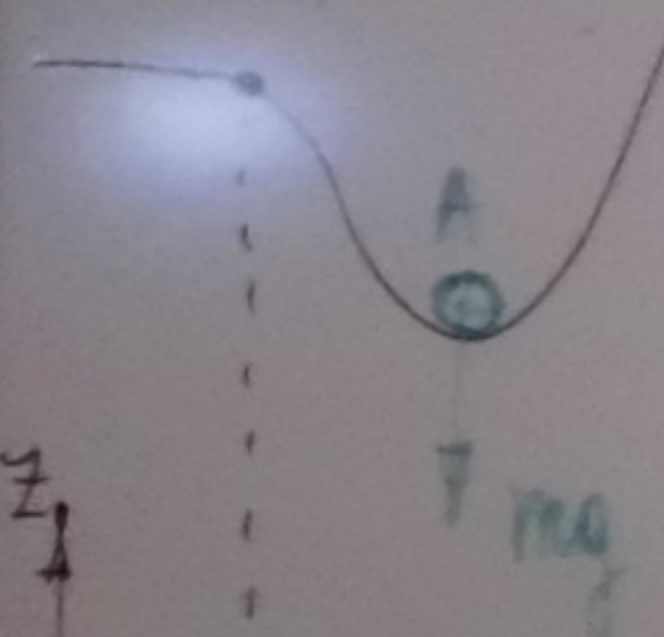


Energy approach (to instability problems)

A, C stationary points of the pot. en.

$V=0, V''>0$
 $Z=Z(x) \text{ min}$



$V' = \frac{\partial V}{\partial x}$

horizontal station

$V=const, V'=0$

$V'=0, V''>0$

$V'=0, V''<0$

$V'=0, V''=0$

$V'=0, V''<0$

$V'=0, V''>0$

$V'=0, V''<0$

$V'=0, V''>0$

$V'=0, V''<0$

$V'=0, V''>0$

$V'=0, V''<0$

$V'=0, V''>0$

$V'=0, V''<0$

$V'=0, V''>0$

$V'=0, V''<0$

$V'=0, V''>0$

$V'=0, V''<0$

$V'=0, V''>0$

$V'=0, V''<0$

$V'=0, V''>0$

$V'=0, V''<0$

$V'=0, V''<0$
 max

$V'=0, V''>0$

$V'=0, V''<0$

$V'=0, V''>0$

$V'=0, V''<0$

$V'=0, V''>0$

$V'=0, V''<0$

$V'=0, V''>0$

$V'=0, V''<0$

$V'=0, V''>0$

$V'=0, V''<0$

$V'=0, V''>0$

$V'=0, V''<0$

$V'=0, V''>0$

$V'=0, V''<0$

$V'=0, V''>0$

$V'=0, V''<0$

$V'=0, V''>0$

$V'=0, V''<0$

$V'=0, V''>0$

$V'=0, V''<0$

$V'=0, V''>0$

- From Lagrange dynamic stability criterion, states A, B and C look much different, in terms of potential effects of little perturbations of these equilibrium confn.

A: STABLE, after little perturbations the ball $V''>0$ tries to get back to the original position

B: INDIFFERENT; perturbations neither lead to system to leave nor to stay in the original configuration. $V''=0$

C: UNSTABLE; little perturbations may let the system leave from the original position and acquire others (like A), that may be even far from the original. $V''<0$

• Purely positional problem, where the potential energy of the system (linked to conservative forces):

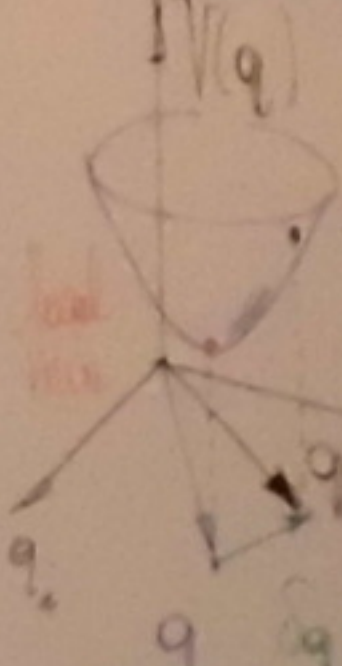
$V(Z(x)) = \frac{mg}{p} Z(x)$

Theorem of Dirichlet (SC for stability)

The equilibrium confn. of a conservative, holonomic system with ideal constraints is stable if, there, the total potential energy of the system is at a (local) minimum

$q_0 \text{ equil. confn.}$
 $V(q_0) \text{ is min} \Rightarrow q_0 \text{ is stable}$

where $V(q) = V_e(q) + V_p(q)$
 elastic energy potential energy of external conservative forces
 $-L_2(q)$

