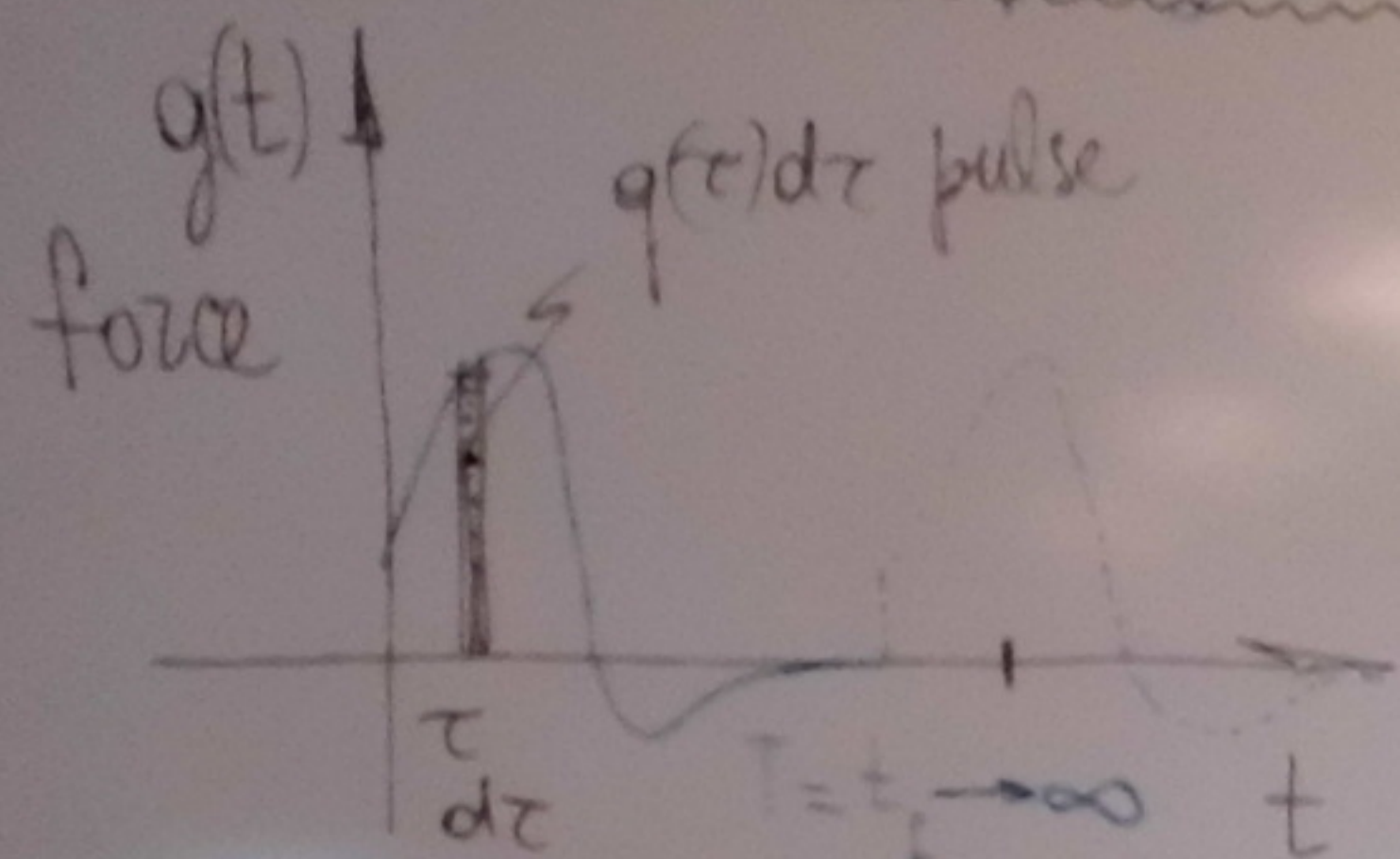


Response in the Frequency Domain



Periodic force
(T)
 $\omega T = 2\pi$

Property of Fourier transform:

$$F\left(\frac{d^n q(t)}{dt^n}\right) = (i\omega)^n F(q(t))$$

based $\frac{d^n e^{i\omega t}}{dt^n} = (i\omega)^n e^{i\omega t}$

int. int $e \rightarrow i\omega e \rightarrow (i\omega)^2 e \rightarrow \dots$

Thus: $\frac{d^n q(t)}{dt^n} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G(\omega) (i\omega)^n e^{i\omega t} d\omega$

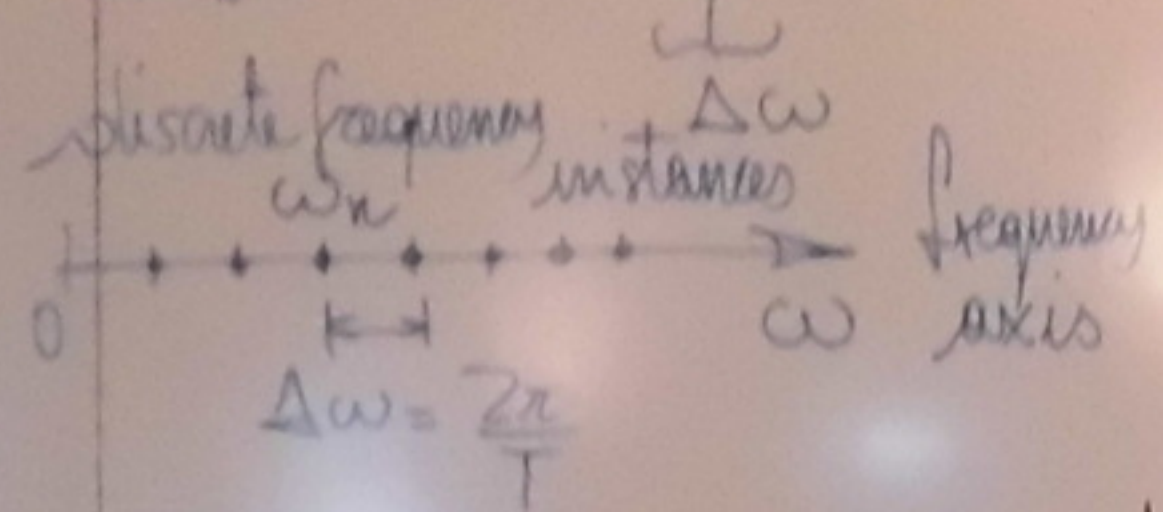
Proof $F\left(\frac{d^n q}{dt^n}\right) = (i\omega)^n F(q)$

