Università degli Studi di Bergamo

Dipartimento di Ingegneria

A.A. 2012-2013

Corso di Meccanica dei Solidi e delle Strutture

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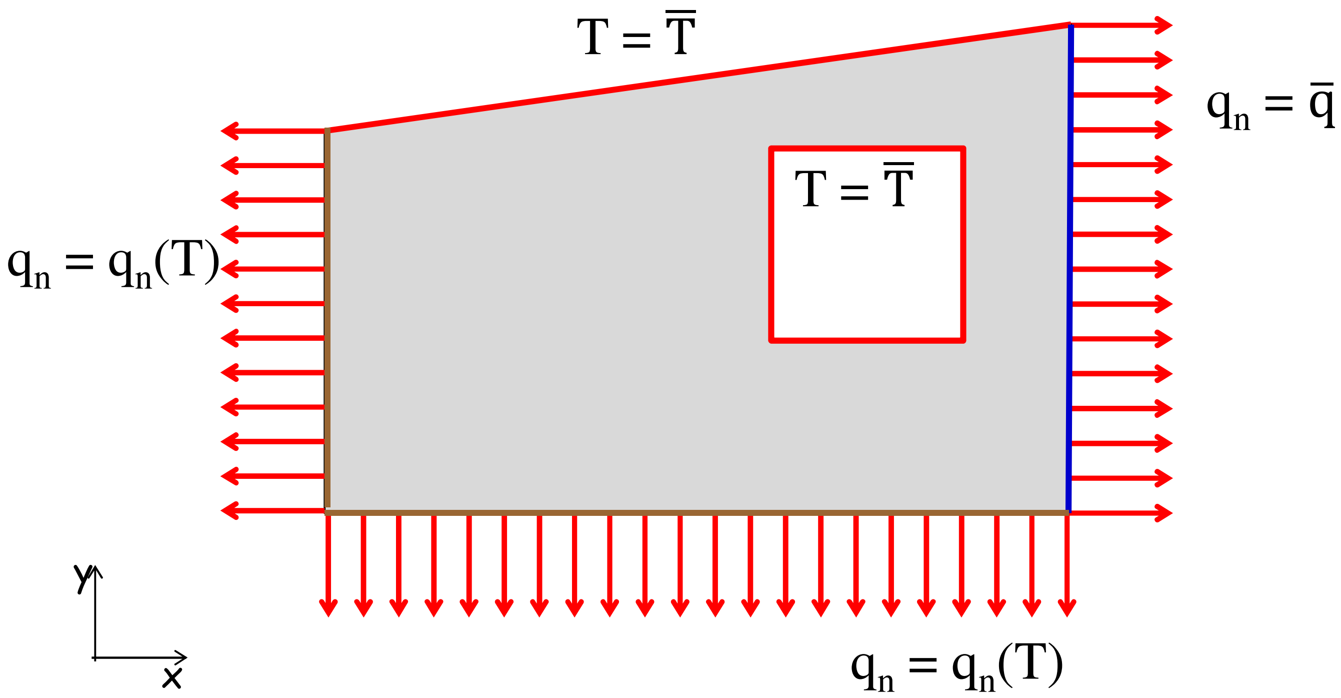
Analisi di problemi termici piani mediante elementi finiti isoparametrici quadrilateri ("a 4 nodi")



Equazioni governanti il problema termico

$$\boxed{\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = \dot{Q} - c\,\rho\,\dot{T}}$$

(conservazione energia/potenza)

in cui:

$$\underline{\mathbf{q}} = \begin{bmatrix} q_x \\ q_y \end{bmatrix}$$ = flusso termico [W/m^2]

$\dot{Q}$ = potenza specifica immessa nel solido [W/m^3]

c = calore specifico [J/kg/K]

ρ = massa specifica [kg/m^3]

T = campo termico [K]

Equazioni governanti il problema termico

$$\boxed{\underline{\mathbf{q}} = -\underline{\underline{\mathbf{k}}}\,\underline{\nabla}T} \quad \text{(legge di Fourier)}$$

in cui:

$$\underline{\underline{\mathbf{k}}} = \begin{bmatrix} k_{xx} & k_{xy} \\ k_{xy} & k_{yy} \end{bmatrix} = \underline{\underline{\mathbf{k}}}^T = \text{matrice dei coefficienti di conducibilità [W/m/K]}$$

$$\underline{\nabla}T = \begin{bmatrix} \dfrac{\partial T}{\partial x} \\ \dfrac{\partial T}{\partial y} \end{bmatrix}$$

Equazioni governanti il problema termico

Condizioni al contorno considerate:

$q_n = \bar{q}$ flusso termico assegnato sul contorno S_q

$T = \bar{T}$ temperatura assegnata sul contorno S_T

$q_n = h\,(T - T_\infty)$ flusso convettivo sul contorno S_h

in cui:

$q_n = q_x\, n_x + q_y\, n_y =$ componente del flusso normale al contorno

h = coefficiente di convezione [W/m^2/K]

T_∞ = temperatura del fluido che lambisce la superficie

Equazioni governanti il problema termico

$$
\begin{cases}
\dfrac{\partial q_x}{\partial x} + \dfrac{\partial q_y}{\partial y} = \dot{Q} - c\,\rho\,\dot{T} \\[2ex]
\underline{\mathbf{q}} = -\underline{\underline{\mathbf{k}}}\,\underline{\nabla} T \\[2ex]
q_n = \bar{q} \qquad \text{su} \quad \mathrm{S_q} \\[2ex]
T = \bar{\bar{T}} \qquad \text{su} \quad \mathrm{S_T} \\[2ex]
q_n = h\,(T - T_\infty) \qquad \text{su} \quad \mathrm{S_h} \\[2ex]
T(x, y, t{=}0) = T_0(x, y) \quad \text{in} \;\; \mathrm{V} \quad \text{(condizione iniziale)}
\end{cases}
$$

Formulazione "debole" del problema termico

$f\,(x,y)$ = "funzione test" (arbitraria, purché sufficientemente regolare)

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} - \dot{Q} + c\,\rho\,\dot{T} = 0$$

$$\Downarrow$$

$$\boxed{\int_V f(\underline{\mathbf{x}})\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} - \dot{Q} + c\,\rho\,\dot{T}\right) dV = 0 \qquad \forall f(\underline{\mathbf{x}})}$$

Formulazione "debole" del problema termico

Applicando il lemma di Green si può scrivere:

$$\int_V f(\underline{\mathbf{x}})\frac{\partial q_x}{\partial x}dV \equiv \int_S f(\underline{\mathbf{x}})\,q_x\,n_x dS - \int_V \frac{\partial f}{\partial x}q_x dV$$

$$\int_V f(\underline{\mathbf{x}})\frac{\partial q_y}{\partial y}dV \equiv \int_S f(\underline{\mathbf{x}})\,q_y\,n_y dS - \int_V \frac{\partial f}{\partial y}q_y dV$$

da cui si ottiene:

$$\int_V f(\underline{\mathbf{x}})\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}\right)dV=\int_S f(\underline{\mathbf{x}})\underbrace{\left(q_x\,n_x+q_y\,n_y\right)}_{q_n}dS-\int_V \underbrace{\frac{\partial f}{\partial x}q_x+\frac{\partial f}{\partial y}q_y}_{(\underline{\nabla} f)^T\underline{\mathbf{q}}}\,dV$$

Formulazione "debole" del problema termico

Ne consegue che l'equazione di conservazione si può scrivere come:

$$\int_S f\, q_n dS - \int_V \left(\underline{\nabla} f\right)^T \underline{\mathbf{q}}\, dV + \int_V f(\underline{\mathbf{x}})\left(-\dot{Q} + c\,\rho\,\dot{T}\right) dV = 0 \qquad \forall f(\underline{\mathbf{x}})$$

