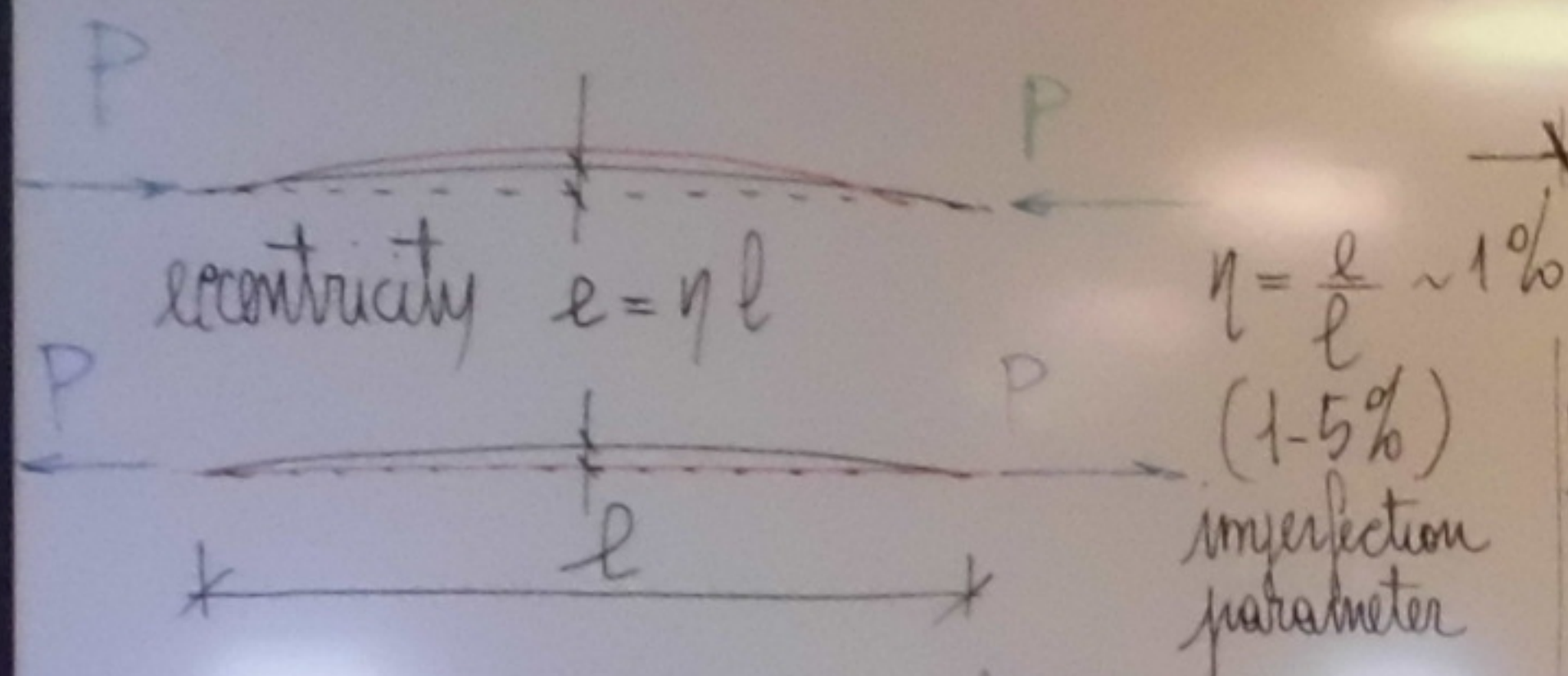
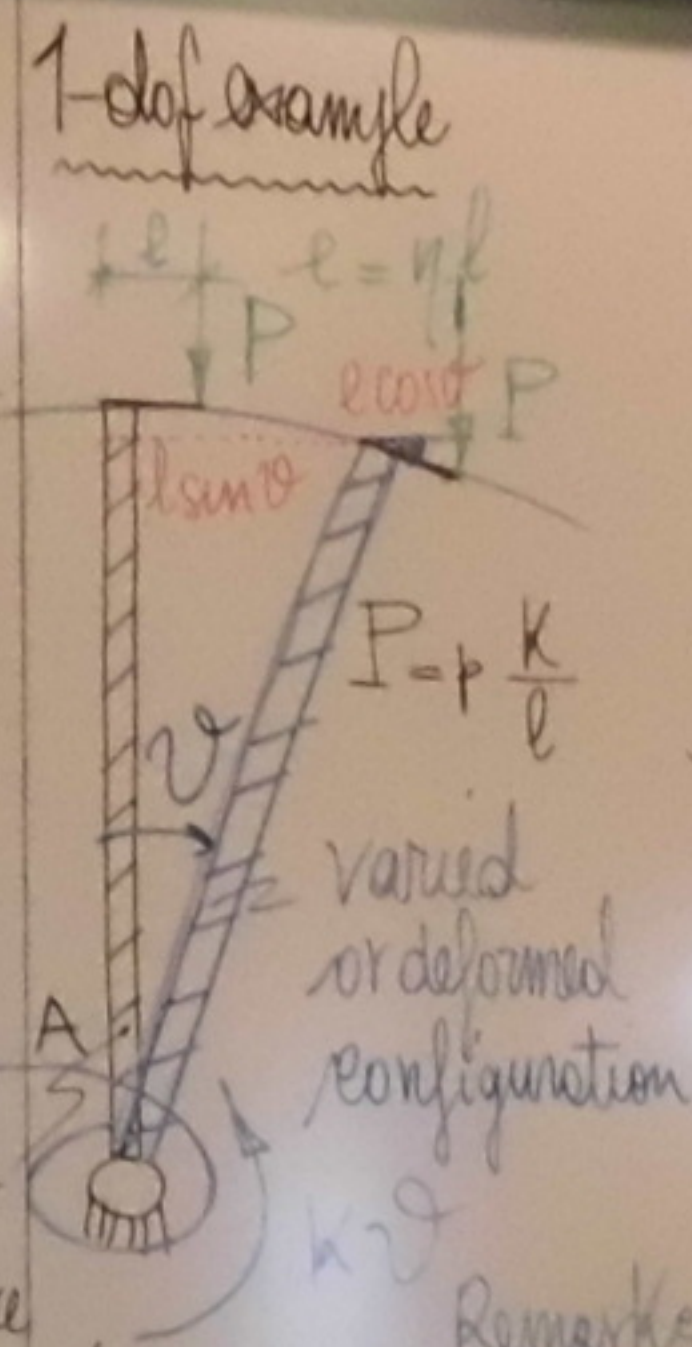


Instability analysis of real systems



- Geometrical (or other) imperfections in real systems (for instance eccentricities) may be prone to produce instability effects under compression-like loading

The question arises about how this may affect the instability phenomenon, for instance in terms of allowable compression loading.