$d\omega = F^{-1}((i\omega)^n G(\omega))$

Fourier series:

$$q(t) = \sum_{-\infty}^{+\infty} C_n e^{i\omega_n t}$$

$$\omega_n = n\omega = n \frac{2\pi}{T} = n \Delta\omega$$



$$C_n = \frac{1}{2} (A_n - iB_n)$$

$$= \sum_{-\infty}^{+\infty} \frac{\Delta\omega}{2\pi} \int_{-T/2}^{T/2} q(t) e^{-i\omega_n t} dt e^{i\omega_n t}$$

$T \rightarrow \infty$ (general non periodic function)

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} q(t) e^{-i\omega t} dt e^{i\omega t} d\omega$$

Fourier transform: $G(\omega) = F(q(t))$

$$q(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G(\omega) e^{i\omega t} d\omega$$

Inverse Fourier transform

$$q(t) = F^{-1}(G(\omega))$$

Eq of motion:

$$F(m\ddot{u} + c\dot{u} + ku = q(t))$$

$$[m(\omega^2 + c\omega + k)]U(\omega) = G(\omega)$$

$$H(\omega) = \frac{1}{[k - m\omega^2 + i c\omega]} U(\omega) = G(\omega)$$

Response in the frequency domain $U(\omega) = H(\omega) \cdot G(\omega)$

SC Dirichlet Condition $\int_{-\infty}^{+\infty} |q(t)| dt < \infty$
is actually the Fourier transform of unit pulse response $\delta(t)$

Check: $F^{-1}(F(\delta(t))) = \delta(t)$

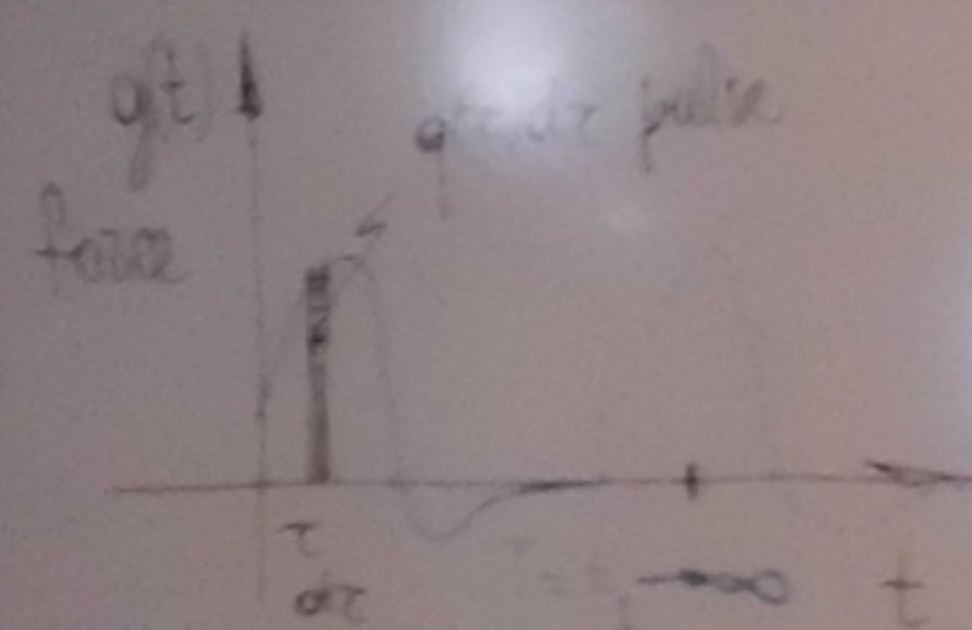
In fact, recall that response to harmonic force

$$q(t) = \sin(\omega t)$$

TIME DOMAIN $q(t)$

FREQUENCY DOMAIN $Q(\omega)$

Resonance in the Frequency Domain



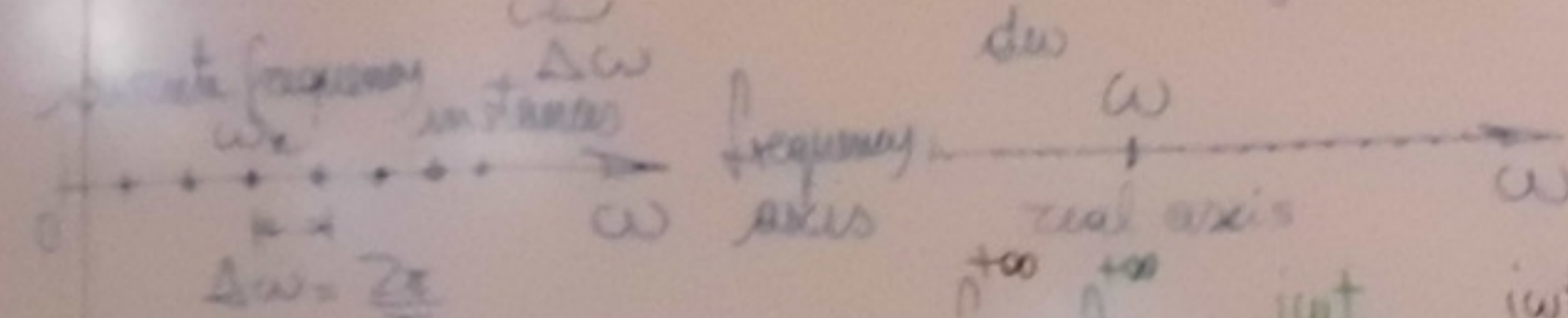
Property of Fourier transform

$$F\left(\frac{d}{dt}q(t)\right) = (i\omega)F(q(t))$$

$$\lim_{t \rightarrow \infty} \frac{d}{dt}q(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) i\omega e^{i\omega t} d\omega = F'(G(\omega))$$