Inserendo le restanti equazioni si ricava l'equazione governante il problema termico in forma debole:

$$\int_{S_q} f\, \overline{q} dS + \int_{S_h} f\, h\left(T - T_\infty\right) dS + \int_{S_T} f\, q_n dS + \int_V \left(\underline{\nabla} f\right)^T \underline{\underline{\mathbf{k}}}\, \underline{\nabla} T\, dV +$$

$$+ \int_V \left(-\dot{Q} + c\,\rho\,\dot{T}\right) f\, dV = 0 \qquad \forall f(\underline{\mathbf{x}})$$

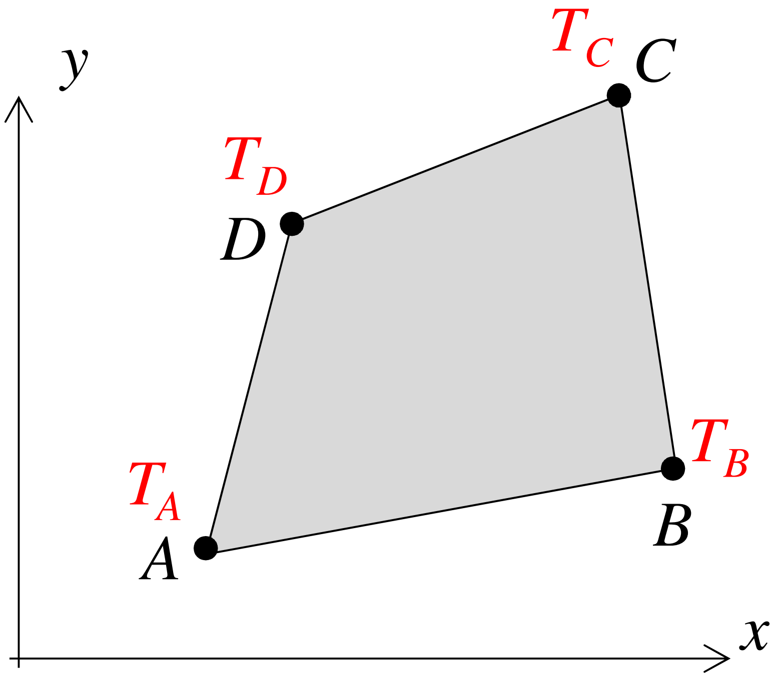
Formulazione "debole" del problema termico

Ri-ordinando i termini si ottiene:

$$\int_V f\, c\,\rho\,\dot{T} dV + \int_V \left(\underline{\nabla} f\right)^T \underline{\underline{\mathbf{k}}}\,\underline{\nabla} T\, dV + \int_{S_h} f\, h T dS =$$

$$= \int_V f\, \dot{Q} dV - \int_{S_q} f\, \overline{q} dS + \int_{S_h} f\, h T_\infty dS - \int_{S_T} f\, q_n dS \qquad \forall f\left(\underline{\mathbf{x}}\right)$$

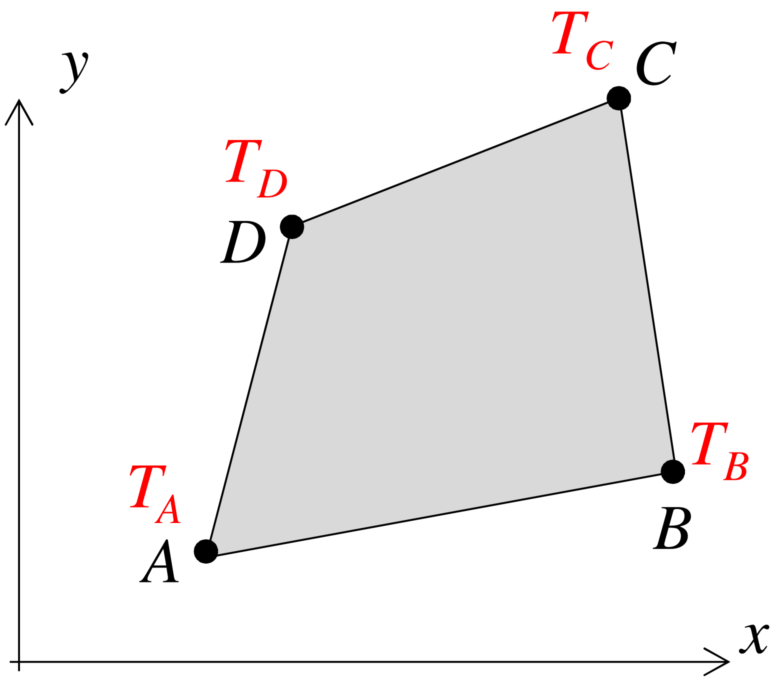
Elemento finito quadrilatero ("a 4 nodi")



$$T\left(x,y,t\right)=T\left(\underline{\mathbf{x}},t\right)=\Psi_{\mathrm{A}}\left(\underline{\mathbf{x}}\right)T_{\mathrm{A}}\left(t\right)+\Psi_{\mathrm{B}}\left(\underline{\mathbf{x}}\right)T_{\mathrm{B}}\left(t\right)+$$

$$+\Psi_{\mathrm{C}}\left(\underline{\mathbf{x}}\right)T_{\mathrm{C}}\left(t\right)+\Psi_{\mathrm{D}}\left(\underline{\mathbf{x}}\right)T_{\mathrm{D}}\left(t\right)$$

Elemento finito quadrilatero ("a 4 nodi")

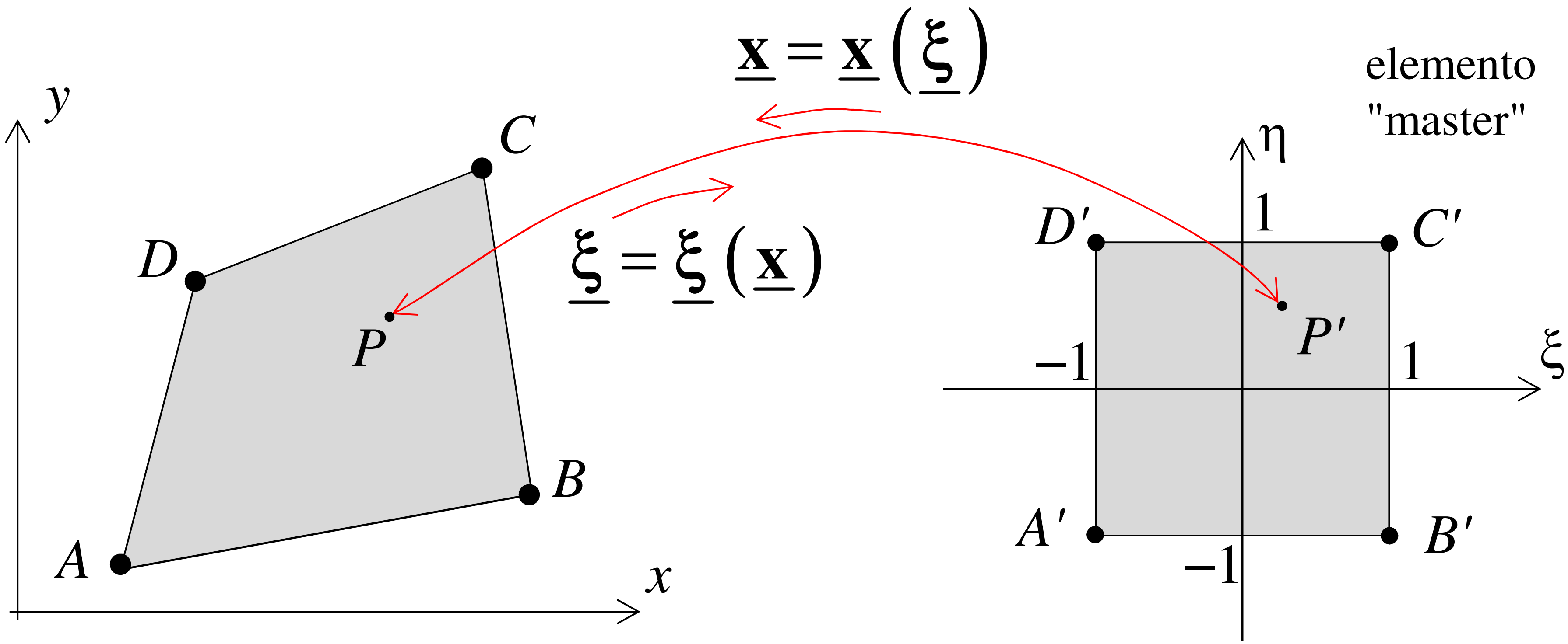


$$\underline{\Psi} = \begin{bmatrix} \Psi_A(\underline{\mathbf{x}}) \\ \Psi_B(\underline{\mathbf{x}}) \\ \Psi_C(\underline{\mathbf{x}}) \\ \Psi_D(\underline{\mathbf{x}}) \end{bmatrix}$$

