z_1 = N u_{st} \cos\xi = \frac{1}{\sqrt{D}} u_{st} \frac{1-\beta^2}{\sqrt{D}} = \frac{1-\beta^2}{D} u_{st} \\ Z_2 = N u_{st} \sin\xi = \frac{1}{\sqrt{D}} u_{st} \frac{2\zeta\beta}{\sqrt{D}} = \frac{2\zeta\beta}{D} u_{st} \end{cases}$ $\sqrt{Z_1^2 + Z_2^2} = N u_{st}$

- General integral:

  $u(t) = e^{-\zeta\omega_1 t} (A \sin\omega_d t + B \cos\omega_d t) + Z_1 \sin\omega t - Z_2 \cos\omega t$

- By imposing the i.cs.

  $\begin{cases} u_0 = B - Z_2 \Rightarrow \boxed{B = u_0 + Z_2} \\ \dot{u}_0 = -\zeta\omega_1 B + \omega_d A + \omega Z_1 \end{cases}$

  $\rightarrow A = \frac{1}{\omega_d} (\dot{u}_0 + \zeta\omega_1 (u_0 + Z_2) - \omega Z_1)$

  $= \frac{\dot{u}_0 + \zeta\omega_1 u_0}{\omega_d} + \frac{\zeta\omega_1 Z_2 - \omega Z_1}{\omega_d}$ $\frac{\omega_1}{\omega_d} (\zeta Z_2 - \beta Z_1) = \boxed{\frac{\dot{u}_0 + \zeta\omega_1 u_0}{\omega_d} + \frac{\zeta Z_2 - \beta Z_1}{\sqrt{1-\zeta^2}} = A}$

- Final solution:

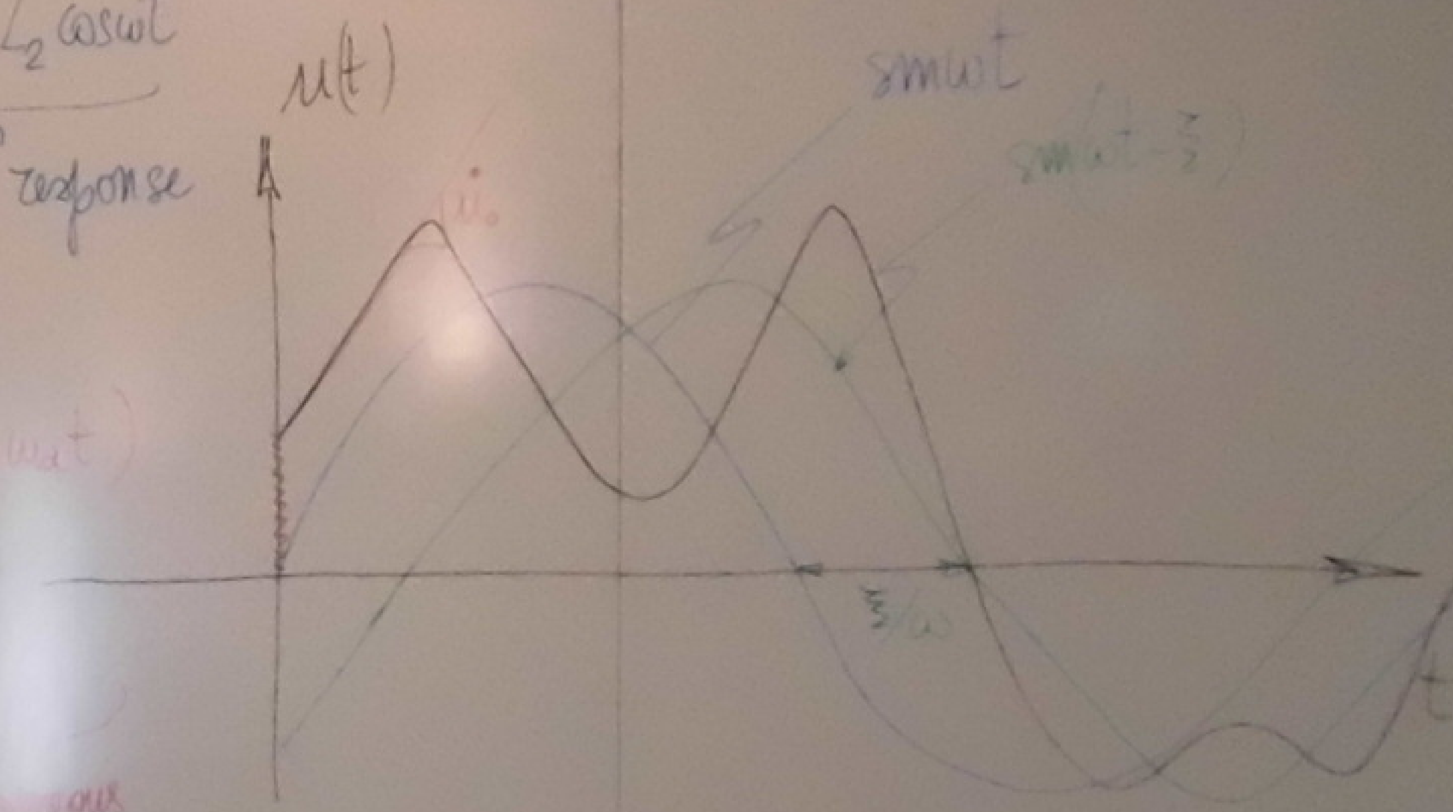
$$u(t) = \underbrace{e^{-\zeta\omega_1 t} \left(\left(\frac{\dot{u}_0 + \zeta\omega_1 u_0}{\omega_d} + \frac{\zeta Z_2 - \beta Z_1}{\sqrt{1-\zeta^2}} \right) \sin\omega_d t + (u_0 + Z_2) \cos\omega_d t \right)}_{\text{"transient response" } (\to 0,\ t \to \infty)} + \underbrace{Z_1 \sin\omega t - Z_2 \cos\omega t}_{\text{"steady-state" response}}$$