Periodic force

$$q(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 t} = \sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} \int_{-\infty}^{\infty} q(t) e^{-i\omega t} dt e^{in\omega_0 t}$$



Fourier transform

$$G(\omega) = F(q(t))$$

$$q(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{i\omega t} d\omega$$

Inverse Fourier transform

$$q(t) = F^{-1}(G(\omega))$$

Eq of motion:

$$F(m\ddot{u} + c\dot{u} + ku = q(t))$$

$$[m(i\omega)^2 + c(i\omega) + K]U(\omega) = G(\omega)$$

$$H(\omega) = \frac{1}{[K - m\omega^2 + i c\omega]} U(\omega) = G(\omega)$$

Response in the frequency domain

$$U(\omega) = H(\omega) \cdot G(\omega)$$

Check: $F'(2\pi\delta(\omega - \omega_0)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0) e^{i\omega t} d\omega = e^{i\omega_0 t}$

In fact, recall that response to harmonic force

$$q(t) = \frac{1}{2} e^{i\omega_0 t}$$

$$Z_{ss}(t) = \frac{1}{K} \frac{1}{1 - \frac{c^2}{4K}} e^{i\omega_0 t}$$

and represent the compliance to harmonic loading

Contribution to integral

$$\int_{-\infty}^{\infty} q(t) h(t-\tau) d\tau = \frac{1}{2} \int_{-\infty}^{\infty} e^{i\omega_0 t} h(t-\tau) d\tau$$

unit pulse response

$$h(t) = \frac{1}{K} \frac{1}{1 - \frac{c^2}{4K}} e^{i\omega_0 t}$$

INPUT (force)

$$q(t)$$

TIME DOMAIN

$$G(\omega)$$

FREQUENCY DOMAIN

$$1$$

OUTPUT (response)

$$u(t)$$

TIME DOMAIN

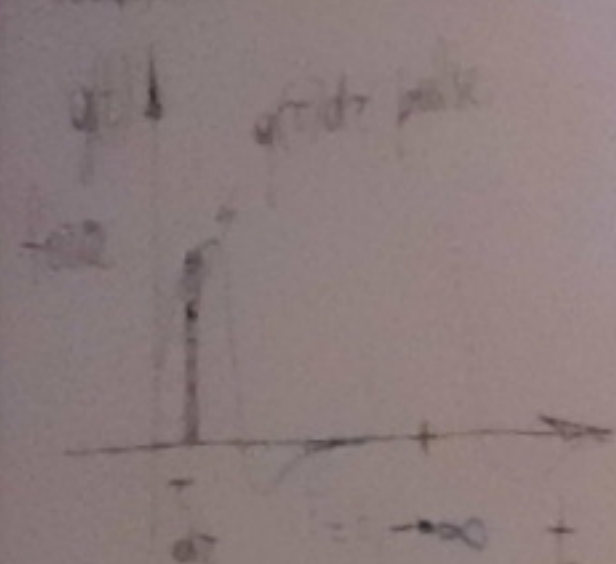
$$H(\omega)$$

FREQUENCY DOMAIN

$$H(\omega)$$

$$H \cdot G$$

Resonance in the Frequency Domain



Property of Fourier Transform

$$F\left\{\frac{d}{dt}f(t)\right\} = i\omega F\{f(t)\}$$

$$F\{t f(t)\} = i \frac{d}{d\omega} F(\omega)$$

Fourier series

$$q(t) = \sum_{n=-\infty}^{+\infty} C_n e^{i n \omega_0 t}$$

$$C_n = \frac{1}{T} \int_{-T/2}^{T/2} q(t) e^{-i n \omega_0 t} dt$$

non periodic function

$$C_n \rightarrow \Delta\omega \rightarrow \frac{\Delta\omega}{2\pi} \rightarrow \frac{1}{2\pi}$$

$$q(t) = \int_{-\infty}^{+\infty} G(\omega) e^{i\omega t} d\omega$$

$$G(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} q(t) e^{-i\omega t} dt$$

Fourier transform

$$G(\omega) = F\{q(t)\}$$

$$q(t) = F^{-1}\{G(\omega)\}$$

Inverse Fourier transform

Eq of motion

$$F(m\ddot{u} + c\dot{u} + ku = q(t))$$

$$[m(i\omega)^2 + c i\omega + k] U(\omega) = G(\omega)$$

$$H(\omega) = \frac{1}{k - m\omega^2 + i c \omega}$$

$$U(\omega) = H(\omega) \cdot G(\omega)$$

Response in the frequency domain

$$H(\omega) = \frac{1}{k} \frac{1}{1 - (\frac{\omega}{\omega_1})^2 + i 2\zeta \frac{\omega}{\omega_1}}$$

compliance in the frequency domain

$$H(\omega) = \frac{1}{k} \frac{1}{1 - (\frac{\omega}{\omega_1})^2 + i 2\zeta \frac{\omega}{\omega_1}}$$

$$F(2\pi \delta(\omega - \omega_0)) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} 2\pi \delta(\omega - \omega_0) e^{i\omega t} d\omega = e^{i\omega_0 t}$$