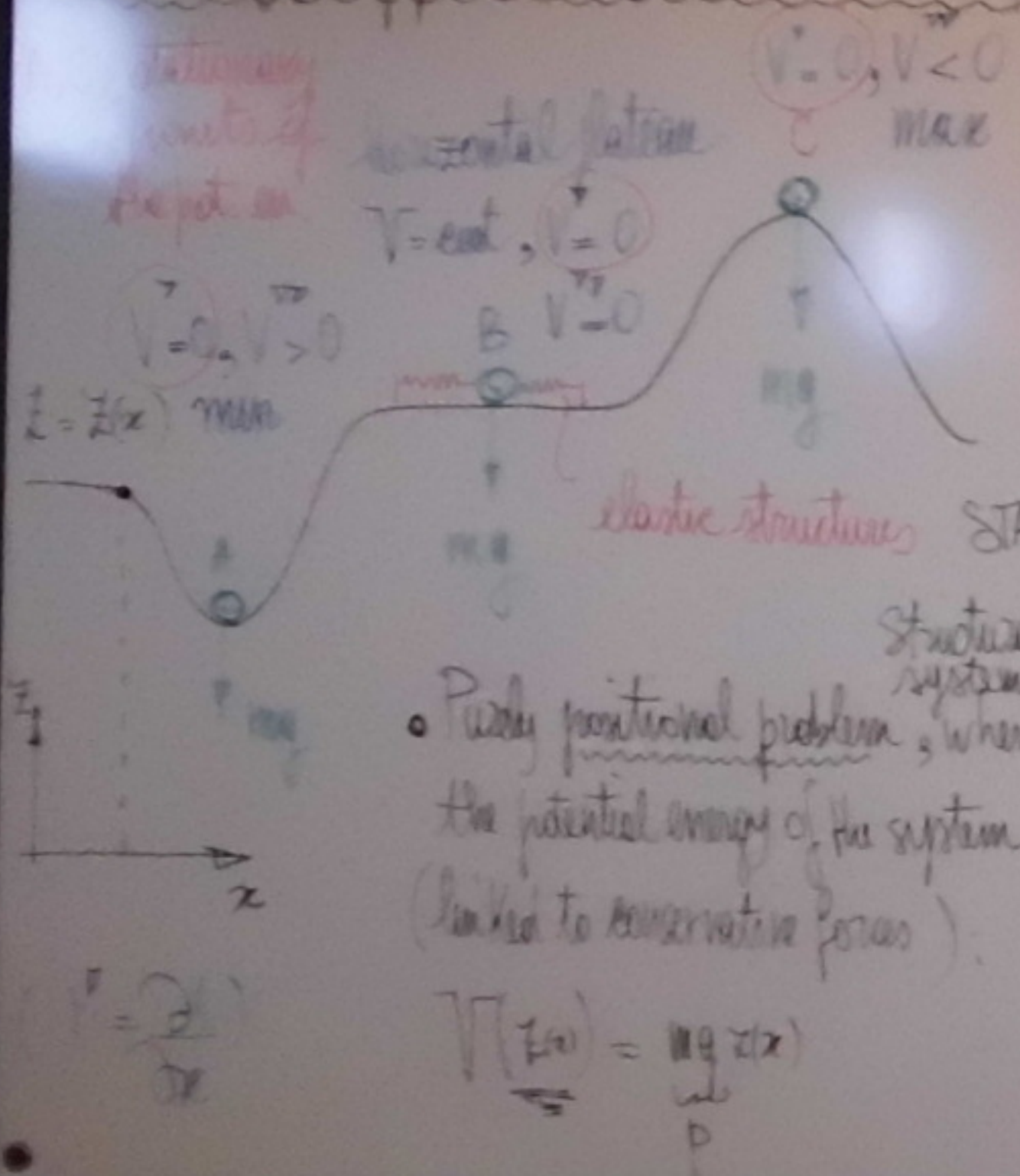


$\Delta V = V(q) - V(q_0)$
 $\Delta V > 0$
 $\Delta V = \frac{\partial V}{\partial q} \Delta q + \frac{1}{2} \frac{\partial^2 V}{\partial q^2} \Delta q^2 + \dots$
 $\frac{\partial V}{\partial q} = 0$
 $\Delta V = \frac{1}{2} V'' \Delta q^2$
 $V'' > 0 \Rightarrow \Delta V > 0$
 $V'' < 0 \Rightarrow \Delta V < 0$
 $V'' = 0 \Rightarrow \Delta V = 0$

$q = q_0 + \Delta q$
 $\Delta q = q - q_0$

+ Obviously, to be on a local minimum, and then to achieve the + position ΔV , which would be negative to the original position, leading to the stationary equilibrium.

Energy approach (to instability problems)



From Lyapunov dynamic stability criterion, states A, B and C look much different, in terms of potential effects of little perturbations of these equilibrium confn.

A: STABLE, after little perturbations the system $V'' > 0$ tries to get back to the original position

B: INDIFFERENT; perturbations neither lead to system to leave nor to stay in the original configuration. $V'' = 0$