Static approach

$$P(l \sin \theta + e \cos \theta) = K \theta$$

$$\frac{Pl}{K} (\sin \theta + \eta \cos \theta) = \theta$$

$$f(\theta) = \frac{\theta}{\sin \theta + \eta \cos \theta}$$

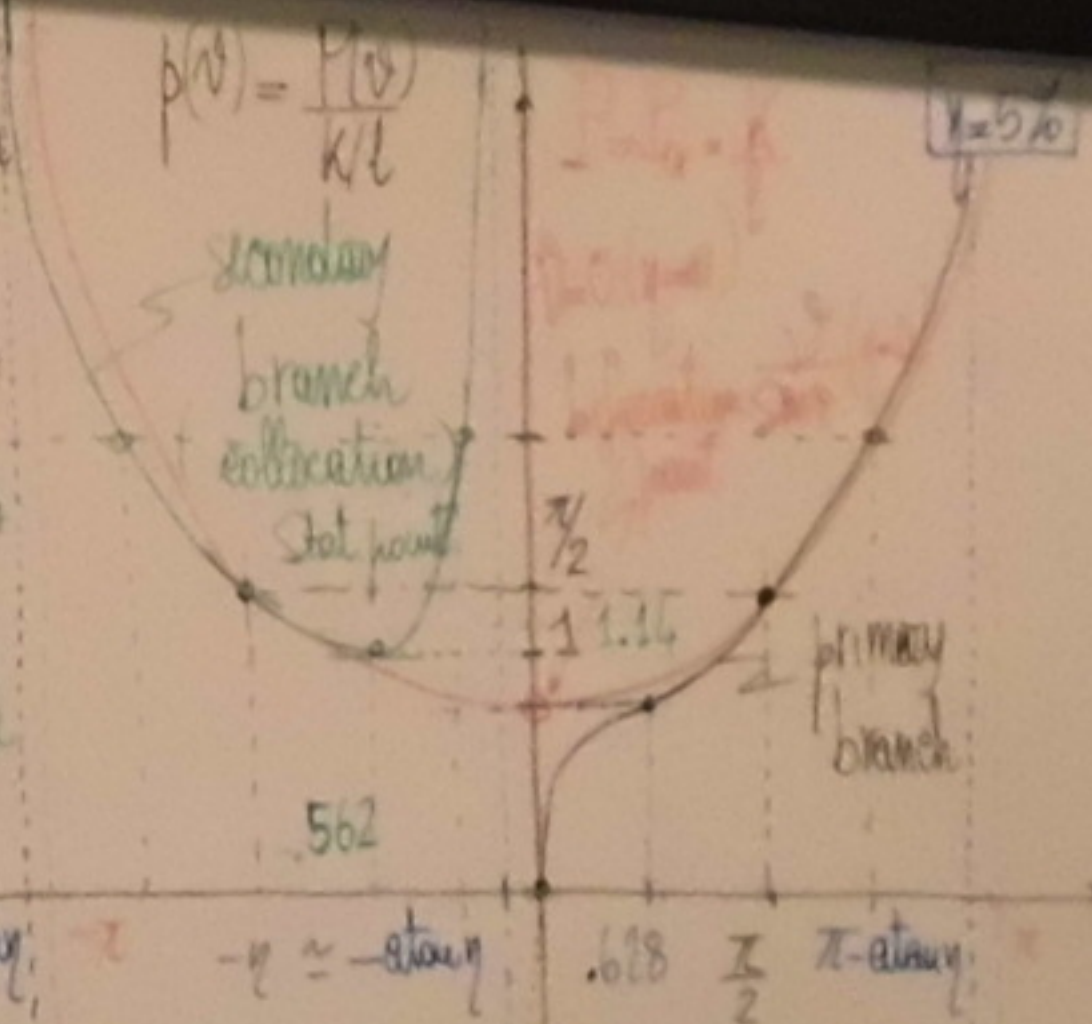
equilibrium configurations (through geometric non-linearity or P-Δ effect)

Remarks:

- for $\eta = 0$ (ideal system), one gets back the original ideal relation $p(\theta) = \frac{\theta}{\sin \theta}$ (particular case)
- for $\eta \neq 0$ (real system) $\theta = 0$ is an equilibrium config. only for $P = 0$
- denominator vanishes for $\tan \theta = -\eta \Rightarrow \theta = -\pi - \arctan \eta$
($\eta < 1 \Rightarrow \tan \theta = -\eta \Rightarrow \theta = -\pi - \arctan \eta$)

map of equilibrium paths (equal config)

$p \geq p_s = 1$
($\eta > 1$)
non-critical



$$\text{Stet } p(\theta) = \frac{1}{(\sin \theta + \eta \cos \theta)^2} \Rightarrow \text{tan } \theta = -\eta \Rightarrow \theta = -\pi - \arctan \eta$$

- for $\theta = \pm \pi/2$, no matter the η , one has $p = \pi/2$
- as θ keeps small, the primary branch may move near $\theta = 0$. However it immediately leaves it as soon as $P > 0$
- at $p = 1$, $\theta = \sin \theta + \eta \cos \theta \Rightarrow \theta \approx 0.688 \approx 39^\circ$
- later on, the real system almost follows the bifurcated branch of the ideal one