In fact, recall that response to harmonic force

$$q(t) = e^{i\omega_0 t}$$

$$u(t) = \frac{1}{k} \frac{1}{1 - (\frac{\omega_0}{\omega_1})^2 + i 2\zeta \frac{\omega_0}{\omega_1}} e^{i\omega_0 t}$$

and represents the compliance to harmonic loading

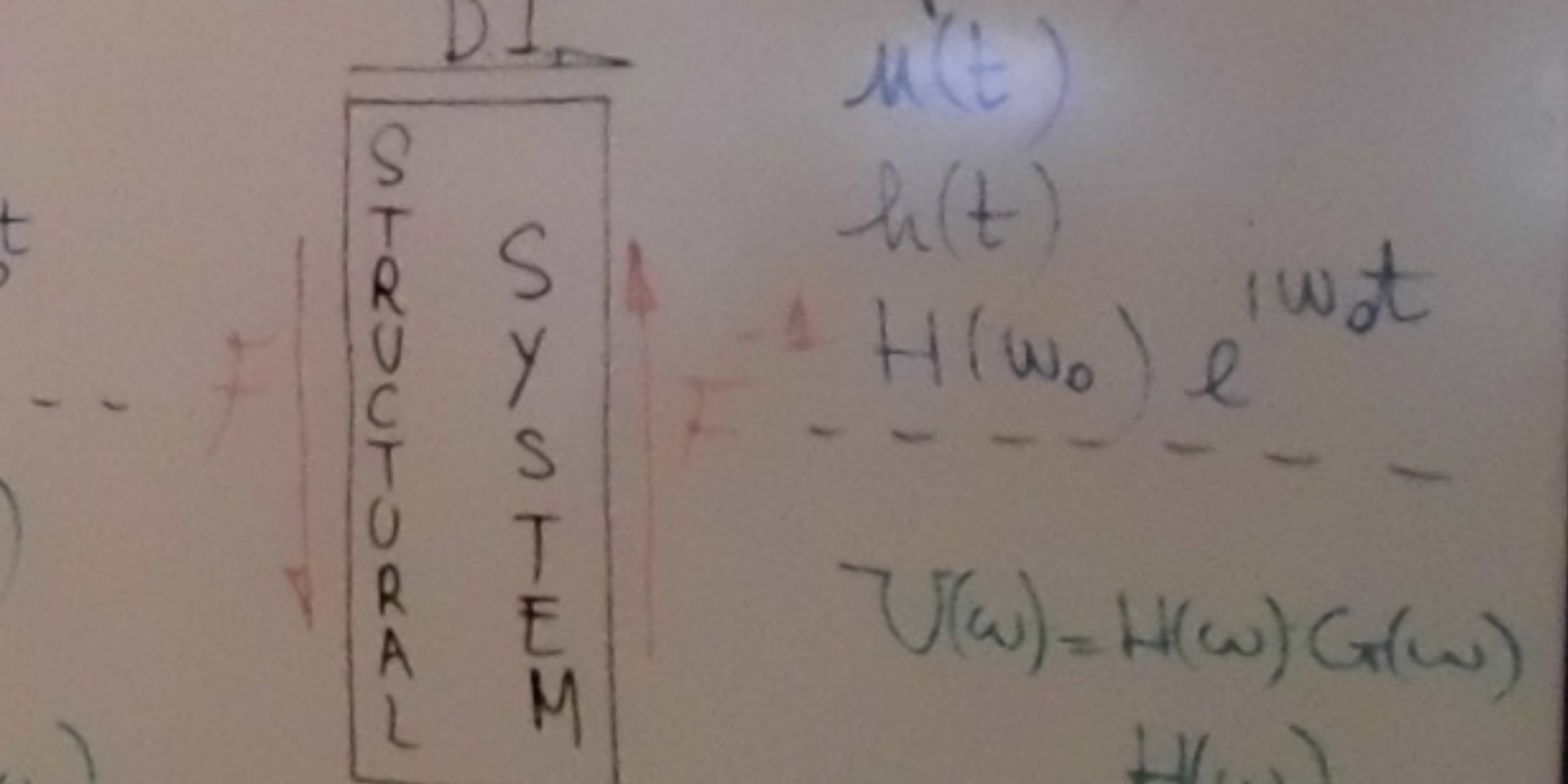
TIME DOMAIN	INPUT (force)
	$q(t)$
	impulse $\delta(t)$
	harmonic $e^{i\omega_0 t}$
FREQUENCY DOMAIN	OUTPUT (response)
	$G(\omega)$
	1
	$2\pi \delta(\omega - \omega_0)$

Convolution integral (Duhamel Integral)

$$\int_{-\infty}^{+\infty} g(\tau) h(t-\tau) d\tau = u(t)$$

where $h(t) = \frac{1}{m\omega_1} e^{-\zeta\omega_1 t} \sin\omega_1 t$

unit pulse response ($\delta(t)$)



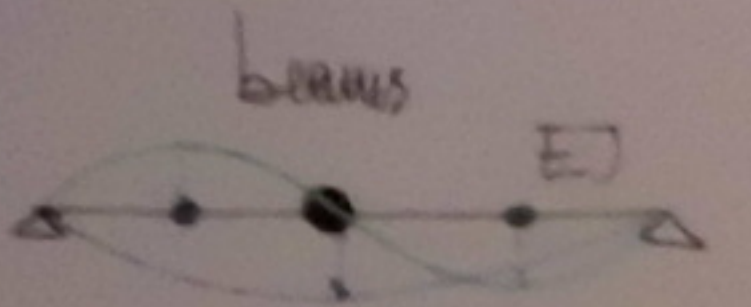
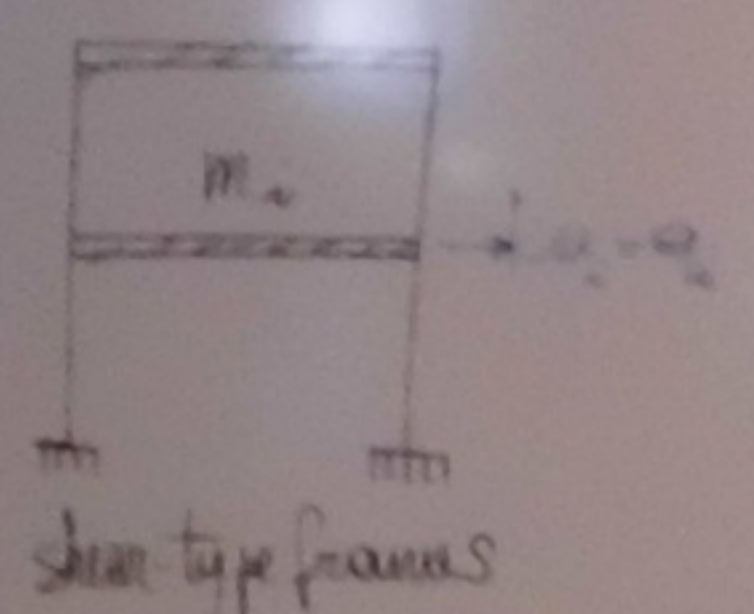
$$U(\omega) = H(\omega) G(\omega)$$