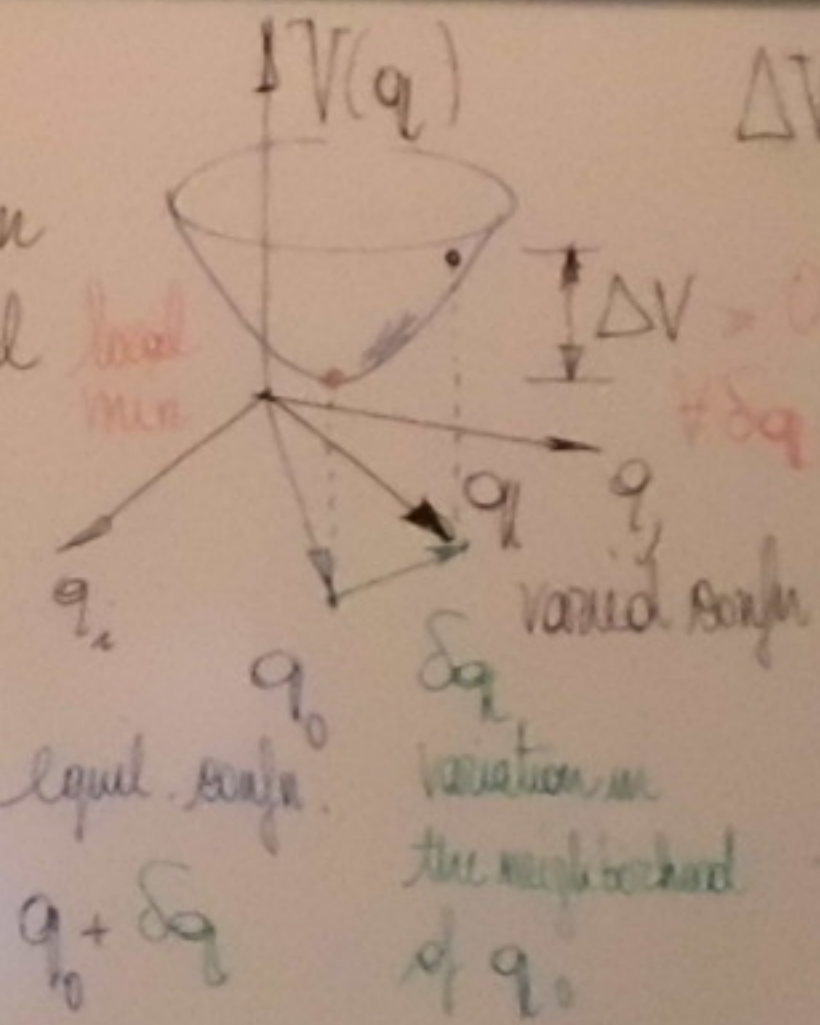
C: UNSTABLE; little perturbations might the system leave from the original position and acquire other ones (like A), that may be even far from the original. $V'' < 0$

Theorem of Dirichlet (SC for stability)

The equilibrium confn. of a conservative, holonomic system with ideal constraints is stable if, there, the total potential energy of the system is at a (local) minimum.

q_0 equil. confn. $\Rightarrow q_0$ is stable
 $V(q_0)$ is min

where: $V(q) = \underbrace{V_e(q)}_{\text{elastic energy}} + \underbrace{V_f(q)}_{\text{potential energy of external conservative forces}} - \underbrace{L_c(q)}$



$$\Delta V = V(q) - V(q_0) = \delta V + \frac{1}{2} \delta^2 V + \dots$$

where:

$$\delta V = \frac{\partial V}{\partial q} \bigg|_{q_0} \delta q$$

1. order variation

$$\frac{1}{2} \delta^2 V = \frac{1}{2} \delta q^T \frac{\partial^2 V}{\partial q \partial q} \bigg|_{q_0} \delta q = \frac{1}{2} \delta q^T K \delta q$$

2. order variation

+ Obviously, to be on a local minimum and then to achieve $\Delta V > 0 \forall \delta q$ the 1. variation δV , which would be opposite to the the sign of δq , shall vanish, leading to the stationary condition: $\delta V = 0 \Rightarrow \frac{\partial V}{\partial q} = 0$ (NC for equilibrium configurations)

Then, at a stationary point, the 2. order $\Delta V = \frac{1}{2} \delta V$ is the only term that matters. Thus, $\delta^2 V > 0 \Rightarrow$ stable, and $\delta^2 V < 0 \Rightarrow$ unstable.

Energy approach to stability problem

Consider a system with a stable equilibrium state, q_0 , and a small displacement, δq , from this equilibrium state. The potential energy of the system is at a local minimum at q_0 . The total potential energy of the system is at a local minimum at q_0 . The system is stable if, after a small displacement, it will return to its original position.

• Stable equilibrium problem: When the potential energy of a system is at a local minimum, the system is stable.

• Unstable equilibrium problem: When the potential energy of a system is at a local maximum, the system is unstable.

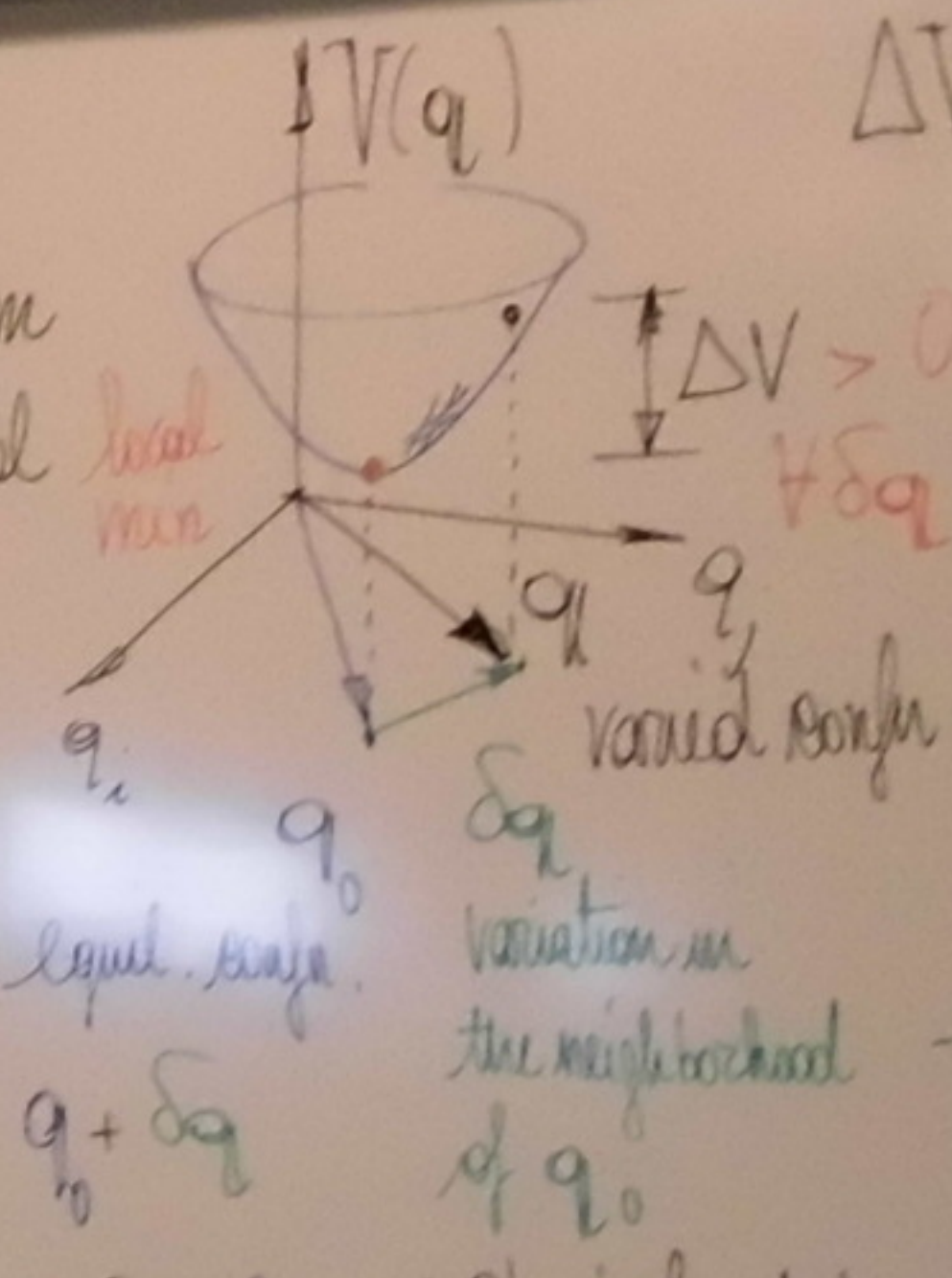
• Indifferent equilibrium problem: When the potential energy of a system is constant, the system is indifferent.