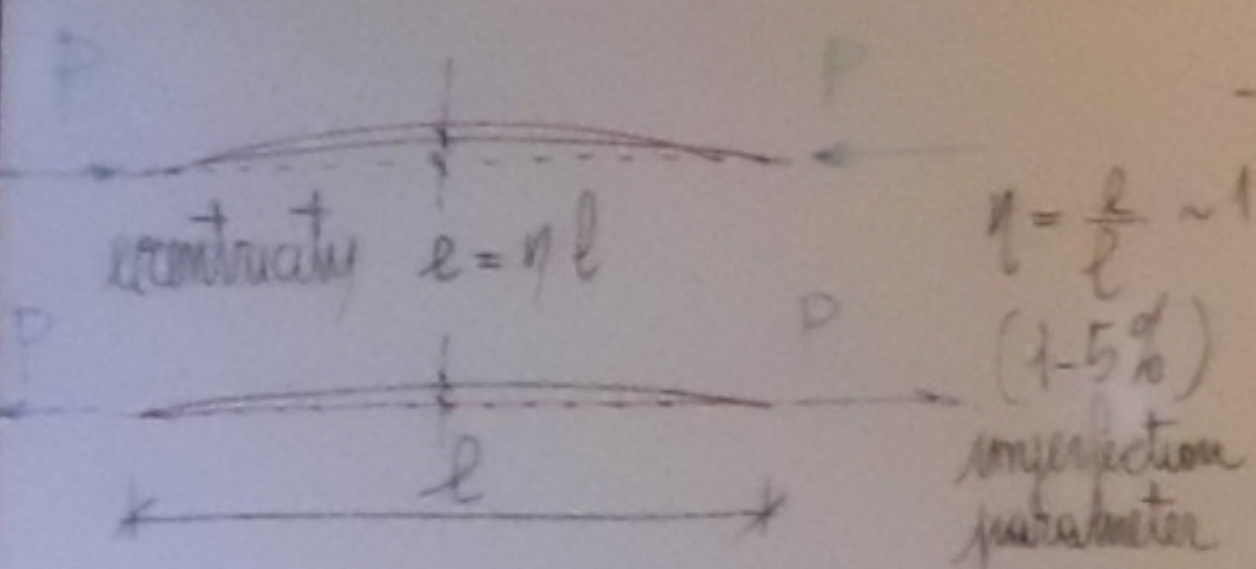
- bifurcation point no longer appears in $\eta \neq 0$ real system and system undergoes a continuous transition from the state $(\theta=0, P=0)$ on the side of $e = \eta l$, towards the bifurcated branch of the ideal one
- the value of P for on the real system plays a fundamental role in quantifying the amount of buckling load that may be allowed as a real system based on the accepted level of fabrication and experimental load

Analysis under geometric imperfections

$\theta = 0 \Rightarrow \dots$

$\dots$

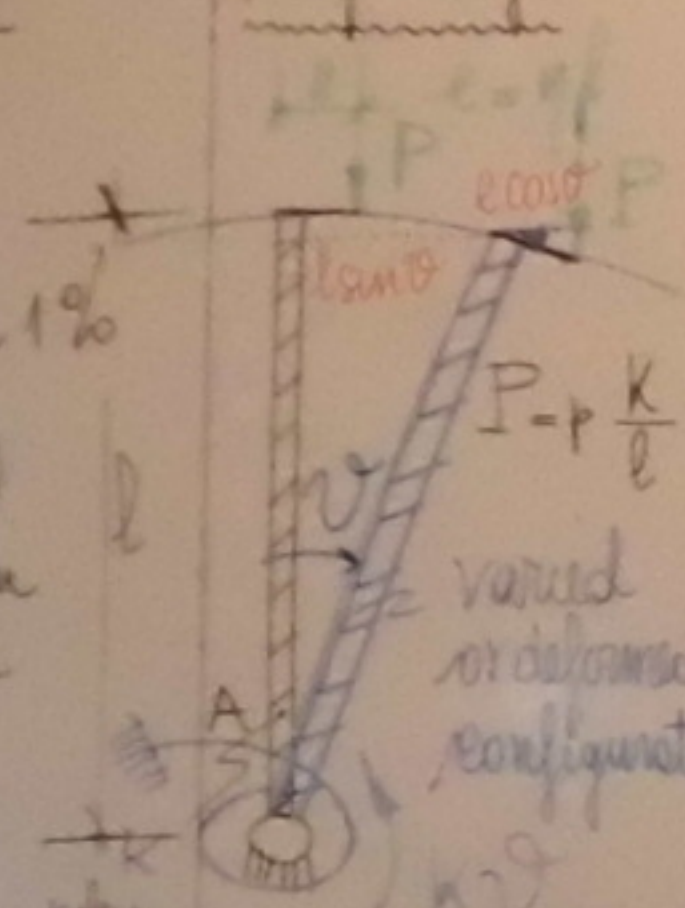
Instability analysis of real systems



Geometrical (or other) imperfections in real systems (for instance crookedness) may be prone to produce instability effects under compression-like loading.

In column, we must have the max effect of the imperfection parameter, in order to find the critical load.