$$H(\omega) = \frac{1}{k - m\omega^2 + i c \omega}$$

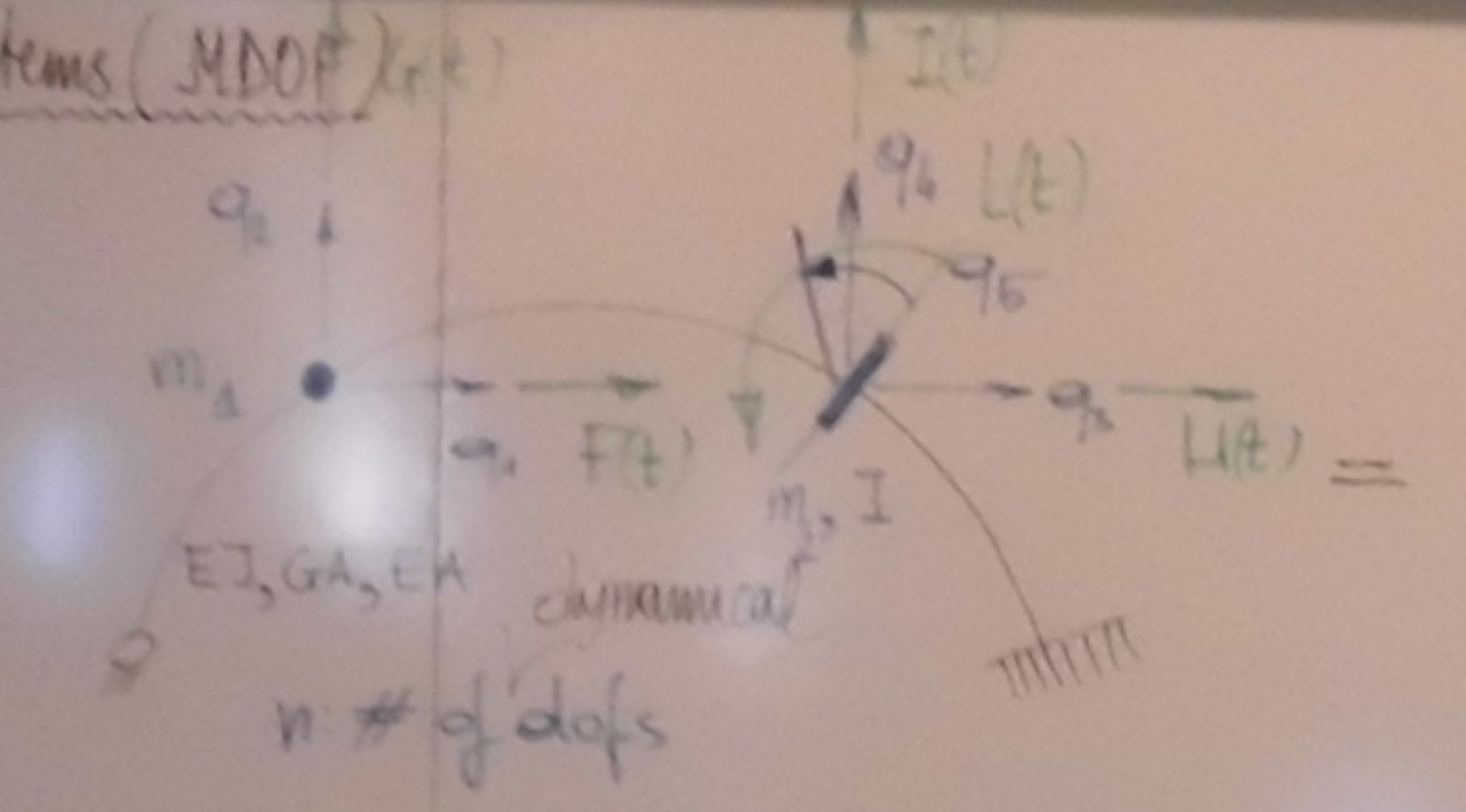
$$H(\omega) 2\pi \delta(\omega - \omega_0)$$

it could be proved that $H(\omega) = F(h(t))$ otherwise $\rightarrow$ gives $H(\omega) \sin(\Delta)$

Multiple degree of freedom systems (MDOF) (r.k.)