$$\underline{\mathbf{T}} = [T_A, T_B, T_C, T_D]^T$$

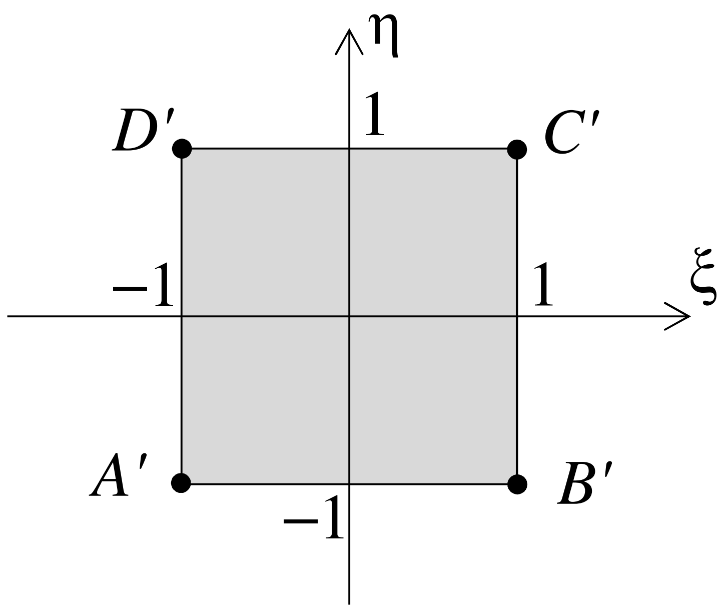
$$\boxed{T(x, y, t) = \underline{\Psi}^T(\underline{\mathbf{x}})\,\underline{\mathbf{T}}(t)}$$

Descrizione della geometria ("mapping")



$$\begin{cases} x = x(\xi, \eta) \\ y = y(\xi, \eta) \end{cases} \quad \Rightarrow \quad \underline{\mathbf{x}} = \underline{\mathbf{x}}(\underline{\xi})$$

Funzioni di forma dell'elemento "master"



$$\varphi_A(\underline{\xi}) = (1-\xi)(1-\eta)/4$$

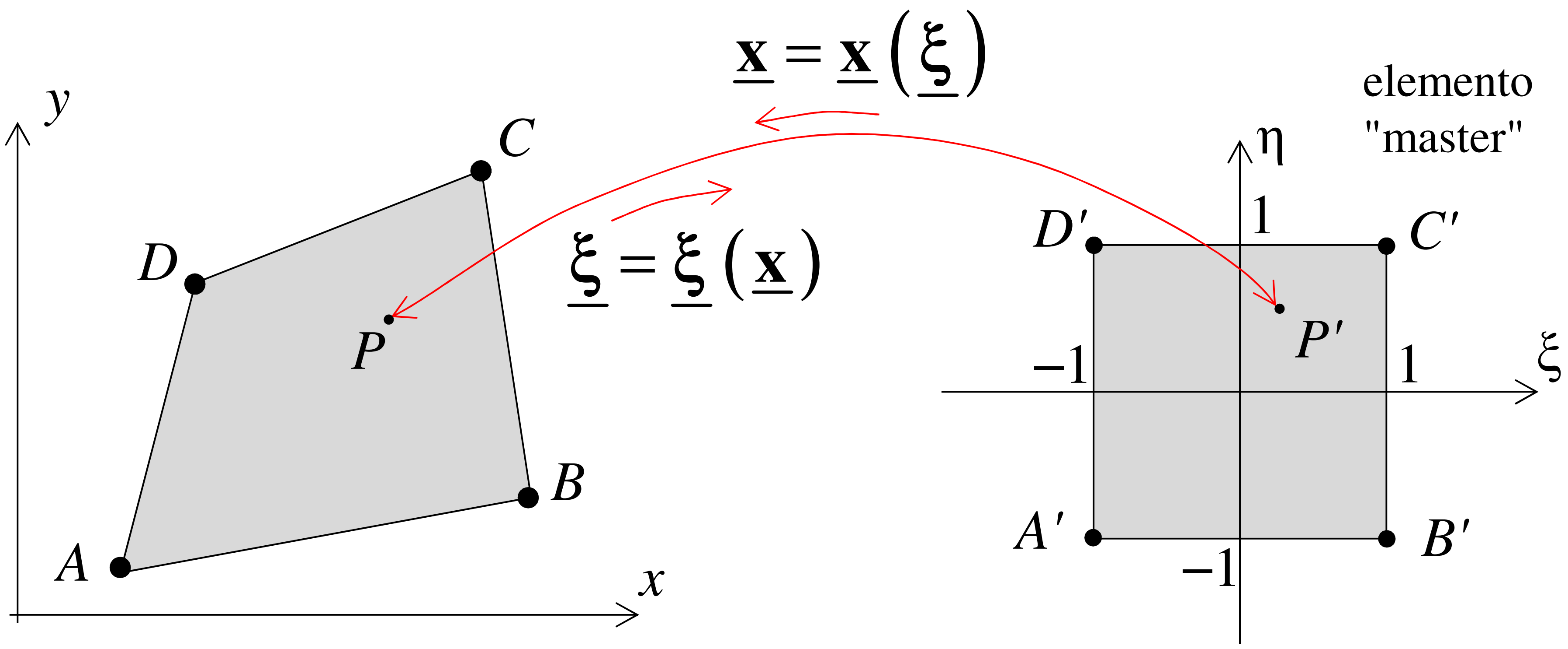
$$\varphi_B(\underline{\xi}) = (1+\xi)(1-\eta)/4$$