Theorem of Dirichlet (SC for stability)

The equilibrium confn. of a conservative, holonomic system with ideal constraints is stable if, there, the total potential energy of the system is at a (local) minimum.

$$q_0 \text{ equil. confn.} \left. \begin{array}{l} V(q_0) \text{ is min} \end{array} \right\} \Rightarrow q_0 \text{ is stable}$$

$$V(q) = \underbrace{V_e(q)}_{\text{elastic energy}} + \underbrace{V_f(q)}_{\text{potential energy of external conservative forces}} - \underbrace{L_2(q)}$$



$$\Delta V = V(q) - V(q_0) = \delta V + \frac{1}{2} \delta^2 V + \dots$$

where: $\delta V = \frac{\partial V}{\partial q} \bigg|_{q_0} \delta q$ (1-order variation)

$\delta^2 V = \frac{1}{2} \delta q^T \frac{\partial^2 V}{\partial q \partial q} \bigg|_{q_0} \delta q = \frac{1}{2} \delta q^T K \delta q$ (2-order variation)

+ Obviously, to be on a local minimum and then to achieve $\Delta V > 0 \forall \delta q$, the 1-variation δV , which would be sensitive to the sign of δq , shall vanish, leading to the stationary condition:

$$\delta V = 0 \Rightarrow \frac{\partial V}{\partial q} \bigg|_{q_0} = 0$$

(NC for equilibrium configurations!)

+ Then, at a stationary point, the sign of $\Delta V = \frac{1}{2} \delta^2 V$ is ruled by the sign of the 2-order variation. Thus, $\frac{1}{2} \delta^2 V > 0$ is a SC for stability and this is ruled by the eigenvalues of K .

Example (SDOF discrete system, with concentrated elastic deformation)

Diagram of a vertical rod of length l fixed at the bottom (A) and free at the top (B). A horizontal force P is applied at the top. The displacement at the top is v . The potential energy is $V(v) = V_e(v) + V_f(v)$.

Spring constant $K > 0$. Load multiplication $p = \frac{Pl}{K}$. $P = p \frac{K}{l}$.

Equilibrium equation: $M = K\theta$; $\theta = \frac{1}{2} M$. $V = \frac{1}{2} M\theta = \frac{1}{2} \frac{1}{K} M^2$. Obviously $\theta = 0$ is an equl. confn.

Stability condition: $V''(v) = \frac{d^2 V(v)}{dv^2} = K(1 - p \cos \theta) > 0$. In the equl. confn $p = \frac{1}{\sin \theta}$. $V''(v_{equl}) = \frac{1}{K} - \frac{1}{\sin^3 \theta}$.