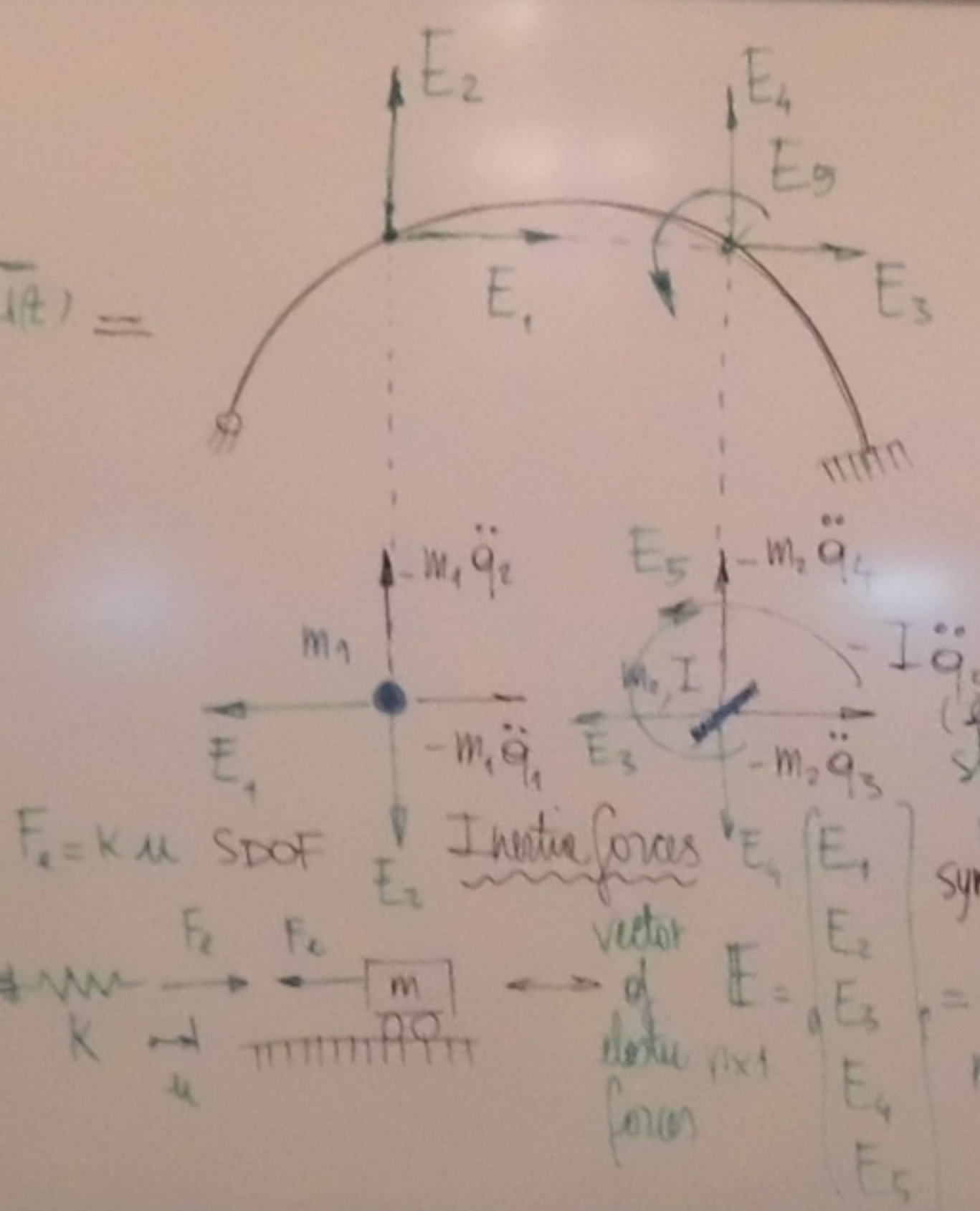


Continuous system $\rightarrow$ discrete system
 distributed (infinite) $\rightarrow$ lumped (finite)
 ∞ d.o.f $\rightarrow$ finite d.o.f



Vector of displacements of freedom
 $q = [q_1, q_2, q_3, q_4, q_5]^T$
 $n \times 1$
 (Lagrange variables)