1-dof example



reference configuration (with geometrical imperfection)
 $v=0$

Static approach

$$P(l \sin \theta + \eta \cos \theta) = kv$$

$$\frac{Pl}{k} (\sin \theta + \eta \cos \theta) = v$$

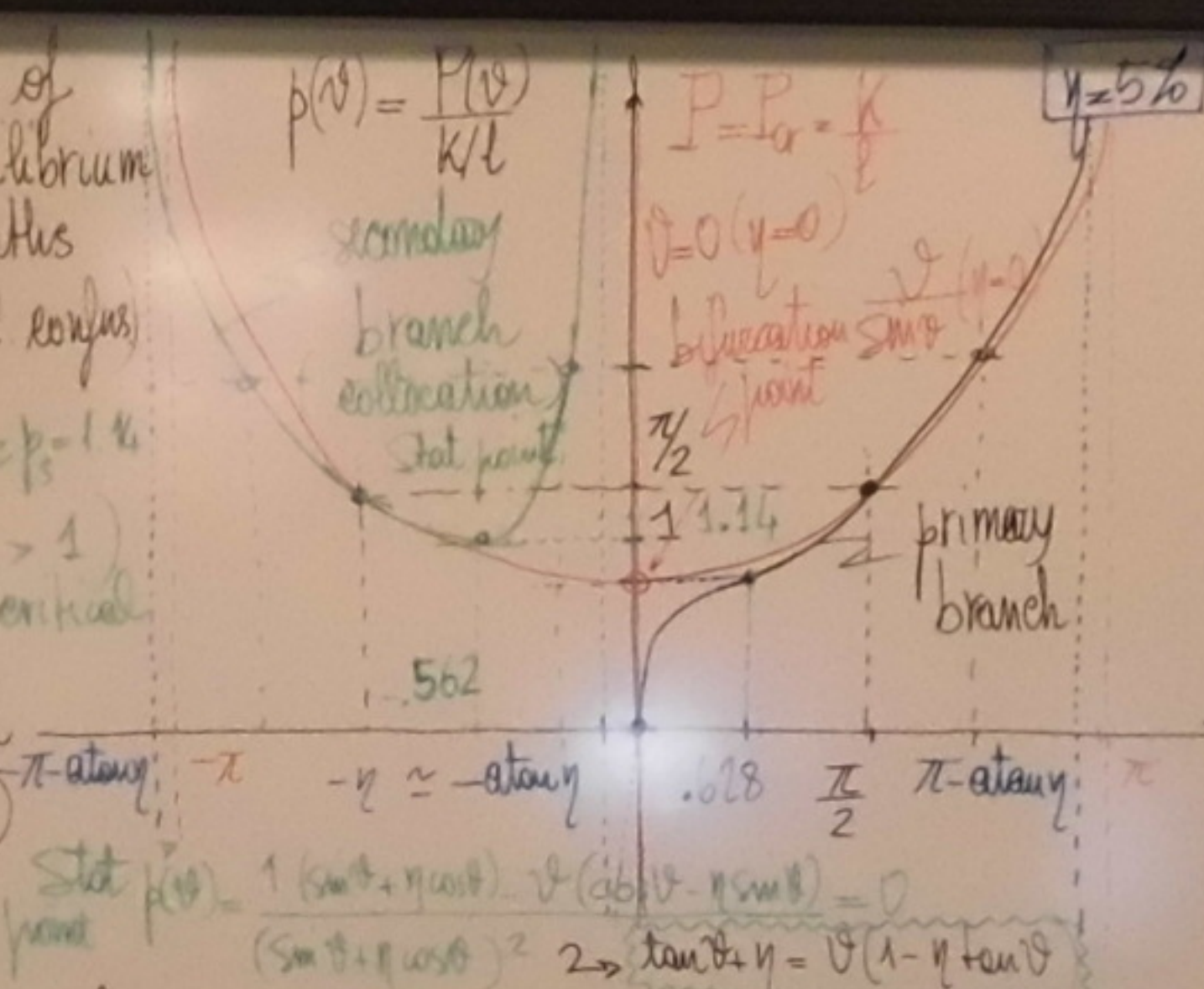
$$p(\theta) = \frac{v}{\sin \theta + \eta \cos \theta}$$

equilibrium configurations (through geometric non-linearity or P-Δ effect)

Remarks:

- for $\eta=0$ (ideal system), one gets back the original ideal relation $p(\theta) = \frac{v}{\sin \theta}$ (particular case)
- for $\eta \neq 0$ (real system) $v=0$ is an equilibrium only for $P=0$
- denominator vanishes for $\tan \theta = -\eta \Rightarrow \theta = -\pi - \arctan \eta$
 $\theta = -\arctan \eta$
 $\theta = \pi - \arctan \eta$

map of equilibrium paths (equil config)
 $p \geq p_c = 1$
($\eta > 1$)
not critical



- for $\theta = \pm \pi/2$, no matter the η , one has $p = \pi/2$
- as η keeps small, the primary branch is very much near $v=0$. However it immediately leaves it as soon as $P > 0$
- at $p=1$, $v = \sin \theta + \eta \cos \theta \Rightarrow \theta \approx 0.678 \approx 36^\circ$
- later on, the real system almost follows the bifurcated branch of the ideal one

- bifurcation point no longer appears as $\eta \neq 0$ (real system) and system undergoes a continuous transition from the state ($\theta=0, P=0$) on the side of $e = \eta l$, towards the bifurcated branch of the ideal sys.
- the evaluation of $P=P_c$ in the ideal system keeps of fundamental importance in quantifying the amount of compression load that may be allowed in a real system based on the accepted level of deformation and corresponding load P^* in a real system may be stated as P^*_{ideal} + safety factor 1.2-1.5

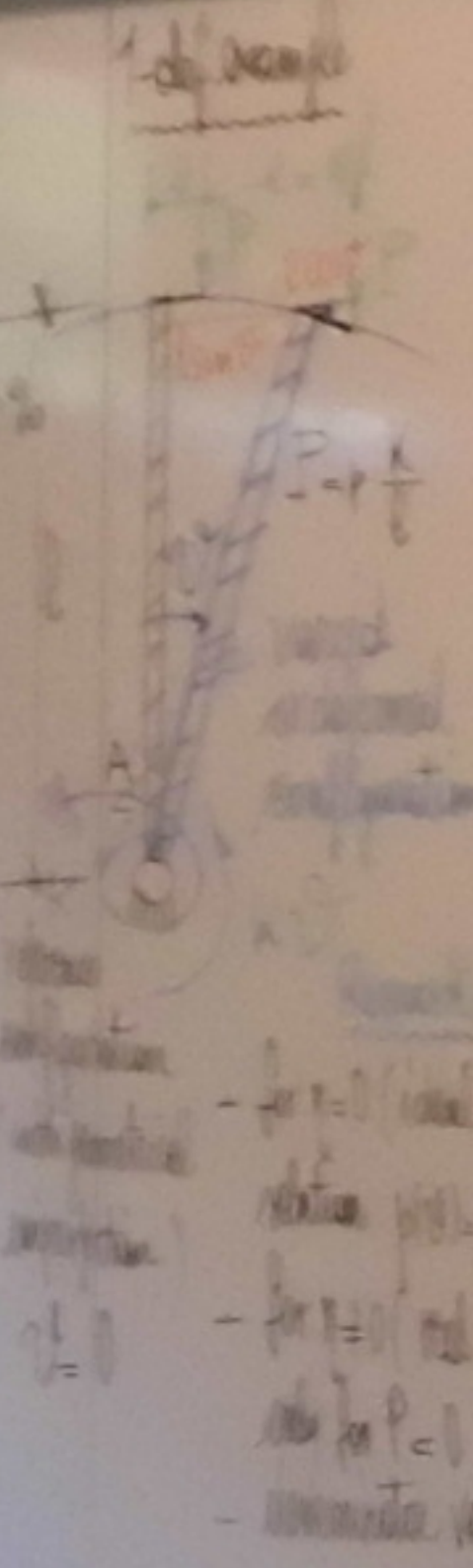
- Analysis under 'geometrically small' displacements
 $|\eta| \ll 1 \Rightarrow \sin \theta \approx \theta; \cos \theta \approx 1$
 $\frac{Pl}{k} (\theta + \eta) = v$
 $p(\theta) = \frac{v}{\theta + \eta} = \frac{1}{1 + \eta/\theta}$
 $\theta \rightarrow \infty, p \rightarrow 1$
secondary branch
primary branch
not critical
stat. point
equivalent to v the bifurcated branch of the ideal system marking P_c