$$\varphi_C(\underline{\xi}) = (1+\xi)(1+\eta)/4$$

$$\varphi_D(\underline{\xi}) = (1-\xi)(1+\eta)/4$$

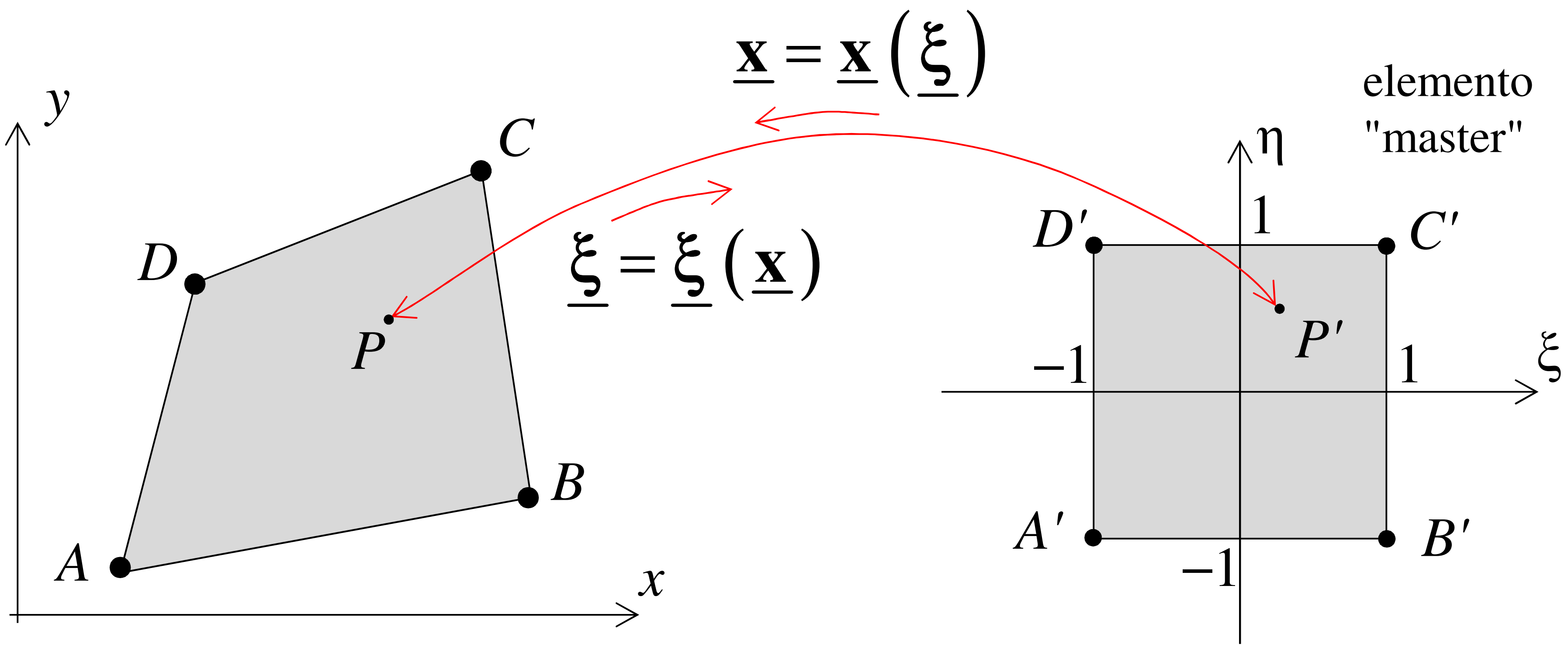
$$\underline{\varphi} = \left[\varphi_A(\underline{\xi}),\ \varphi_B(\underline{\xi}),\ \varphi_C(\underline{\xi}),\ \varphi_D(\underline{\xi})\right]^T$$

Funzioni di forma dell'elemento quadrilatero



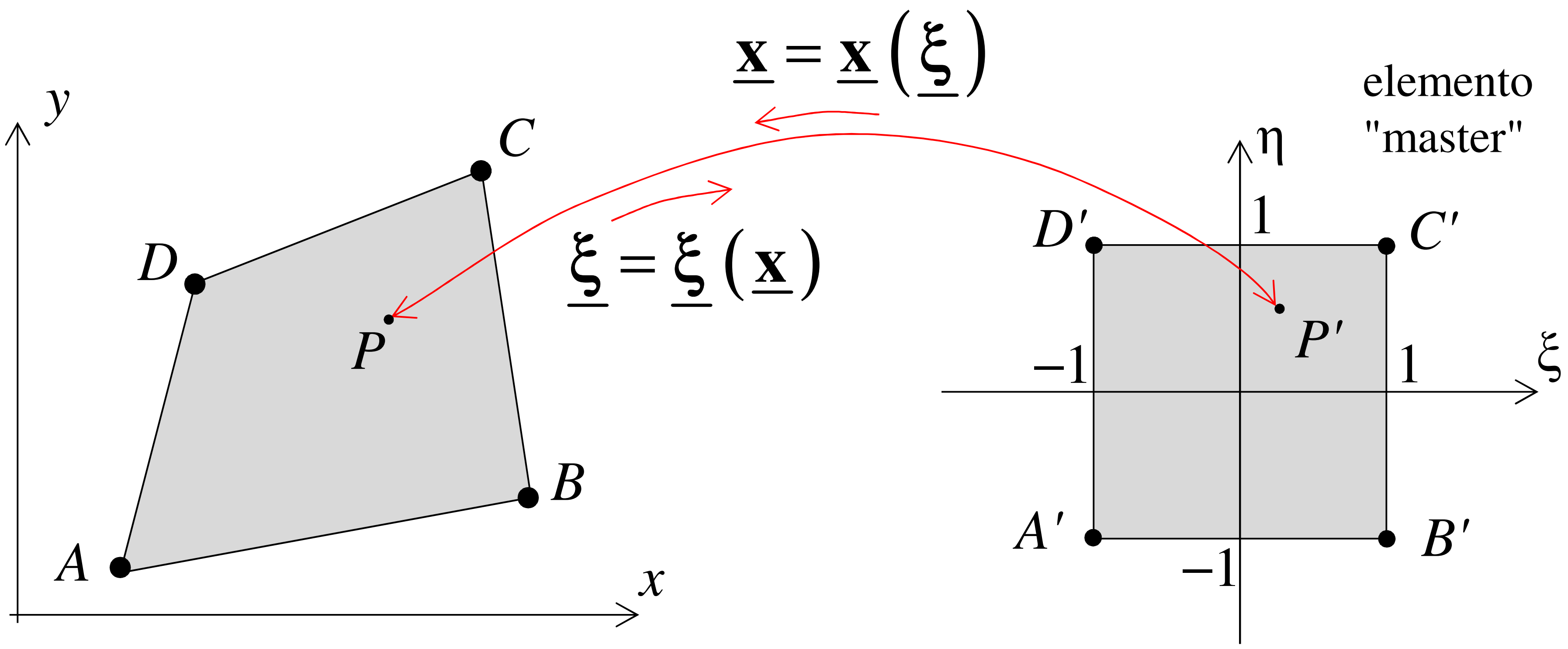
$$\Psi_j(\underline{\mathbf{x}}) \triangleq \varphi_j\left[\underline{\xi}(\underline{\mathbf{x}})\right] \quad j = A, B, C, D \qquad \Rightarrow \qquad \boxed{\underline{\Psi}(\underline{\mathbf{x}}) \triangleq \underline{\varphi}\left[\underline{\xi}(\underline{\mathbf{x}})\right]}$$

Descrizione della geometria ("mapping")



$$\underline{\mathbf{x}} = \underline{\mathbf{x}}(\underline{\xi}) \iff \begin{cases} x = x(\xi,\eta) = \varphi_A(\underline{\xi})\, x_A + \varphi_B(\underline{\xi})\, x_B + \varphi_C(\underline{\xi})\, x_C + \varphi_D(\underline{\xi})\, x_D \\ y = y(\xi,\eta) = \varphi_A(\underline{\xi})\, y_A + \varphi_B(\underline{\xi})\, y_B + \varphi_C(\underline{\xi})\, y_C + \varphi_D(\underline{\xi})\, y_D \end{cases}$$

Descrizione della geometria ("mapping")



$$\left.\begin{aligned}\underline{\mathbf{X}} &\triangleq [x_A, x_B, x_C, x_D]^T \\ \underline{\mathbf{Y}} &\triangleq [y_A, y_B, y_C, y_D]^T\end{aligned}\right\} \Rightarrow \quad \underline{\mathbf{x}} = \underline{\mathbf{x}}(\underline{\xi}) = \begin{cases} x = x(\xi, \eta) = \underline{\varphi}^T(\underline{\xi})\,\underline{\mathbf{X}} \\ y = y(\xi, \eta) = \underline{\varphi}^T(\underline{\xi})\,\underline{\mathbf{Y}} \end{cases}$$

Descrizione della geometria ("mapping")

$$\begin{cases} dx = \dfrac{\partial x(\xi,\eta)}{\partial \xi} d\xi + \dfrac{\partial x(\xi,\eta)}{\partial \eta} d\eta \\[2ex] dy = \dfrac{\partial y(\xi,\eta)}{\partial \xi} d\xi + \dfrac{\partial y(\xi,\eta)}{\partial \eta} d\eta \end{cases} \Rightarrow \begin{bmatrix} dx \\ dy \end{bmatrix} = \underbrace{\left[\begin{array}{c:c} \dfrac{\partial x(\xi,\eta)}{\partial \xi} & \dfrac{\partial x(\xi,\eta)}{\partial \eta} \\[2ex] \hdashline \dfrac{\partial y(\xi,\eta)}{\partial \xi} & \dfrac{\partial y(\xi,\eta)}{\partial \eta} \end{array}\right]}_{\underline{\underline{\mathbf{j}}} = \text{matrice jacobiana della trasformazione}} \begin{bmatrix} d\xi \\ d\eta \end{bmatrix}$$

$$\det\left(\underline{\underline{\mathbf{j}}}\right) = \underline{\mathbf{X}}^T \underline{\underline{\mathbf{Q}}}\ \underline{\mathbf{Y}}, \qquad \underline{\underline{\mathbf{Q}}} \triangleq \frac{\partial \underline{\varphi}(\underline{\xi})}{\partial \xi} \frac{\partial \underline{\varphi}^T(\underline{\xi})}{\partial \eta} - \frac{\partial \underline{\varphi}(\underline{\xi})}{\partial \eta} \frac{\partial \underline{\varphi}^T(\underline{\xi})}{\partial \xi} = -\underline{\underline{\mathbf{Q}}}^T$$