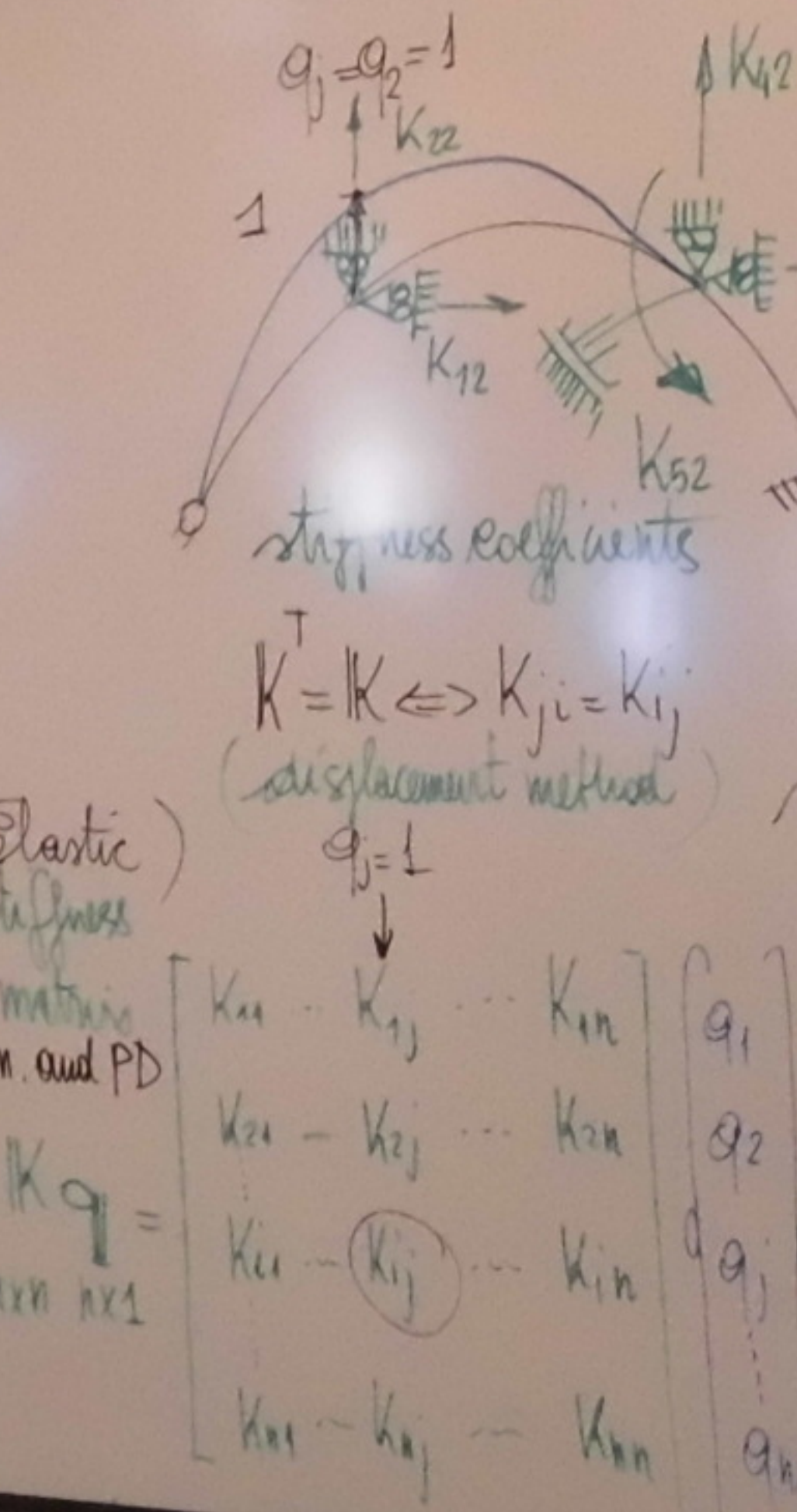
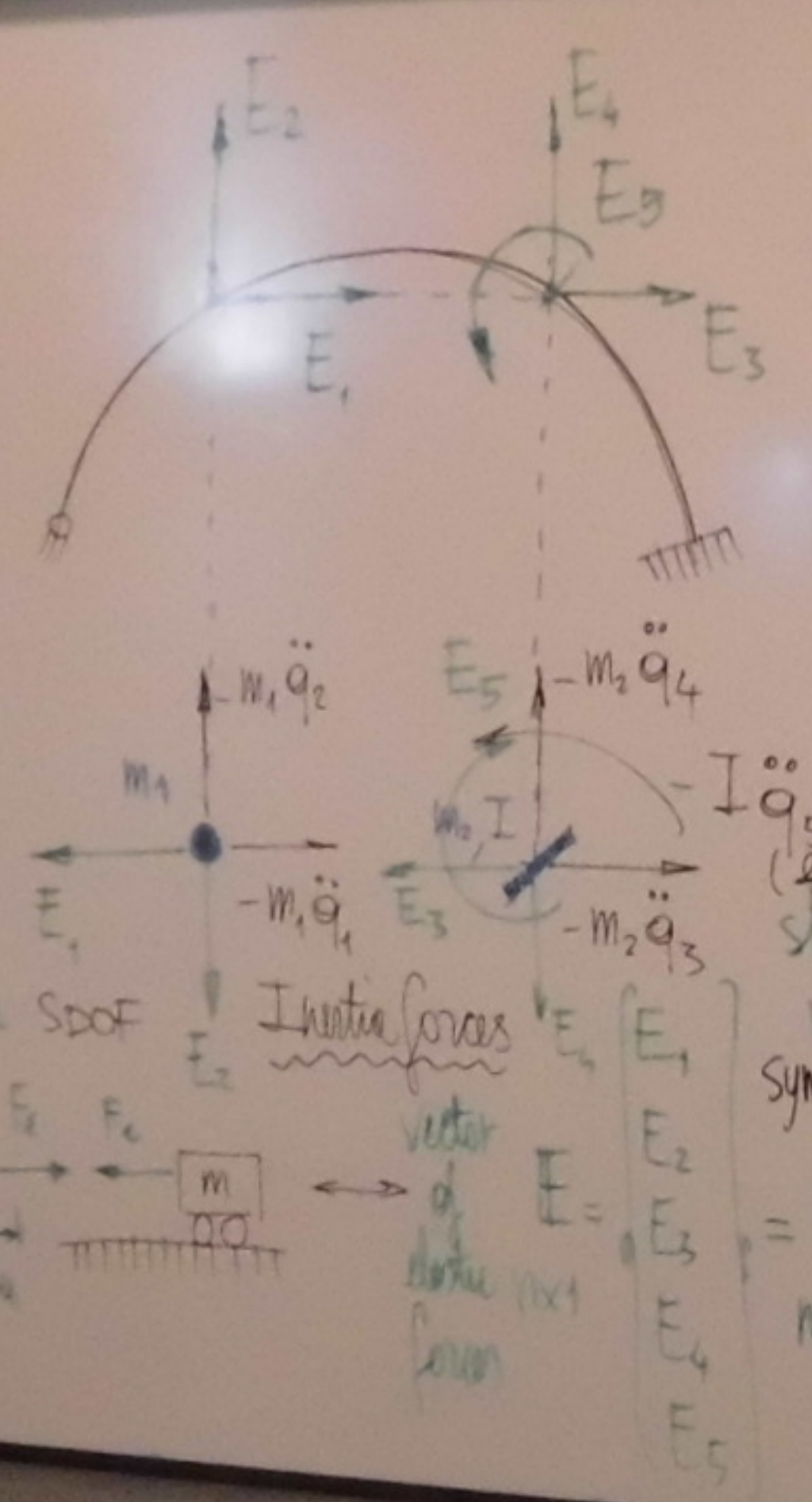
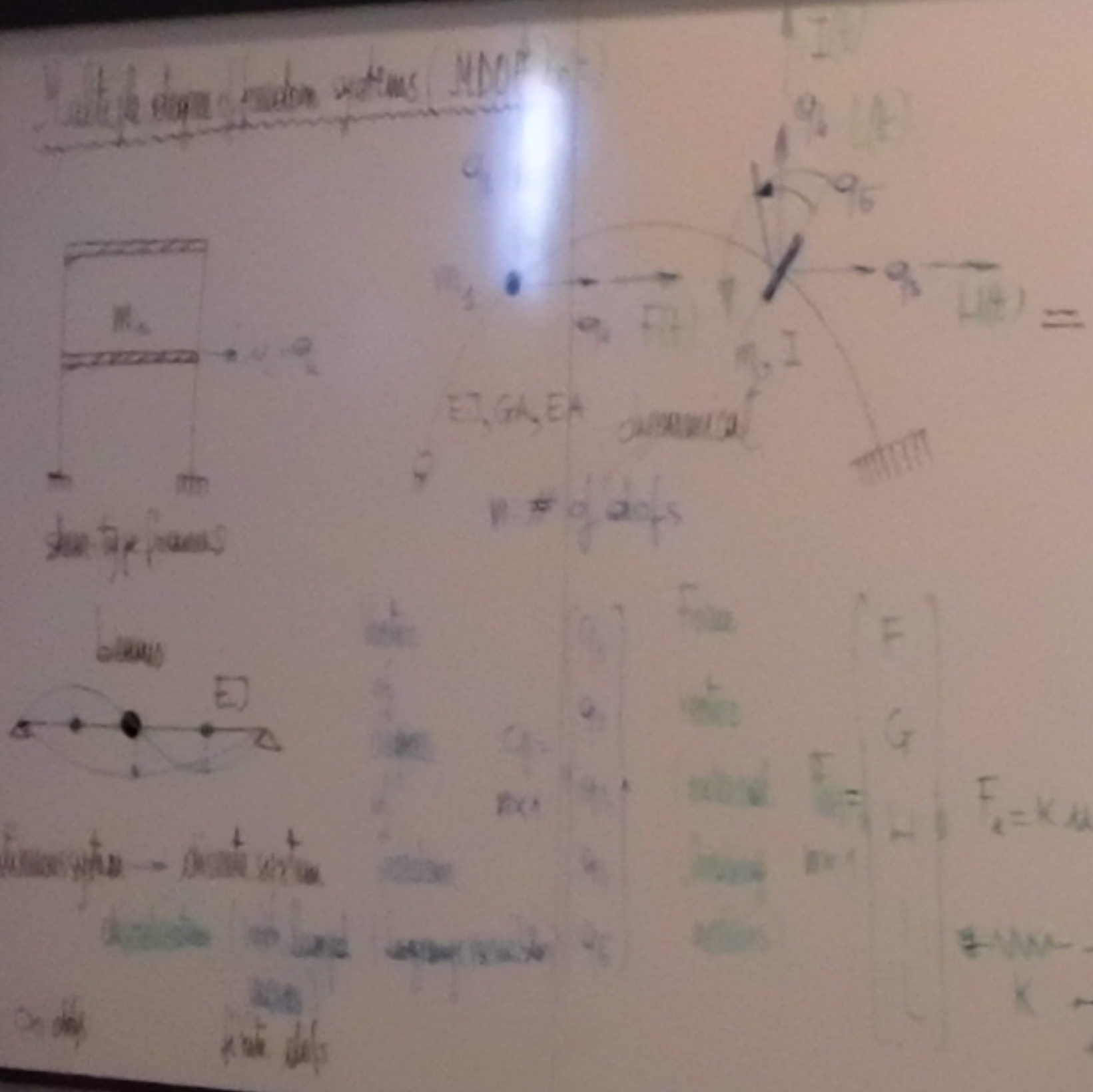
Force vector
 $F = [F_1, F_2, F_3, F_4, F_5]^T$
 $n \times 1$
 (internal forces, external forces, actions)