Stability condition

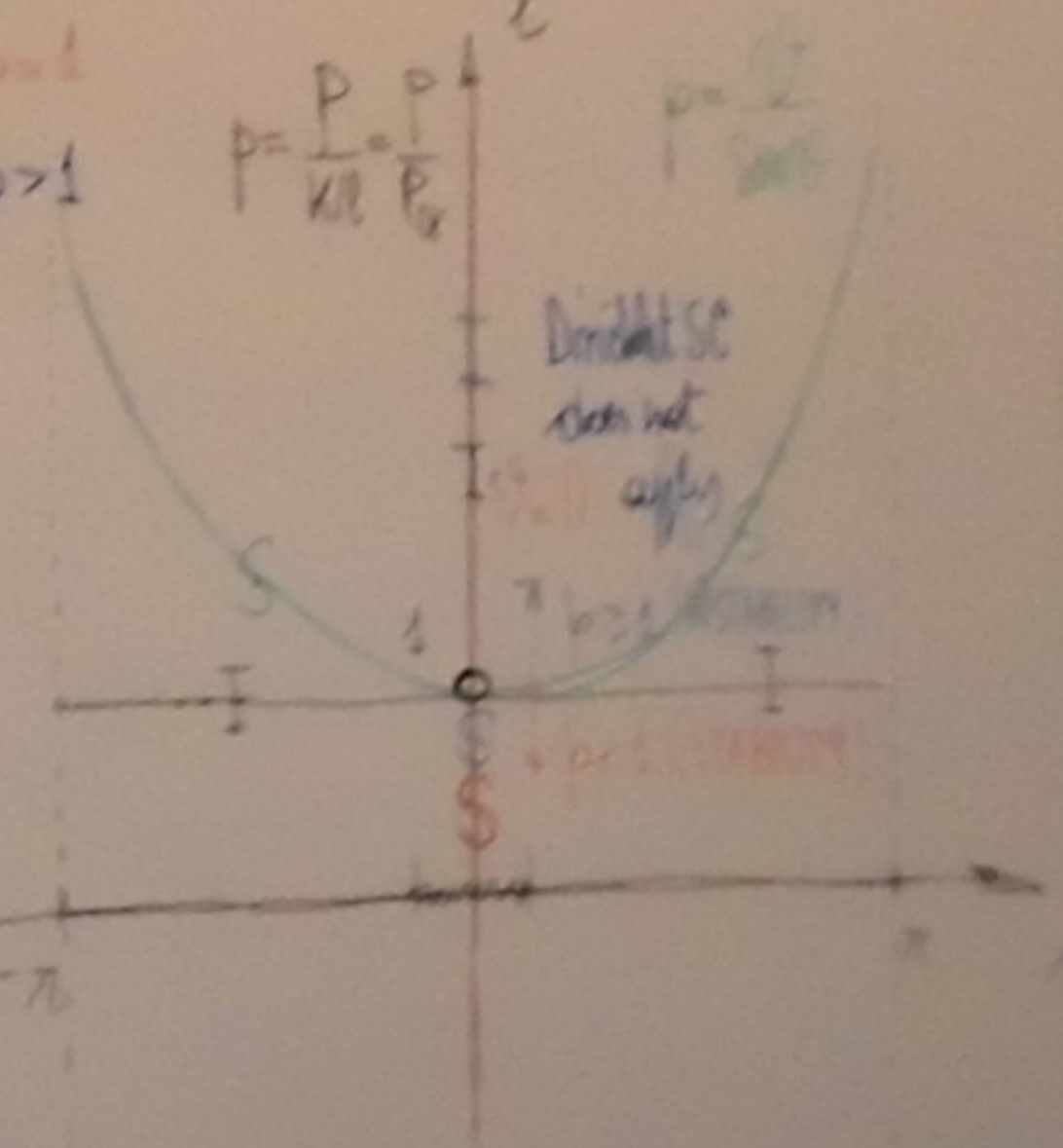
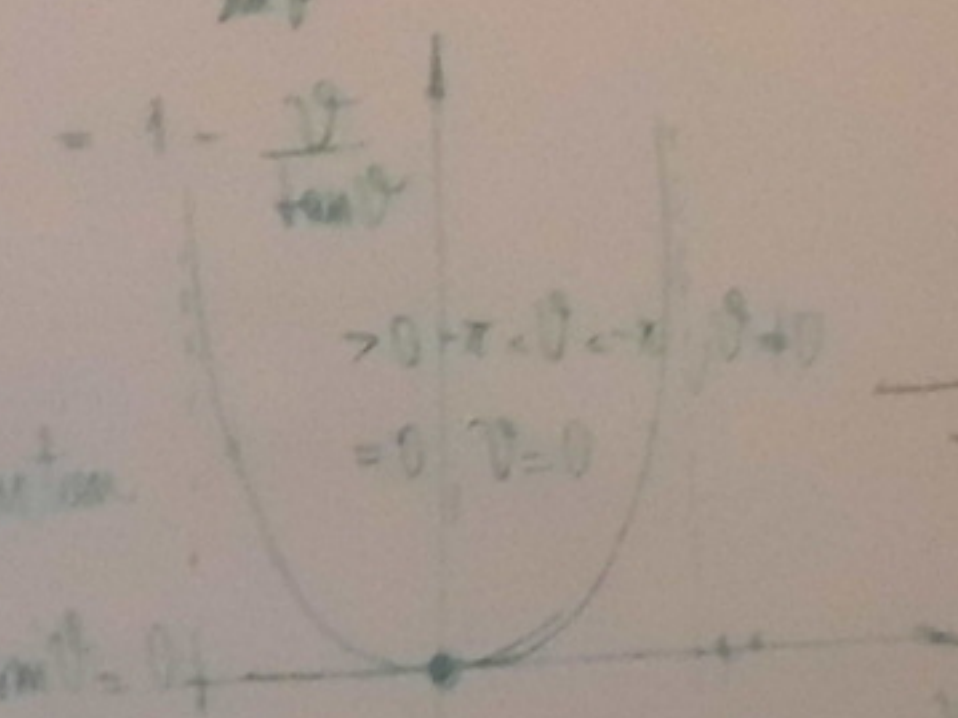
$$-V''(v) = \frac{d^2 V(v)}{dv^2} = K(1 - p \cos \theta) > 0$$

In the equl. confn $p = \frac{1}{\sin \theta}$

$$\frac{V''(v_{equl})}{K} = 1 - \frac{1}{\sin^3 \theta}$$

+ $\theta = 0$

- $1 - p > 0, p < 1$ stable $P < P_{cr} = \frac{K}{l}$
- $1 - p = 0, p = 1$ $P = P_{cr}$
- $1 - p < 0, p > 1$ unstable