Instability analysis of a column

control eq: $\ddot{v} + \frac{P}{K} \dot{v} = 0$

instability parameter $\lambda = \frac{P}{K}$

practical: in order to prevent buckling, one can use a viscoelastic material or a damper



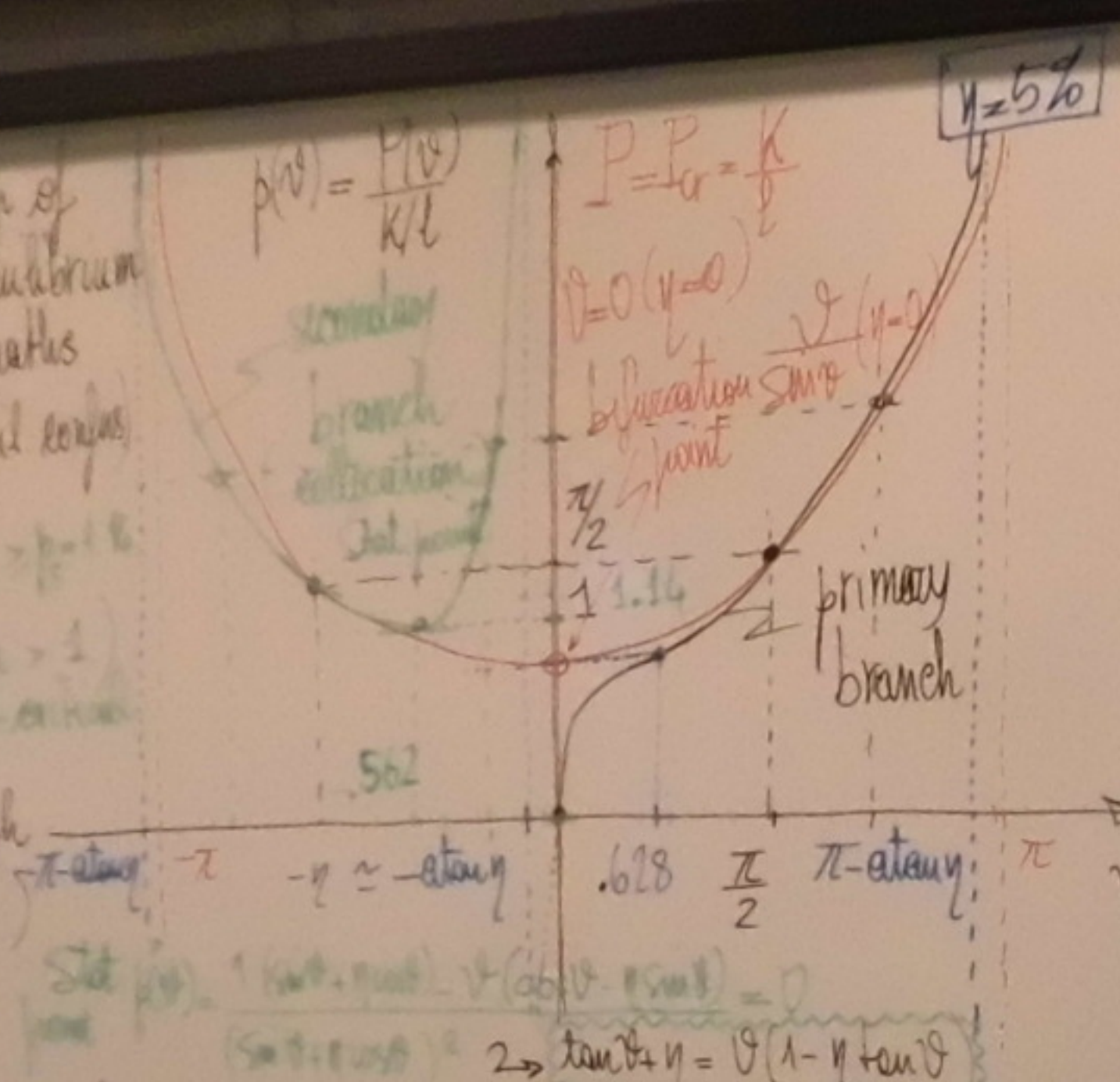
State approach

Map of equilibrium paths (equal length)

$P_L \sin(\theta + \eta \cos \theta) = K v$

$\frac{P}{K} = \frac{v}{\sin(\theta + \eta \cos \theta)}$

equilibrium configurations (through geometric non-linearity or P-Δ effect)



- for $\theta = \pi/2$, no matter the η , one has $p = \pi/2$
- as η keeps small, the primary branch is very much near $\theta = 0$. However it immediately leaves it as soon as $P > 0$
- at $p = 1$, $\theta = \sin \theta + \eta \cos \theta \rightarrow \theta \approx .628 \approx 36^\circ$
- later on, the real system almost follows the bifurcated branch of the ideal one

- bifurcation point no longer appears as $\eta \neq 0$ (real system) and system undergoes a continuous transition from the state $(\theta = 0, P = 0)$ on the side of $e = \eta l$ towards the bifurcated branch of the ideal sys.

- the evaluation of $P = P_{cr}$ in the ideal system keeps of fundamental importance in quantifying the amount of compression load that may be allowed in a real system based on the accepted level of deformation and corresponding load P

" P_{cr} " in a real system may be stated as $\frac{P_{cr, ideal}}{\gamma}$ with γ safety factor $\gamma \approx 2-5$

- Analysis under "geometrically-small" displacements

linearization of the equl eqn $|\eta| \ll 1 \Rightarrow \sin \theta \approx \theta; \cos \theta \approx 1$ (to the 1st order)

