
Instability analysis of real systems
eccentricity e = η l
η = e/l ~ 1%
(1-5%)
imperfection parameter
- Geometrical (or other) imperfections in real systems (for instance eccentricities) may be prone to produce instability effects under compression-like loading
The question arises about how this may affect the instability phenomenon, for instance in terms of allowable compression loading.
1-dof example
P = p K/l
varied or deformed configuration
reference configuration (with geometrical imperfection)
ϑ = 0
Static approach
P(l sin ϑ + e cos ϑ) = K ϑ
Pl/K (sin ϑ + η cos ϑ) = ϑ
p(ϑ) = ϑ / (sin ϑ + η cos ϑ)
equilibrium configurations (through geometric non-linearity or P-Δ effect)
Remarks:
- for η = 0 (ideal system), one gets back the original ideal solution p(ϑ) = ϑ / sin ϑ (particular case)
- for η ≠ 0 (real system) ϑ = 0 is an equilibrium config. only for P = 0
- denominator vanishes for tan ϑ = -η
Map of equilibrium paths (equil. configs)
primary branch
.562
.618
π/2
- for ϑ = ±π/2, no matter the η, one has p = π/2
- at p = 1
P > 0

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1-dof example
e cos ϑ
l sin ϑ
P = p K/l
varied or deformed configuration
reference configuration (with geometrical imperfection)
ϑ = 0
Remarks:
- for η = 0 (ideal system), one gets back the original ideal relation p(ϑ) = ϑ / sin ϑ (particular case)
- for η ≠ 0 (real system) ϑ = 0 is an equilibrium config. only for P = 0
- denominator vanishes for tan ϑ = -η ⇒ ϑ = -π - atan η = - atan η = π - atan η
Static approach
P l (sin ϑ + η cos ϑ) = K ϑ
p(ϑ) = ϑ / (sin ϑ + η cos ϑ)
equilibrium configurations (through geometric non-linearity or P-Δ effect)
map of equilibrium paths (equil. configs)
p(ϑ) = P(ϑ) / (K/l)
secondary branch
bifurcation point
primary branch
η = 5%
π/2
1
1.14
.562
-π - atan η
-η ≃ -atan η
.628
π/2
π - atan η
ϑ
Stat point
tan ϑ + η = ϑ (1 - η tan ϑ)
- for ϑ = ±π/2, no matter the η, one has p = π/2
- as ϑ keeps small, the primary branch is very much near ϑ = 0. However it immediately leaves it as soon as P > 0
- at p = 1, ϑ = sin ϑ + η cos ϑ → ϑ ≃ .628 ≃ 36°
- later on, the real system almost follows the bifurcated branch of the ideal one
- bifurcation point no longer appears as η ≠ 0 (real system) and system undergoes a continuous transition from the state (ϑ = 0, P = 0) on the side of e = η l, towards the bifurcated branch of the ideal sys.
- the evaluation of P = Pcr in the ideal system keeps of fundamental importance in quantifying the amount of compression load that may be allowed in a real system based on the accepted level of deformation and corresponding load
f safety factor
f = 2-5
- Analysis under geometrically-small displacements
|ϑ| << 1 ⇒
linearization of the equil. eqn.
= K ϑ
p(ϑ) = ϑ / (ϑ + η) = 1 / (1 + η/ϑ)
ϑ → ±∞, p → 1
p(ϑ)
secondary branch
primary branch
1
ϑ