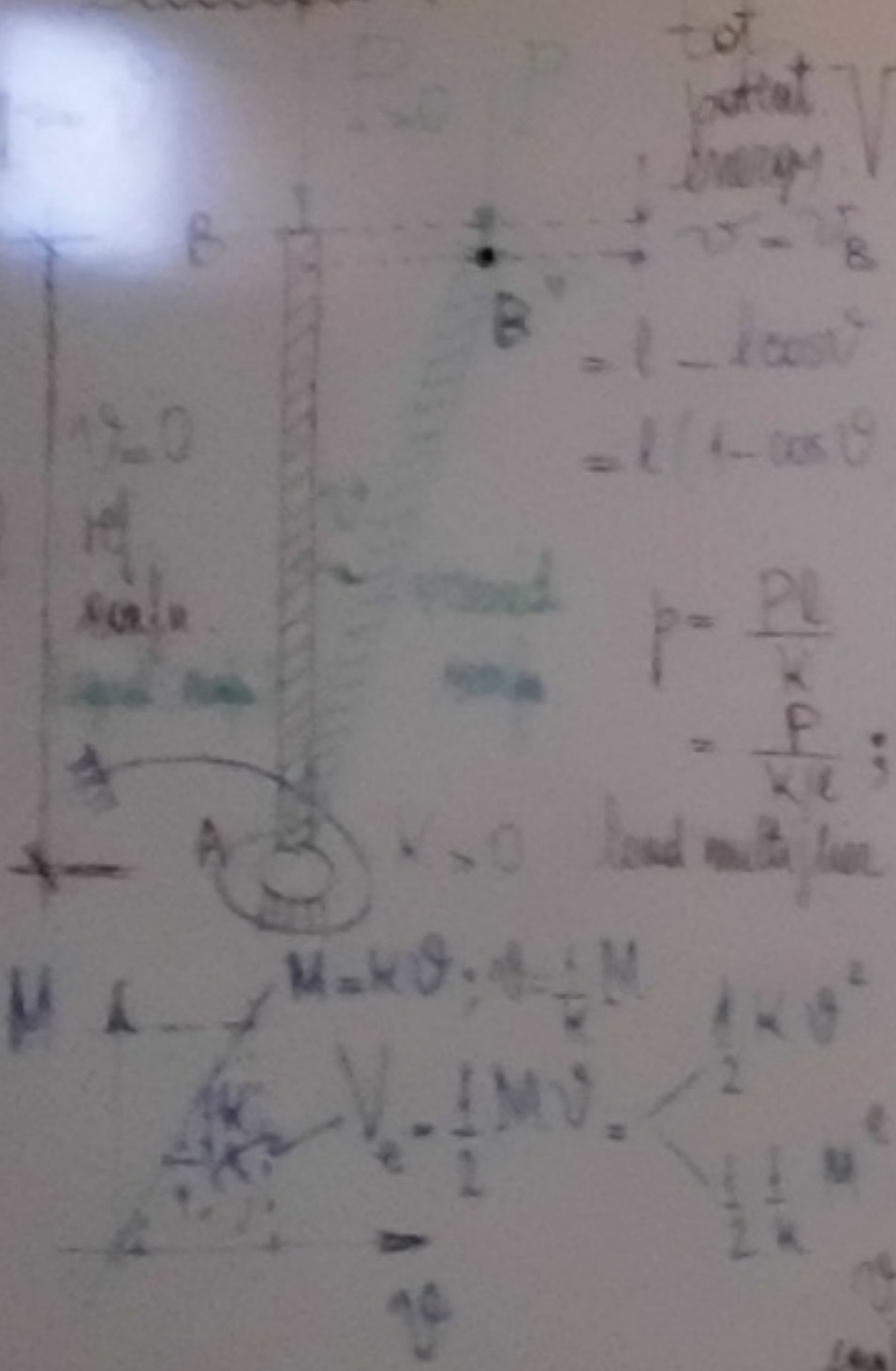


Graphically - not dependent on θ

Stability condition: $V''(v) = \frac{d^2 V(v)}{dv^2} = K(1 - p \cos \theta) > 0$

Stability condition: $V''(v) = \frac{d^2 V(v)}{dv^2} = K(1 - p \cos \theta) > 0$

Example (SDOF discrete system, with concentrated elastic deformation)