$\frac{P}{K} \left(\theta + \frac{\eta}{2} \right) = K v$

$p(\theta) = \frac{\theta}{\theta + \eta} = \frac{1}{1 + \eta/\theta}$

$\theta \rightarrow +\infty, p \rightarrow 1$

secondary branch η

primary branch

"post-critical" $p > 1$

asymptotic to v the bifurcated branch of the ideal system, marking for

Energy approach

Total Potential Energy

$$V(\theta) = \frac{1}{2} k \theta^2 - P \cdot v(\theta)$$

$$V_f = -L_{ef}$$

$$= k \left(\frac{1}{2} \theta^2 - p(1 - \cos \theta + \eta \sin \theta) \right)$$

$$V'(\theta) = k \left(\theta - p(\sin \theta + \eta \cos \theta) \right) \stackrel{S.C.}{=} 0$$

$$\theta = p(\sin \theta + \eta \cos \theta) \text{ equib. config.}$$

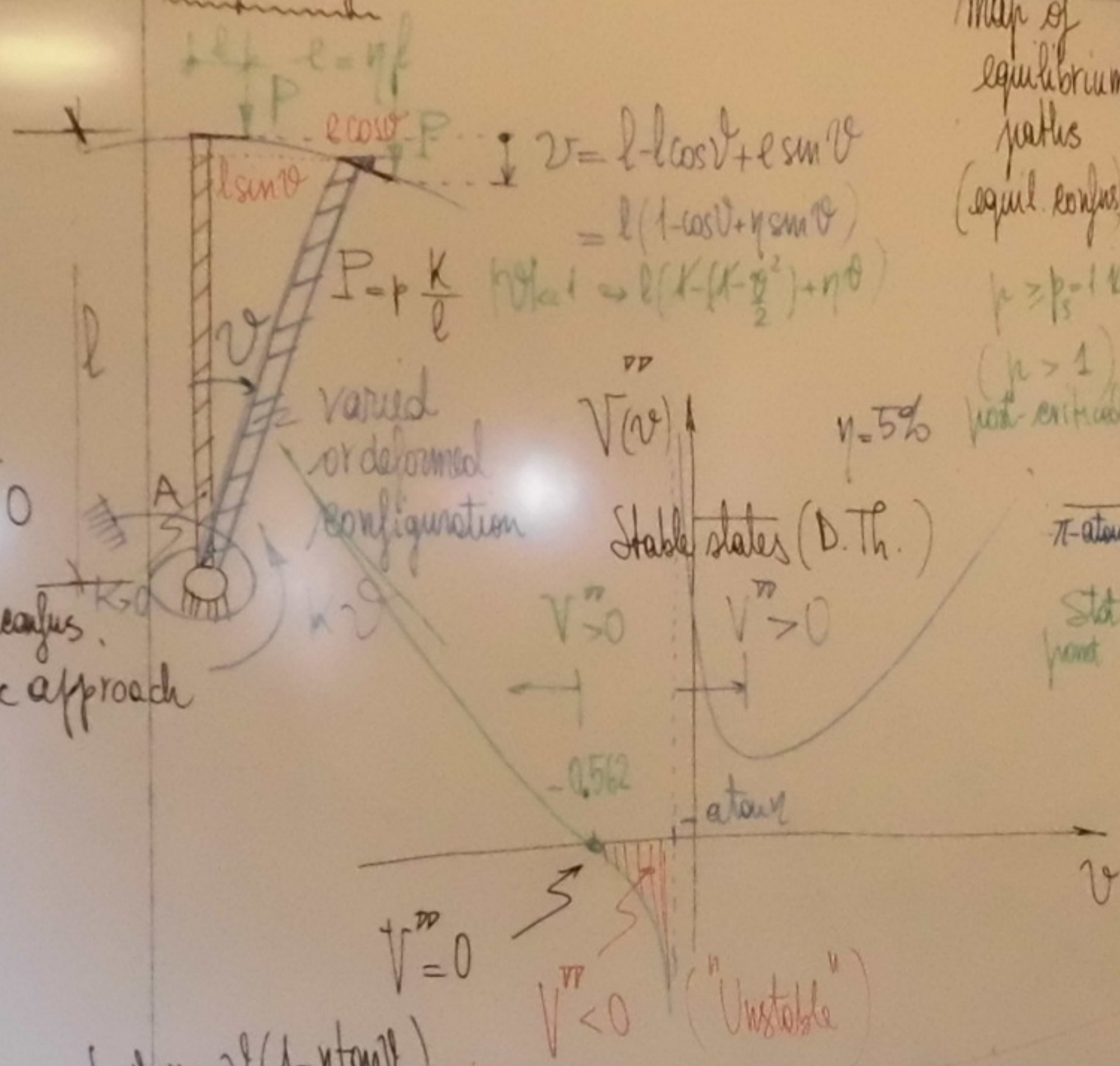
as determined from the static approach

$$V''(\theta) = k(1 - p(\cos \theta - \eta \sin \theta))$$

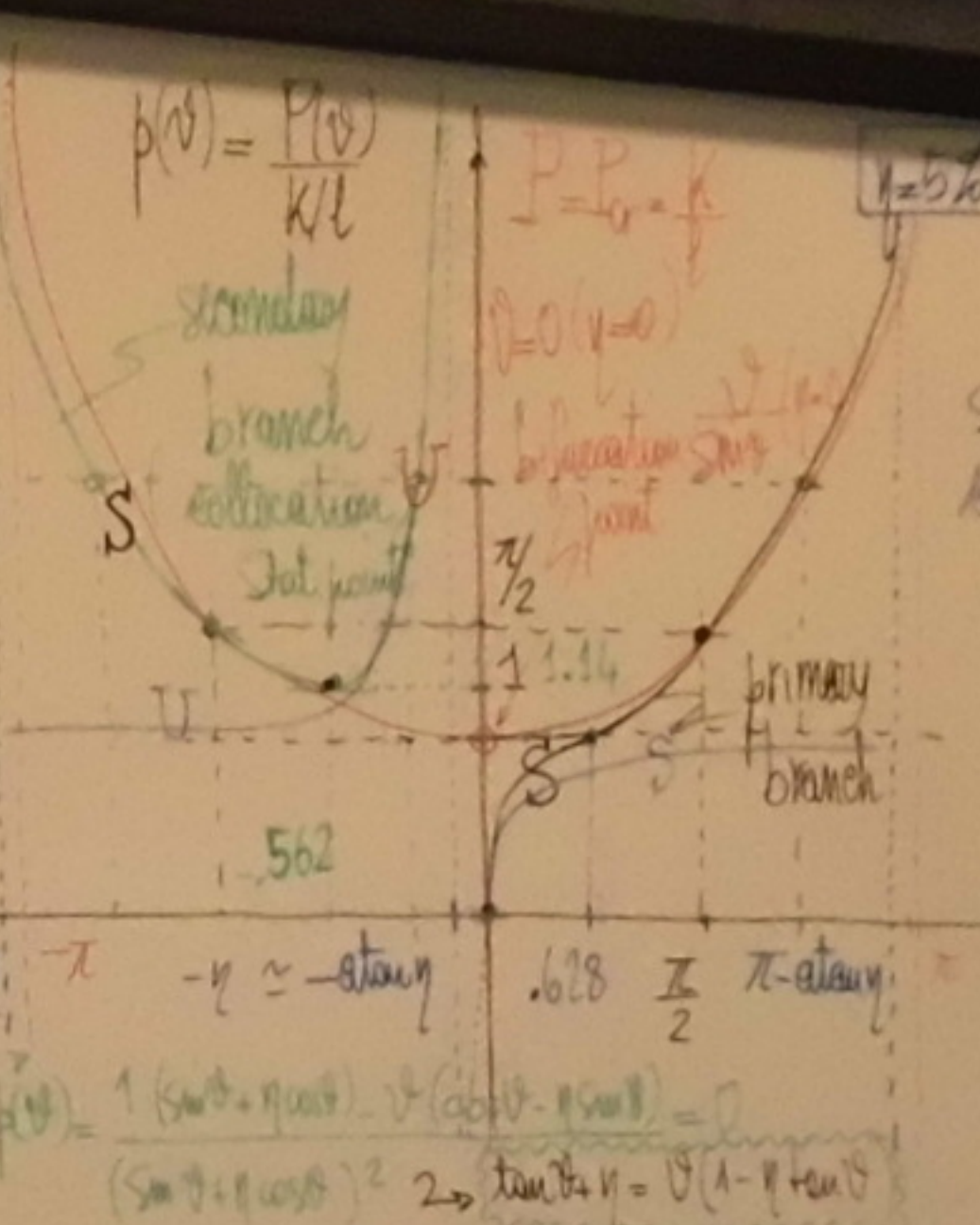
in the equib. config. where

$$p(\theta) = \frac{\theta}{\sin \theta + \eta \cos \theta} = k \left(1 - \frac{\cos \theta - \eta \sin \theta}{\sin \theta + \eta \cos \theta} \right) = 0 \Rightarrow \tan \theta + \eta = \theta(1 - \eta \tan \theta)$$

1-dof example



Map of equilibrium paths (equib. config.)