$$= \underbrace{e^{-\zeta\omega_1 t} \left(\frac{\dot{u}_0 + \zeta\omega_1 u_0}{\omega_d} \sin\omega_d t + u_0 \cos\omega_d t \right)}_{\text{linked to non-homogeneous i.cs.}}$$

$(0,$ if $u_0 = 0, \dot{u}_0 = 0)$

$$+ \underbrace{e^{-\zeta\omega_1 t} \left(\frac{\zeta Z_2 - \beta Z_1}{\sqrt{1-\zeta^2}} \sin\omega_d t + Z_2 \cos\omega_d t \right) + Z_1 \sin\omega t - Z_2 \cos\omega t}_{\text{response to harmonic force for homogeneous initial conditions}}$$

$e^{-\zeta\omega_1 t} \rightarrow 0, t \rightarrow \infty$



• Final solution

$$u(t) = e^{-\zeta\omega_1 t}\left(\frac{\dot{u}_0+\zeta\omega_1 u_0}{\omega_d}\sin\omega_d t + u_0\cos\omega_d t\right) + e^{-\zeta\omega_1 t}\left(\frac{\zeta Z_2-\beta Z_1}{\sqrt{1-\zeta^2}}\sin\omega_d t + Z_2\cos\omega_d t\right) + \underbrace{Z_1\sin\omega t - Z_2\cos\omega t}_{\text{"steady-state" response}}$$

$e^{-\zeta\omega_1 t} \to 0,\ t\to\infty$

$$= \underbrace{e^{-\zeta\omega_1 t}\left(\frac{\dot{u}_0+\zeta\omega_1 u_0}{\omega_d}\sin\omega_d t + u_0\cos\omega_d t\right)}_{\text{linked to non-homogeneous i.c.}}$$

(0, if $u_0=0$, $\dot{u}_0=0$)

$$+ \underbrace{e^{-\zeta\omega_1 t}\left(\frac{\zeta Z_2-\beta Z_1}{\sqrt{1-\zeta^2}}\sin\omega_d t + Z_2\cos\omega_d t\right) + Z_1\sin\omega t - Z_2\cos\omega t}_{\text{response to harmonic force for homogeneous initial conditions}}$$

$$\boxed{\frac{\dot{u}_0+\zeta\omega_1 u_0}{\omega_d} + \frac{\zeta Z_2-\beta Z_1}{\sqrt{1-\zeta^2}} = A}$$