Gradienti e matrice di compatibilità

$$\frac{\partial T}{\partial x} = \frac{\partial \underline{\mathbf{\Psi}}^T(\underline{\mathbf{x}})}{\partial x}\underline{\mathbf{T}}, \qquad \frac{\partial T}{\partial y} = \frac{\partial \underline{\mathbf{\Psi}}^T(\underline{\mathbf{x}})}{\partial y}\underline{\mathbf{T}}$$

$$\begin{bmatrix} \dfrac{\partial T}{\partial x} \\ \hdashline \dfrac{\partial T}{\partial y} \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial \underline{\mathbf{\Psi}}^T(\underline{\mathbf{x}})}{\partial x} \\ \hdashline \dfrac{\partial \underline{\mathbf{\Psi}}^T(\underline{\mathbf{x}})}{\partial y} \end{bmatrix}}_{\underline{\underline{\mathbf{B}}}}\underline{\mathbf{T}}$$

Matrice di compatibilità

$$\frac{\partial \underline{\varphi}^T(\underline{\xi})}{\partial \xi} = \frac{\partial \underline{\Psi}^T\left[\underline{\mathbf{x}}(\underline{\xi})\right]}{\partial \xi} = \frac{\partial \underline{\Psi}^T}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial \underline{\Psi}^T}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial \underline{\varphi}^T(\underline{\xi})}{\partial \eta} = \frac{\partial \underline{\Psi}^T\left[\underline{\mathbf{x}}(\underline{\xi})\right]}{\partial \eta} = \frac{\partial \underline{\Psi}^T}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial \underline{\Psi}^T}{\partial y}\frac{\partial y}{\partial \eta}$$

$$\begin{bmatrix} \dfrac{\partial \underline{\varphi}^T(\underline{\xi})}{\partial \xi} \\[2ex] \dfrac{\partial \underline{\varphi}^T(\underline{\xi})}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\[2ex] \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \underline{\Psi}^T}{\partial x} \\[2ex] \dfrac{\partial \underline{\Psi}^T}{\partial y} \end{bmatrix}$$

Matrice di compatibilità

$$\begin{bmatrix} \dfrac{\partial \underline{\boldsymbol{\Psi}}^T}{\partial x} \\ \dfrac{\partial \underline{\boldsymbol{\Psi}}^T}{\partial y} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{\partial \underline{\boldsymbol{\varphi}}^T(\underline{\xi})}{\partial \xi} \\ \dfrac{\partial \underline{\boldsymbol{\varphi}}^T(\underline{\xi})}{\partial \eta} \end{bmatrix} = \frac{1}{\det\left(\underline{\underline{\mathbf{j}}}^T\right)} \begin{bmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial y}{\partial \xi} \\ -\dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \xi} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \underline{\boldsymbol{\varphi}}^T(\underline{\xi})}{\partial \xi} \\ \dfrac{\partial \underline{\boldsymbol{\varphi}}^T(\underline{\xi})}{\partial \eta} \end{bmatrix}$$

$$\Rightarrow \begin{cases} \dfrac{\partial \underline{\boldsymbol{\Psi}}^T}{\partial x} = \dfrac{1}{\det\left(\underline{\underline{\mathbf{j}}}^T\right)} \left[\dfrac{\partial y}{\partial \eta} \dfrac{\partial \underline{\boldsymbol{\varphi}}^T(\underline{\xi})}{\partial \xi} - \dfrac{\partial y}{\partial \xi} \dfrac{\partial \underline{\boldsymbol{\varphi}}^T(\underline{\xi})}{\partial \eta} \right] = -\dfrac{\underline{\mathbf{Y}}^T \underline{\underline{\mathbf{Q}}}}{\underline{\mathbf{X}}^T \underline{\underline{\mathbf{Q}}}\,\underline{\mathbf{Y}}} \\[2ex] \dfrac{\partial \underline{\boldsymbol{\Psi}}^T}{\partial y} = \dfrac{1}{\det\left(\underline{\underline{\mathbf{j}}}^T\right)} \left[-\dfrac{\partial x}{\partial \eta} \dfrac{\partial \underline{\boldsymbol{\varphi}}^T(\underline{\xi})}{\partial \xi} + \dfrac{\partial x}{\partial \xi} \dfrac{\partial \underline{\boldsymbol{\varphi}}^T(\underline{\xi})}{\partial \eta} \right] = \dfrac{\underline{\mathbf{X}}^T \underline{\underline{\mathbf{Q}}}}{\underline{\mathbf{X}}^T \underline{\underline{\mathbf{Q}}}\,\underline{\mathbf{Y}}} \end{cases}$$

Matrice di compatibilità

$$\begin{bmatrix} \dfrac{\partial T}{\partial x} \\ \hdashline \dfrac{\partial T}{\partial y} \end{bmatrix} = \frac{1}{\underline{\mathbf{X}}^T \underline{\underline{\mathbf{Q}}}\,\underline{\mathbf{Y}}} \begin{bmatrix} -\underline{\mathbf{Y}}^T \underline{\underline{\mathbf{Q}}} \\ \hdashline \underline{\mathbf{X}}^T \underline{\underline{\mathbf{Q}}} \end{bmatrix} \underline{\mathbf{T}} = \frac{1}{\underline{\mathbf{X}}^T \underline{\underline{\mathbf{Q}}}\,\underline{\mathbf{Y}}} \begin{bmatrix} -\underline{\mathbf{Y}}^T \\ \hdashline \underline{\mathbf{X}}^T \end{bmatrix} \underline{\underline{\mathbf{Q}}}\,\underline{\mathbf{T}} = \underline{\underline{\mathbf{B}}}\left(\underline{\xi}\right) \underline{\mathbf{T}}(t)$$

$$\boxed{\underline{\underline{\mathbf{B}}}\left(\underline{\xi}\right) = \frac{1}{\underline{\mathbf{X}}^T \underline{\underline{\mathbf{Q}}}\,\underline{\mathbf{Y}}} \begin{bmatrix} -\underline{\mathbf{Y}}^T \\ \hdashline \underline{\mathbf{X}}^T \end{bmatrix} \underline{\underline{\mathbf{Q}}}}$$

Formulazione "debole" del problema termico

Dalla slide numero 9:

$$
\boxed{
\begin{aligned}
&\int_V f\, c\rho \dot{T} dV + \int_V \left(\underline{\nabla} f\right)^T \underline{\underline{\mathbf{k}}}\, \underline{\nabla} T\, dV + \int_{S_h} f\, h T dS = \\
&\qquad = \int_V f\, \dot{Q} dV - \int_{S_q} f\, \bar{q} dS + \int_{S_h} f\, h T_\infty dS - \int_{S_T} f\, q_n dS \qquad \forall f\left(\underline{\mathbf{x}}\right)
\end{aligned}
}
$$