Stiffness matrix K
 $K = K^T \Rightarrow K_{ji} = K_{ij}$
 (elastic stiffness matrix)
 sym and PD
 $K = [K_{11} \ K_{12} \ \dots \ K_{1n}]$
 $[K_{21} \ K_{22} \ \dots \ K_{2n}]$
 $[K_{31} \ K_{32} \ \dots \ K_{3n}]$
 $[K_{n1} \ K_{n2} \ \dots \ K_{nn}]$

Elastic energy
 $\mathcal{E} = \frac{1}{2} E^T q = \frac{1}{2} q^T E > 0$
 scalar
 $= \frac{1}{2} q^T K q = \frac{1}{2} E^T \eta E$
 $= \frac{1}{2} q^T K q = \frac{1}{2} E^T \eta E$
 ; inverse relation:
 $q = K^{-1} E = \eta E$
 compliance matrix ($\eta = K^{-1}; K = \eta^{-1}$)

Compliance matrix
 $\eta = \eta^T \Rightarrow \eta_{ji} = \eta_{ij}$ (Maxwell's theorem)
 Eqs of motion
 $M \ddot{q} + K q = F$
 $M \ddot{q}_1 + E_1 = F_1$
 $M \ddot{q}_2 + E_2 = F_2$
 $M \ddot{q}_3 + E_3 = F_3$
 $M \ddot{q}_4 + E_4 = F_4$
 $M \ddot{q}_5 + E_5 = F_5$



elastic energy

$$E = \frac{1}{2} \mathbf{E}^T \mathbf{q} = \frac{1}{2} \mathbf{q}^T \mathbf{E} > 0$$

scalar

$$= \frac{1}{2} \mathbf{q}^T \mathbf{K} \mathbf{q} = \frac{1}{2} \mathbf{E}^T \boldsymbol{\eta} \mathbf{E}$$

$$= \frac{1}{2} q_i K_{ij} q_j = \frac{1}{2} E_i \eta_{ij} E_j$$

inverse relation:

$$\mathbf{q} = \mathbf{K}^{-1} \mathbf{E} = \boldsymbol{\eta} \mathbf{E}$$

compliance matrix ($\boldsymbol{\eta} = \mathbf{K}^{-1}; \mathbf{K} = \boldsymbol{\eta}^{-1}$)

compliance coeff. (influence)

Maxwell Theorem

Eqs. of motion ($n=5$)

$$\begin{aligned} m_1 \ddot{q}_1 + E_1 &= F(t) \\ m_1 \ddot{q}_2 + E_2 &= G(t) \\ m_2 \ddot{q}_3 + E_3 &= H(t) \\ m_2 \ddot{q}_4 + E_4 &= I(t) \\ I \ddot{q}_5 + E_5 &= L(t) \end{aligned}$$