$$V(\theta) = V_2(\theta) = \frac{1}{2} k \theta^2 - P \cdot v(\theta)$$

$$V_2(\theta) = k \left(\frac{1}{2} \theta^2 - p(1 - \cos \theta + \eta \sin \theta) \right)$$

$$V_2'(\theta) = k(\theta - p(\sin \theta + \eta \cos \theta)) = 0$$

$$\theta = p(\sin \theta + \eta \cos \theta)$$

$$p(\theta) = \frac{\theta}{\sin \theta + \eta \cos \theta}$$

$$p'(\theta) = \frac{1 - \eta \cos \theta}{(\sin \theta + \eta \cos \theta)^2}$$

$$p'(\theta) = 0 \Rightarrow \tan \theta = \eta$$

$$p'(\theta) > 0 \Rightarrow p < 1 \text{ STABLE}$$

$$p'(\theta) = 0 \Rightarrow p = 1$$

$$p'(\theta) < 0 \Rightarrow p > 1$$



Energy approach

Total Potential Energy

$$V(\theta) = \frac{1}{2} k \theta^2 - P v(\theta)$$

$$= k \left(\frac{1}{2} \theta^2 - p \ell (\cos \theta + \eta \sin \theta) \right)$$

$$V'(\theta) = k (\theta - p \sin \theta + \eta \cos \theta) \stackrel{S.C.}{=} 0$$

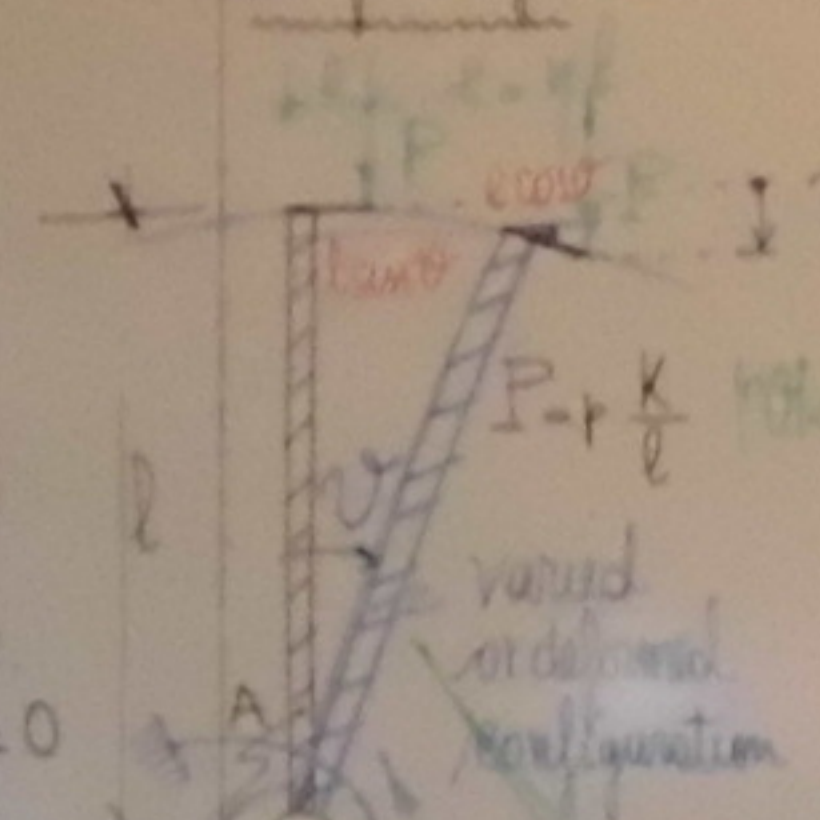
$\theta = p \sin \theta + \eta \cos \theta$ equal radius, as determined from the static approach

$$V''(\theta) = k (1 - p \cos \theta + \eta \sin \theta)$$

on the small angle value

$$V''(0) = k \left(1 - p + \eta \right) = 0 \Rightarrow \tan \theta + \eta = p(1 - \eta \tan \theta)$$