Formulazione "debole" del problema termico

Introducendo una suddivisione ("discretizzazione") in N_{EF} elementi finiti si ha:

$$T_e(x,y,t) = \underline{\boldsymbol{\Psi}}_e^T(\underline{\mathbf{x}})\,\underline{\mathbf{T}}_e(t)$$

$$\sum_{e=1}^{N_{EF}} \int_{V_e} f\, c\,\rho\, \dot{T}_e\, dV + \sum_{e=1}^{N_{EF}} \int_{V_e} (\underline{\nabla} f)^T \underline{\underline{\mathbf{k}}}\, \underline{\nabla} T_e\, dV + \sum_{e=1}^{N_{EF}} \int_{S_h^e} f\, h T_e\, dS =$$

$$= \int_V f\, \dot{Q} dV - \int_{S_q} f\, \bar{q} dS + \int_{S_h} f\, h T_\infty dS - \int_{S_T} f\, q_n dS \qquad \forall f(\underline{\mathbf{x}})$$

Formulazione "debole" del problema termico

Introducendo una suddivisione ("discretizzazione") in N_{EF} elementi finiti si ha:

$$T_e(x,y,t) = \underline{\Psi}_e^T(\underline{\mathbf{x}})\,\underline{\mathbf{T}}_e(t)$$

$$\sum_{e=1}^{N_{EF}} \int_{V_e} f\, c\, \rho\, \underline{\Psi}_e^T(\underline{\mathbf{x}})\, dV\, \dot{\underline{\mathbf{T}}}_e + \sum_{e=1}^{N_{EF}} \int_{V_e} (\underline{\nabla} f)^T \underline{\underline{\mathbf{k}}}\, \underbrace{\underline{\nabla}\underline{\Psi}_e^T(\underline{\mathbf{x}})}_{\underline{\underline{\mathbf{B}}}_e} dV\, \underline{\mathbf{T}}_e +$$

$$+ \sum_{e=1}^{N_{EF}} \int_{S_h^e} f\, h\, \underline{\Psi}_e^T(\underline{\mathbf{x}})\, dS\, \underline{\mathbf{T}}_e =$$

$$= \int_V f\, \dot{Q}\, dV - \int_{S_q} f\, \bar{q}\, dS + \int_{S_h} f\, h\, T_\infty\, dS - \int_{S_T} f\, q_n\, dS \qquad \forall f(\underline{\mathbf{x}})$$

Scelta delle funzioni test

L'arbitrarietà della funzione $f(\underline{\mathbf{x}})$ può suggerire, ad esempio, una scelta di tipo polinomiale, ottenendo un'equazione indipendente per ogni termine.
Se il numero di incognite è N_{inc}, per formare un sistema determinato occorre considerare N_{inc} termini polinomiali.

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & x & & y & & \\
 & x^2 & & xy & & y^2 & \\
x^3 & & x^2 y & & xy^2 & & y^3 \\
\ldots & & & & & &
\end{array}
$$

Scelta delle funzioni test

L'arbitrarietà della funzione $f(\underline{\mathbf{x}})$ può suggerire, ad esempio, una scelta di tipo polinomiale, ottenendo un'equazione indipendente per ogni termine.
Se il numero di incognite è N_{inc}, per formare un sistema determinato occorre considerare N_{inc} termini polinomiali.

$$f_{0,0} = 1 = \text{cost.}$$

$$\sum_{e=1}^{N_{EF}} \int_{V_e} c\,\rho\, \underline{\boldsymbol{\Psi}}_e^T(\underline{\mathbf{x}})\, dV\, \dot{\underline{\mathbf{T}}}_e + \sum_{e=1}^{N_{EF}} \int_{S_h^e} h\, \underline{\boldsymbol{\Psi}}_e^T(\underline{\mathbf{x}})\, dS\, \underline{\mathbf{T}}_e =$$

$$= \int_V \dot{Q}\, dV - \int_{S_q} \overline{q}\, dS + \int_{S_h} h\, T_\infty\, dS - \int_{S_T} q_n\, dS$$

Scelta delle funzioni test

L'arbitrarietà della funzione $f(\underline{\mathbf{x}})$ può suggerire, ad esempio, una scelta di tipo polinomiale, ottenendo un'equazione indipendente per ogni termine.
Se il numero di incognite è N_{inc}, per formare un sistema determinato occorre considerare N_{inc} termini polinomiali.

$$f_{1,0} = x$$

$$\sum_{e=1}^{N_{EF}} \int_{V_e} x\, c\, \rho\, \underline{\mathbf{\Psi}}_e^T(\underline{\mathbf{x}})\, dV\, \underline{\dot{\mathbf{T}}}_e + \sum_{e=1}^{N_{EF}} \int_{V_e} [1,0]\, \underline{\underline{\mathbf{k}}}\, \underline{\nabla}\, \underline{\mathbf{\Psi}}_e^T(\underline{\mathbf{x}})\, dV\, \underline{\mathbf{T}}_e +$$

$$+ \sum_{e=1}^{N_{EF}} \int_{S_h^e} x\, h\, \underline{\mathbf{\Psi}}_e^T(\underline{\mathbf{x}})\, dS\, \underline{\mathbf{T}}_e =$$

$$= \int_V x\, \dot{Q}\, dV - \int_{S_q} x\, \overline{q}\, dS + \int_{S_h} x\, h\, T_\infty\, dS - \int_{S_T} x\, q_n\, dS$$

Scelta delle funzioni test

L'arbitrarietà della funzione $f(\underline{\mathbf{x}})$ può suggerire, ad esempio, una scelta di tipo polinomiale, ottenendo un'equazione indipendente per ogni termine.
Se il numero di incognite è N_{inc}, per formare un sistema determinato occorre considerare N_{inc} termini polinomiali.

$$f_{0,1} = y$$

$$\sum_{e=1}^{N_{EF}} \int_{V_e} y\, c\, \rho\, \underline{\boldsymbol{\Psi}}_e^T(\underline{\mathbf{x}})\, dV\, \underline{\dot{\mathbf{T}}}_e + \sum_{e=1}^{N_{EF}} \int_{V_e} [0,1]\, \underline{\underline{\mathbf{k}}}\, \underline{\nabla}\, \underline{\boldsymbol{\Psi}}_e^T(\underline{\mathbf{x}})\, dV\, \underline{\mathbf{T}}_e +$$

$$+ \sum_{e=1}^{N_{EF}} \int_{S_h^e} y\, h\, \underline{\boldsymbol{\Psi}}_e^T(\underline{\mathbf{x}})\, dS\, \underline{\mathbf{T}}_e =$$

$$= \int_V y\, \dot{Q}\, dV - \int_{S_q} y\, \overline{q}\, dS + \int_{S_h} y\, h\, T_\infty\, dS - \int_{S_T} y\, q_n\, dS$$

Scelta delle funzioni test

L'arbitrarietà della funzione $f(\underline{\mathbf{x}})$ può suggerire, ad esempio, una scelta di tipo polinomiale, ottenendo un'equazione indipendente per ogni termine.
Se il numero di incognite è N_{inc}, per formare un sistema determinato occorre considerare N_{inc} termini polinomiali.

$$f_{2,0} = x^2$$

$$\sum_{e=1}^{N_{EF}} \int_{V_e} x^2 c \rho \underline{\Psi}_e^T(\underline{\mathbf{x}})\, dV\, \underline{\dot{\mathbf{T}}}_e + \sum_{e=1}^{N_{EF}} \int_{V_e} [2x, 0] \underline{\underline{\mathbf{k}}}\, \underline{\nabla} \underline{\Psi}_e^T(\underline{\mathbf{x}})\, dV\, \underline{\mathbf{T}}_e +$$

$$+ \sum_{e=1}^{N_{EF}} \int_{S_h^e} x^2\, h \underline{\Psi}_e^T(\underline{\mathbf{x}})\, dS\, \underline{\mathbf{T}}_e =$$

$$= \int_V x^2\, \dot{Q} dV - \int_{S_q} x^2\, \overline{q} dS + \int_{S_h} x^2\, h T_\infty dS - \int_{S_T} x^2\, q_n\, dS$$

Scelta delle funzioni test

L'arbitrarietà della funzione $f(\underline{\mathbf{x}})$ può suggerire, ad esempio, una scelta di tipo polinomiale, ottenendo un'equazione indipendente per ogni termine.
Se il numero di incognite è N_{inc}, per formare un sistema determinato occorre considerare N_{inc} termini polinomiali.