total potential energy $V(v) = V_e(v) + V_p(v)$

$$= \frac{1}{2} K v^2 - P v \cos \theta$$

$$= \frac{1}{2} K v^2 - P l (1 - \cos \theta)$$

$$p = \frac{P l}{K}$$

$$V(v) = \frac{1}{2} K v^2 - P l (1 - \cos \theta)$$

$$V'(v) = K v - P l \sin \theta = 0$$

$$V''(v) = K$$

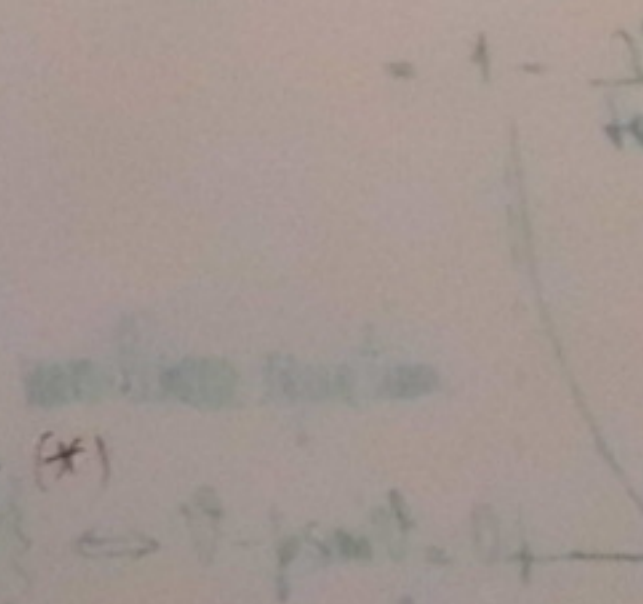
Stability condition

$V''(v) = \frac{d^2 V(v)}{dv^2}$

$$= K(1 - p \cos \theta) > 0$$

In the equil. config $p = \frac{P l}{K}$

$V''(v) = K(1 - p \cos \theta)$

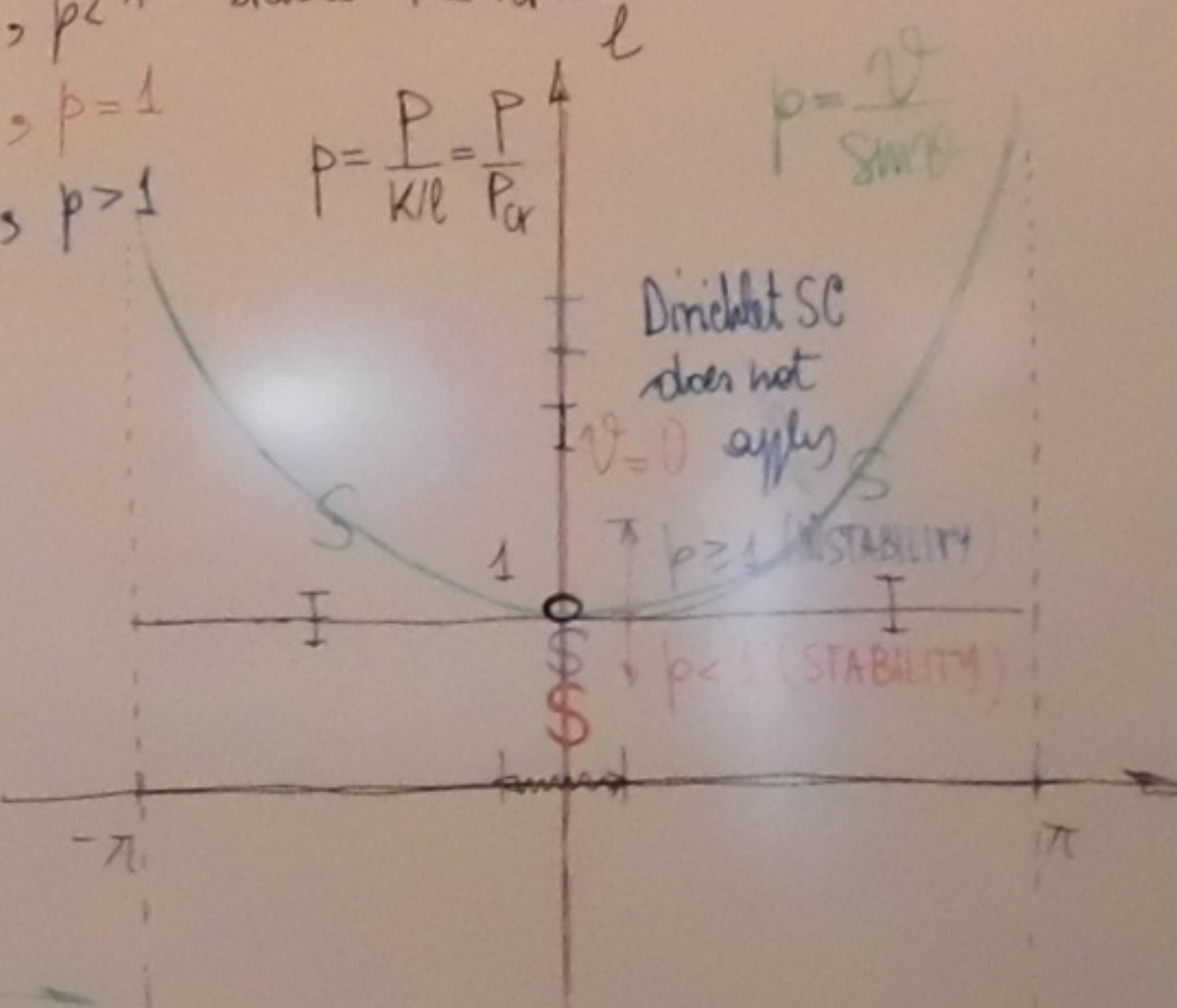


$\theta = 0$

$V''(0) = K(1 - p)$

- $1 - p > 0, p < 1$ Stable $P < P_{cr} = \frac{K}{l}$
- $1 - p = 0, p = 1$
- $1 - p < 0, p > 1$

$p = \frac{P}{K/l} = \frac{P}{P_{cr}}$



Geometrically-small displacements

$\theta \approx 0$

$V(v) \approx V_e(v) = \frac{1}{2} K v^2 - P l (1 - \cos \theta)$

$= \frac{1}{2} K v^2 - P l \frac{1}{2} \theta^2$

$= \frac{1}{2} (K - P l) v^2$

$V_2(v) = (K - P l) v^2 = 0$

$V_2(v) = K - P l > 0 \Rightarrow P < P_{cr}$

$= 0 \Rightarrow P = P_{cr}$

$< 0 \Rightarrow P > P_{cr}$

As θ is small $\sin \theta \approx \theta$

and linearization of (*) leads to linearized equl. equation (*)