1-dof example
Static approach
map of equilibrium paths (equil. configs)
$\frac{Pl}{k}(\sin\vartheta + \eta\cos\vartheta) = \vartheta$
$p(\vartheta) = \frac{\vartheta}{\sin\vartheta + \eta\cos\vartheta}$
equilibrium configurations (through geometric non-linearity or P-Δ effect)
$\eta = 5\%$
$p(\vartheta) = \frac{P(\vartheta)}{k/l}$
$P = P_{cr} = \frac{k}{l}$
$\vartheta = 0\ (\eta = 0)$
bifurcation point
$\frac{\vartheta}{\sin\vartheta}\ (\eta = 0)$
secondary branch
primary branch
$\frac{\pi}{2}$
1
1.14
.562
$-\pi-\mathrm{atan}\eta$
$-\eta \simeq -\mathrm{atan}\eta$
.628
$\pi-\mathrm{atan}\eta$
π
ϑ
$\Rightarrow \tan\vartheta + \eta = \vartheta(1 - \eta\tan\vartheta)$
- for $\vartheta = \pm\pi/2$, no matter the η, one has $p = \pi/2$
- as ϑ keeps small, the primary branch is very much near $\vartheta = 0$. However it immediately leaves it as soon as $P > 0$
- at $p = 1$: $\vartheta = \sin\vartheta + \eta\cos\vartheta \rightarrow \vartheta \simeq .628 \simeq 36°$
- later on, the real system almost follows the bifurcated branch of the ideal one.
- bifurcation point no longer appears as $\eta \neq 0$ (real system) and system undergoes a continuous transition from the state ($\vartheta = 0, P = 0$) on the side of $e = \eta l$, towards the bifurcated branch of the ideal sys.
- the evaluation of $P = P_{cr}$ in the ideal system keeps of fundamental importance in quantifying the amount of compression load that may be allowed in a real system based on the accepted level of deformation and corresponding load P. "P_{cr}" in a real system may be stated as $\frac{P_{cr}^{ideal}}{f}$
f: safety factor $f \simeq 2 \div 5$
- Analysis under "geometrically-small" displacements
linearization of the equil eqn
$|\vartheta| \ll 1 \Rightarrow \sin\vartheta \simeq \vartheta$; $\cos\vartheta \simeq 1$ (to the 1st order)
$\frac{Pl}{k}\left(\vartheta + \frac{e}{l}\right) = \vartheta$
$\frac{e}{l} = \eta$
$p(\vartheta) = \frac{\vartheta}{\vartheta + \eta} = \frac{1}{1 + \eta/\vartheta}$
$\vartheta \rightarrow \pm\infty,\ p \rightarrow 1$
$p(\vartheta)$
secondary branch
primary branch
1
$-\eta$
ϑ small
"post-critical" $p > 1$
ϑ
asymptotic to the bifurcated branch of the ideal system, marking P_{cr}


Energy approach
Total Potential Energy
1-dof example
varied or deformed configuration
equil. configs.
as determined from the static approach
Stable states (D.Th.)
"Unstable"
η=5%
Map of equilibrium paths (equil. configs)
secondary branch
bifurcation
primary branch
STABLE

Energy approach
Total Potential Energy
1 dof example
Stable states (D.Th.)
"Unstable"
Map of equilibrium paths
primary branch
STABLE
Analysis under geometrically small displacements


Analysis under "geometrically-small" displacements
linearization of the equil eqn
|ϑ| << 1 ⇒ sin ϑ ≃ ϑ ; cos ϑ ≃ 1 (to the 1st order)
p(ϑ) = ϑ / (ϑ + η) = 1 / (1 + η/ϑ)
ϑ → ±∞, p → 1
primary branch
secondary branch
"post-critical" p > 1
ϑ small
asymptotic to ϑ the bifurcated branch of the ideal system, marking pcr
V(ϑ) ≃ V2(ϑ) = 1/2 Kϑ² − Pϑ
sin ϑ ≃ ϑ
cos ϑ ≃ 1 − ϑ²/2
= 1/2 Kϑ² − Pl (ϑ²/2 + ηϑ)
V2'(ϑ) = K(ϑ − p(ϑ + η)) = 0
ϑ − p(ϑ + η)
V2''(ϑ) = K(1 − p)
> 0 p < 1 STABLE
1 − p = 0 p = 1 "Unstable"
< 0 p > 1
primary branch
Map of equilibrium paths (equil. config.)
η = 5%
π/2
.628
π − atan η
secondary branch
bifurcation point
Energy approach
stable states