$$f_{1,1} = xy$$

$$\sum_{e=1}^{N_{EF}} \int_{V_e} xy\, c\, \rho\, \underline{\mathbf{\Psi}}_e^T(\underline{\mathbf{x}})\, dV\, \dot{\underline{\mathbf{T}}}_e + \sum_{e=1}^{N_{EF}} \int_{V_e} [y, x]\, \underline{\underline{\mathbf{k}}}\, \underline{\nabla}\, \underline{\mathbf{\Psi}}_e^T(\underline{\mathbf{x}})\, dV\, \underline{\mathbf{T}}_e +$$

$$+ \sum_{e=1}^{N_{EF}} \int_{S_h^e} xy\, h\, \underline{\mathbf{\Psi}}_e^T(\underline{\mathbf{x}})\, dS\, \underline{\mathbf{T}}_e =$$

$$= \int_V xy\, \dot{Q}\, dV - \int_{S_q} xy\, \overline{q}\, dS + \int_{S_h} xy\, h\, T_\infty\, dS - \int_{S_T} xy\, q_n\, dS$$

Scelta delle funzioni test

In generale si può scegliere: $f_{r,s} = x^r y^s$

$$\sum_{e=1}^{N_{EF}} \int_{V_e} x^r y^s c\rho \underline{\boldsymbol{\Psi}}_e^T(\underline{\mathbf{x}})\, dV\, \underline{\dot{\mathbf{T}}}_e +$$

$$+\sum_{e=1}^{N_{EF}} \int_{V_e} \left[r\, x^{r-1} y^s, \, s\, x^r y^{s-1} \right] \underline{\underline{\mathbf{k}}}\, \underline{\nabla} \underline{\boldsymbol{\Psi}}_e^T(\underline{\mathbf{x}})\, dV\, \underline{\mathbf{T}}_e +$$

$$+\sum_{e=1}^{N_{EF}} \int_{S_h^e} x^r y^s h \underline{\boldsymbol{\Psi}}_e^T(\underline{\mathbf{x}})\, dS\, \underline{\mathbf{T}}_e =$$

$$= \int_V x^r y^s \dot{Q} dV - \int_{S_q} x^r y^s \overline{q} dS + \int_{S_h} x^r y^s h T_\infty dS - \int_{S_T} x^r y^s q_n \, dS$$

Metodo dei "residui pesati"

In alternativa, visto che i requisiti di regolarità di $f(\underline{\mathbf{x}})$ sono gli stessi richiesti alle funzioni T_e e che il numero di funzioni test indipendenti deve essere pari a N_{inc} (per avere N_{inc} equazioni), in analogia a quanto fatto per T_e per ogni elemento finito si può assumere:

$$f_e(x, y) = \underline{\phi}_e^T(\underline{\mathbf{x}})\underline{\mathbf{F}}_e = \underline{\mathbf{F}}_e^T\underline{\phi}_e(\underline{\mathbf{x}})$$

con $\underline{\mathbf{F}}_e$ = vettore che raccoglie i valori nodali di $f_e(\underline{\mathbf{x}})$

Metodo dei "residui pesati"

Sostituendo nell'equazione governante si ottiene la seguente espressione:

$$\sum_{e=1}^{N_{EF}} \underline{\mathbf{F}}_e^T \int_{V_e} \underline{\phi}_e(\underline{\mathbf{x}})\, c\, \rho\, \underline{\Psi}_e^T(\underline{\mathbf{x}})\, dV\, \dot{\underline{\mathbf{T}}}_e +$$

$$+\sum_{e=1}^{N_{EF}} \underline{\mathbf{F}}_e^T \int_{V_e} \left(\underline{\nabla}\underline{\phi}_e^T(\underline{\mathbf{x}})\right)^T \underline{\underline{\mathbf{k}}}\, \underline{\underline{\mathbf{B}}}_e\, dV\, \underline{\mathbf{T}}_e +$$

$$+\sum_{e=1}^{N_{EF}} \underline{\mathbf{F}}_e^T \int_{S_h^e} \underline{\phi}_e(\underline{\mathbf{x}})\, h\, \underline{\Psi}_e^T(\underline{\mathbf{x}})\, dS\, \underline{\mathbf{T}}_e =$$

$$=\sum_{e=1}^{N_{EF}} \underline{\mathbf{F}}_e^T \int_{V_e} \underline{\phi}_e(\underline{\mathbf{x}})\, \dot{Q}\, dV - \underline{\mathbf{F}}_e^T \int_{S_q^e} \underline{\phi}_e(\underline{\mathbf{x}})\, \overline{q}\, dS +$$

$$+\sum_{e=1}^{N_{EF}} \underline{\mathbf{F}}_e^T \int_{S_h^e} \underline{\phi}_e(\underline{\mathbf{x}})\, h\, T_\infty\, dS - \underline{\mathbf{F}}_e^T \int_{S_T^e} \underline{\phi}_e(\underline{\mathbf{x}})\, q_n\, dS \qquad \forall \underline{\mathbf{F}}_e$$

Metodo dei "residui pesati"

in cui si possono definire:

$$\underline{\underline{\mathbf{C}}}_e = \int_{V_e} \underline{\phi}_e(\underline{\mathbf{x}})\, c\, \rho\, \underline{\Psi}_e^T(\underline{\mathbf{x}})\, dV$$ = matrice di capacità termica

$$\underline{\underline{\mathbf{K}}}_e = \int_{V_e} \left(\underline{\nabla}\underline{\phi}_e^T(\underline{\mathbf{x}})\right)^T \underline{\underline{\mathbf{k}}}\, \underline{\nabla}\underline{\Psi}_e^T(\underline{\mathbf{x}})\, dV$$ = matrice di conducibilità termica

$$\underline{\underline{\mathbf{H}}}_e = \int_{S_h^e} \underline{\phi}_e(\underline{\mathbf{x}})\, h\, \underline{\Psi}_e^T(\underline{\mathbf{x}})\, dS$$ = matrice di convezione

Metodo dei "residui pesati"

in cui si possono definire:

$$\underline{\dot{\mathbf{Q}}}_e = \int_{V_e} \underline{\phi}_e(\underline{\mathbf{x}})\, \dot{Q}\, dV$$ = vettore della potenza immessa (nell'elemento finito)

$$\underline{\overline{\mathbf{Q}}}_e = \int_{S_q^e} \underline{\phi}_e(\underline{\mathbf{x}})\, \overline{q}\, dS$$ = vettore di flusso assegnato sull'elemento

$$\underline{\mathbf{Q}}_\infty^e = \int_{S_h^e} \underline{\phi}_e(\underline{\mathbf{x}})\, h T_\infty\, dS$$ = vettore di flusso convettivo limite

$$\underline{\mathbf{Q}}_n^e = \int_{S_T} \underline{\phi}_e(\underline{\mathbf{x}})\, q_n\, dS$$ = vettore di flusso incognito sull'elemento

Metodo dei "residui pesati"

da cui si ricava:

$$\sum_{e=1}^{N_{EF}} \underline{\mathbf{F}}_e^T \underline{\underline{\mathbf{C}}}_e \dot{\underline{\mathbf{T}}}_e + \sum_{e=1}^{N_{EF}} \underline{\mathbf{F}}_e^T \underline{\underline{\mathbf{K}}}_e \underline{\mathbf{T}}_e + \sum_{e=1}^{N_{EF}} \underline{\mathbf{F}}_e^T \underline{\underline{\mathbf{H}}}_e \underline{\mathbf{T}}_e =$$

$$= \sum_{e=1}^{N_{EF}} \underline{\mathbf{F}}_e^T \left(\dot{\underline{\mathbf{Q}}}_e - \underline{\overline{\mathbf{Q}}}_e + \underline{\mathbf{Q}}_\infty^e - \underline{\mathbf{Q}}_n^e \right) \qquad \forall \underline{\mathbf{F}}_e$$

Metodo di Galerkin'

Se si assumono le funzioni di interpolazione della funzione test identiche alle funzioni di interpolazione del campo di temperatura:

$$\underline{\phi}_e(\underline{\mathbf{x}}) \equiv \underline{\Psi}_e(\underline{\mathbf{x}})$$