[illegible]
$$-V(\theta) = \frac{d^2 V(\theta)}{d\theta^2} = K(1 - p \cos \theta) > 0$$

In the equl confn $p = \frac{1}{\sin \theta}$

- $1-p > 0, p < 1$
- $1-p = 0, p = 1$
- $1-p < 0, p > 1$

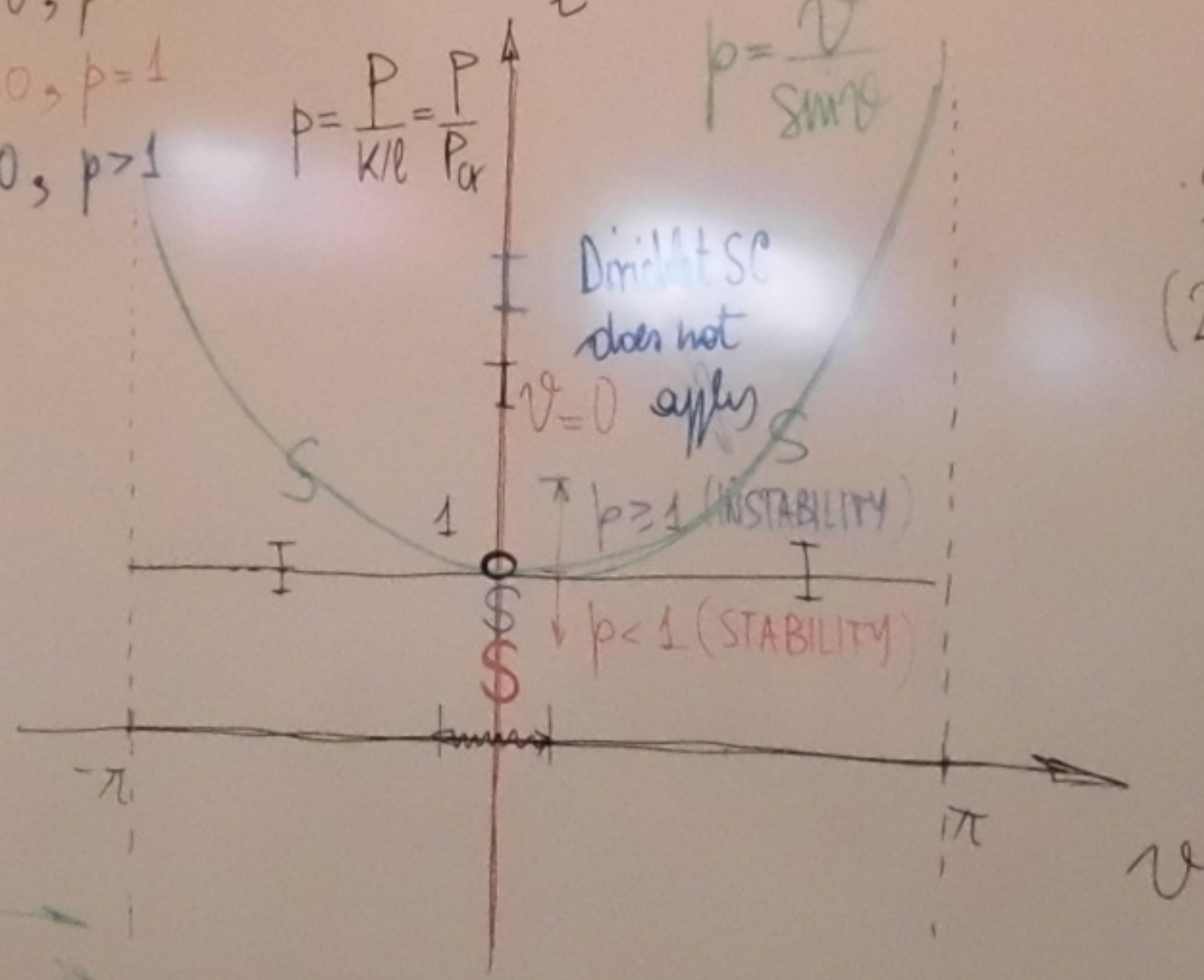
$$P = \frac{P}{K/l} = \frac{P}{P_{cr}}$$
$$p = \frac{V}{S \cdot m^2}$$

Dirichlet SC
does not
apply

$\gamma = 0$ applies

$\uparrow p > 1$ (INSTABILITY)

$\downarrow p < 1$ (STABILITY)



^{VP} Geometrically - small ^{VP} displacements
 $|h| \ll 1$

$\cos \theta$ up to 2nd-order

$$V(\psi) \approx V_2(\psi) = \frac{1}{2} K \psi^2 - Pl \left(1 - \left(1 - \frac{1}{2} \psi^2 + \dots \right) \right)$$

order theory

order effects

$$= \frac{1}{2} K \psi^2 - Pl \frac{1}{2} \psi^2$$

$$= \frac{1}{2} (K - Pl) \psi^2$$

$K - Pl > 0$ ψ^2

2nd-order theory
~~~~~  
(2<sup>nd</sup>-order effects)

(c)  $T_2(v) = (K - Pl)v = 0$

linearized equl. equation  
linear stat. cond

$K - Pl = 0, v = \text{arb.}$

$$V_2(r) = K - \frac{P}{r} \quad \begin{matrix} > 0 & P < P_{cr} \\ \text{const.} & = 0 & P = P_{cr} \\ < 0 & P > P_{cr} \end{matrix}$$

As  $\theta$  is small  $\sin \theta \approx \theta$   
and linearization of (\*) leads to linearized eqn. equation (9)