1-dof example



$$v = l \cos \theta + \eta \sin \theta$$

$$= l(1 - \cos \theta + \eta \sin \theta)$$

$$V(\theta) = \frac{1}{2} k (l(1 - \cos \theta + \eta \sin \theta))^2 - P l (1 - \cos \theta + \eta \sin \theta)$$

$$V'(\theta) = k l (1 - \cos \theta + \eta \sin \theta) (1 - \cos \theta + \eta \sin \theta - p) = 0$$

Stable states (D.Th.)

$$V''(\theta) > 0$$

$$V''(\theta) < 0$$

("Unstable")

Map of equilibrium paths (equal radius)

$$p = \frac{P}{P_c} = \frac{P}{k l}$$

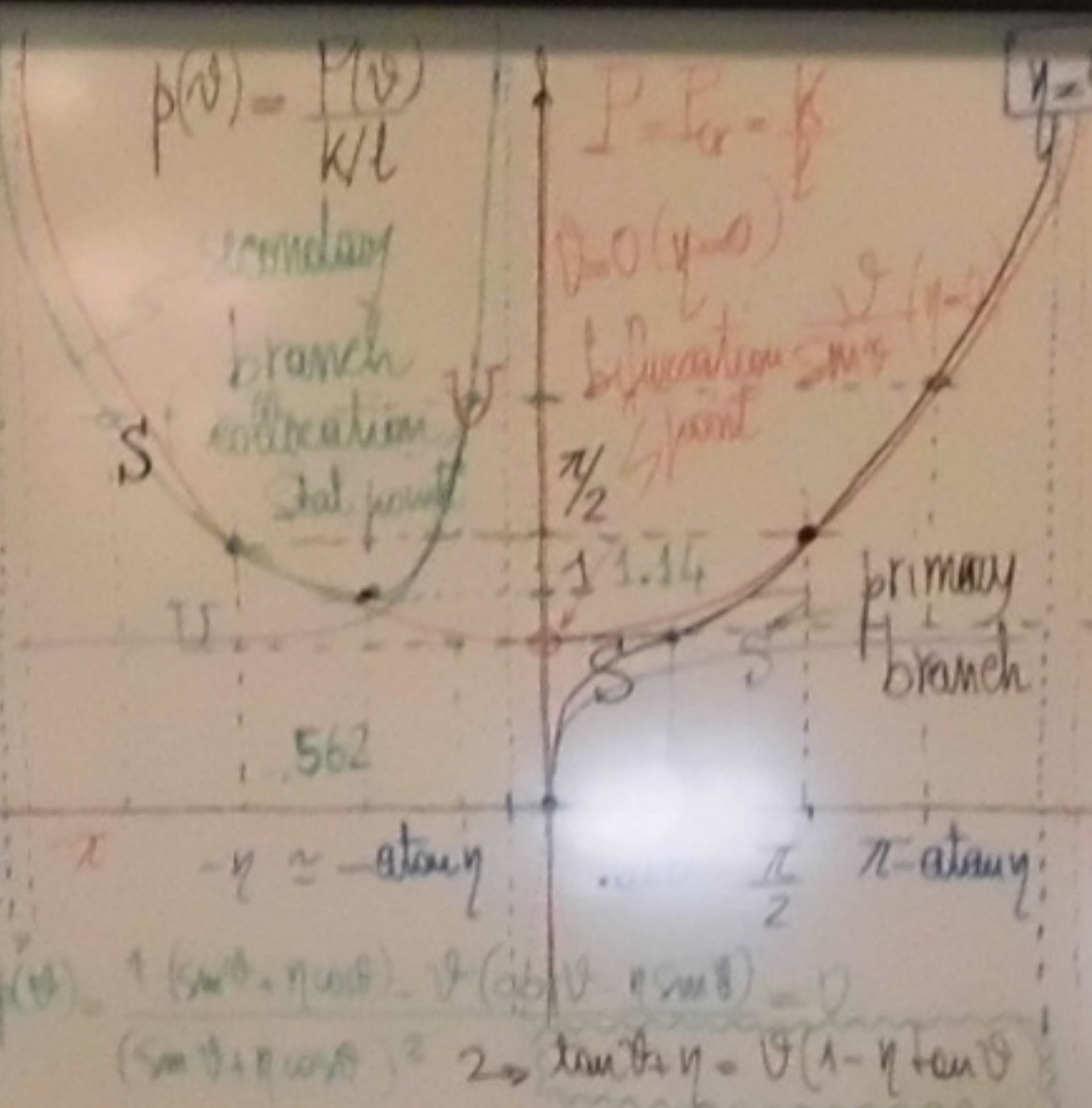
$p > 1$ post-critical

$p < 1$ pre-critical

$p = 1$ critical

$p = 1$ critical

$p = 1$ critical



$$V_2(\theta) = k (\theta - p \theta + \eta) = 0$$

$$\theta = p(\theta + \eta)$$

$$V_2''(\theta) = k(1 - p)$$

$$1 - p \begin{cases} > 0 & p < 1 \text{ STABLE} \\ = 0 & p = 1 \text{ "Unstable"} \\ < 0 & p > 1 \end{cases}$$

$$V(\theta) \approx V_2(\theta) = \frac{1}{2} k \theta^2 - P \theta$$

$$\sin \theta \approx \theta$$

$$\cos \theta \approx 1 - \frac{\theta^2}{2} = \frac{1}{2} k \theta^2 - P \theta$$

$$V_2'(\theta) = k(\theta - p \theta + \eta) = 0$$

$$\theta = p(\theta + \eta)$$

$$V_2''(\theta) = k(1 - p)$$

$$1 - p \begin{cases} > 0 & p < 1 \text{ STABLE} \\ = 0 & p = 1 \text{ "Unstable"} \\ < 0 & p > 1 \end{cases}$$

$$p = 1$$

$$p = 1$$

Analysis under "geometrically small" displacements

$$|\eta| < 1 \Rightarrow \sin \theta \approx \theta; \cos \theta \approx 1 - \frac{\theta^2}{2}$$

$$p(\theta) = \frac{V}{\theta + \eta} = \frac{1}{1 + \eta/\theta}$$

$$p(\theta) = \frac{V}{\theta + \eta} = \frac{1}{1 + \eta/\theta}$$

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