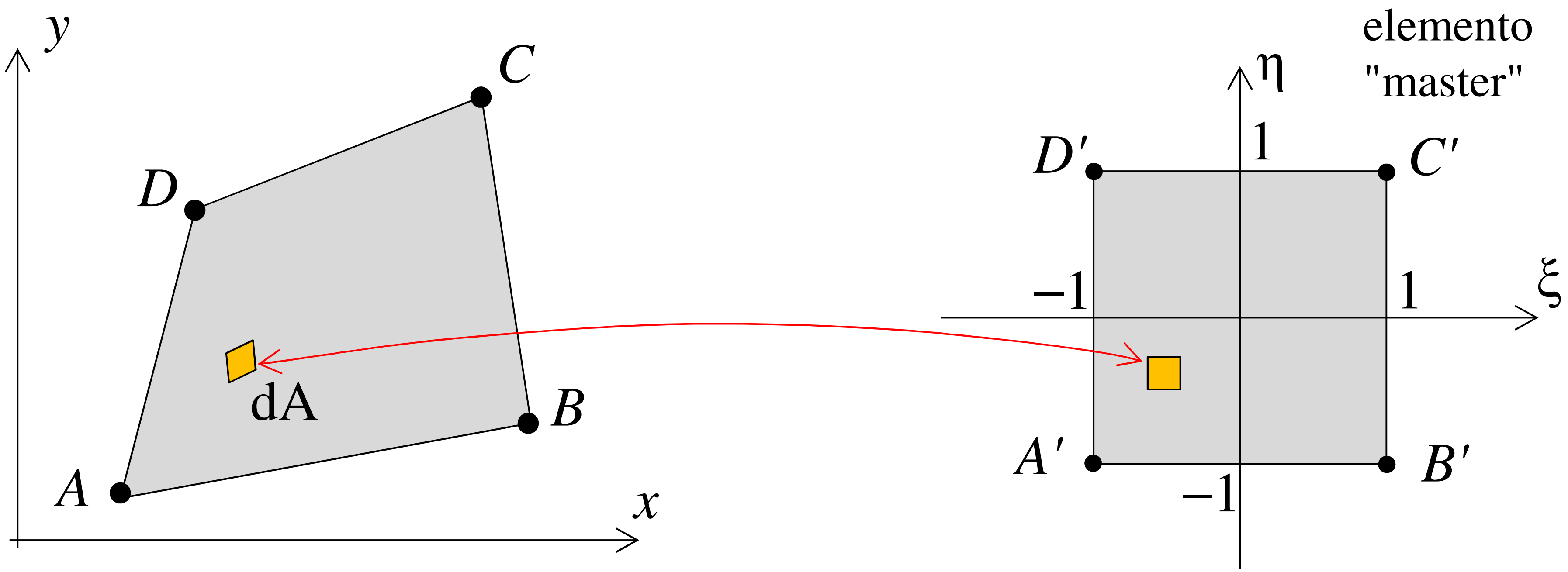
le tre matrici governanti il sistema divengono simmetriche:

$$\underline{\underline{\mathbf{C}}}_e = \int_{V_e} \underline{\Psi}_e(\underline{\mathbf{x}})\, c\, \rho\, \underline{\Psi}_e^T(\underline{\mathbf{x}})\, dV = \underline{\underline{\mathbf{C}}}_e^T$$

$$\underline{\underline{\mathbf{K}}}_e = \int_{V_e} \left(\underline{\nabla}\underline{\Psi}_e^T(\underline{\mathbf{x}})\right)^T \underline{\underline{\mathbf{k}}}\, \underline{\nabla}\underline{\Psi}_e^T(\underline{\mathbf{x}})\, dV = \int_{V_e} \underline{\underline{\mathbf{B}}}_e^T\, \underline{\underline{\mathbf{k}}}\, \underline{\underline{\mathbf{B}}}\, dV = \underline{\underline{\mathbf{K}}}_e^T$$

$$\underline{\underline{\mathbf{H}}}_e = \int_{S_h^e} h\, \underline{\Psi}_e(\underline{\mathbf{x}})\, \underline{\Psi}_e^T(\underline{\mathbf{x}})\, dS = \underline{\underline{\mathbf{H}}}_e^T$$

Calcolo della matrice $\underline{\underline{C}}$

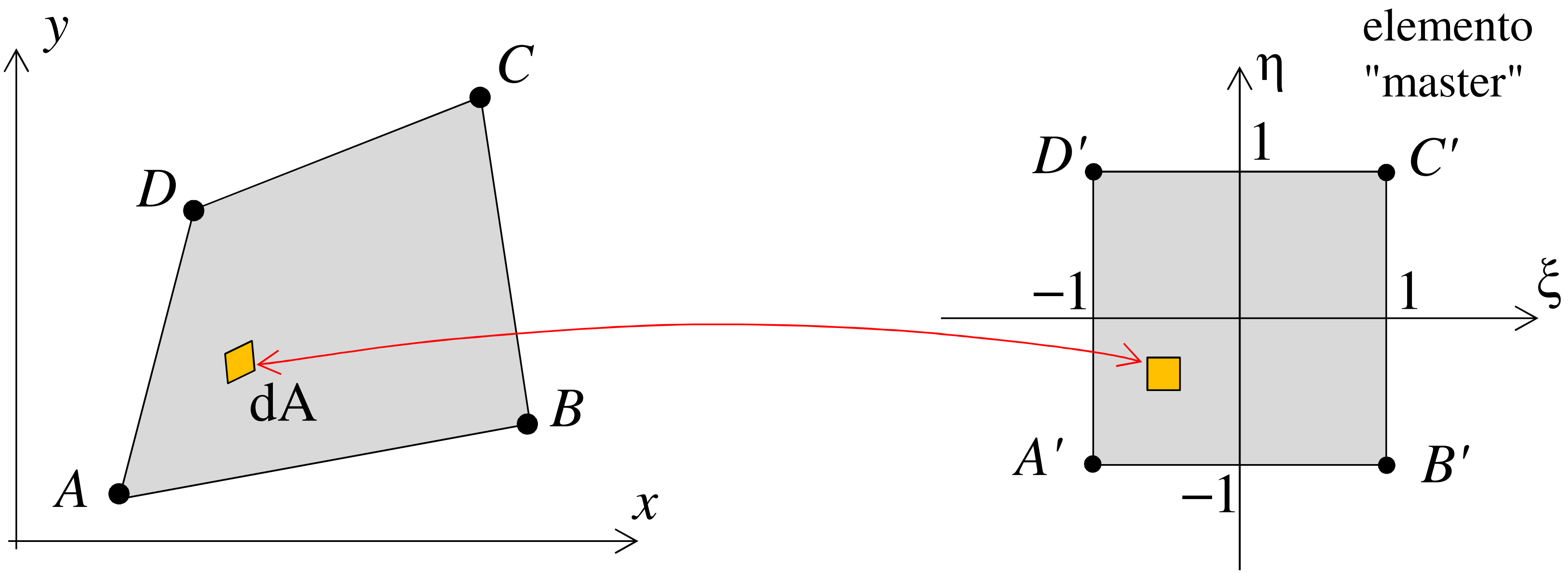


$$\underline{\underline{C}}_e = \int_{V_e} c\,\rho\,\underline{\Psi}_e(\underline{x})\,\underline{\Psi}_e^T(\underline{x})\,dV = \int_{A_e} t_e\,c\,\rho\,\underline{\Psi}_e(\underline{x})\,\underline{\Psi}_e^T(\underline{x})\,dA =$$

$$= \int_{-1}^{1}\int_{-1}^{1} t_e\,c\,\rho\,\underline{\varphi}_e(\underline{\xi})\,\underline{\varphi}_e^T(\underline{\xi})\,\overbrace{\left|\det\left(\underline{\underline{j}}_e\right)\right| d\xi\,d\eta}^{dA}$$

t_e= spessore dell'elemento finito, A_e= area dell'elemento finito

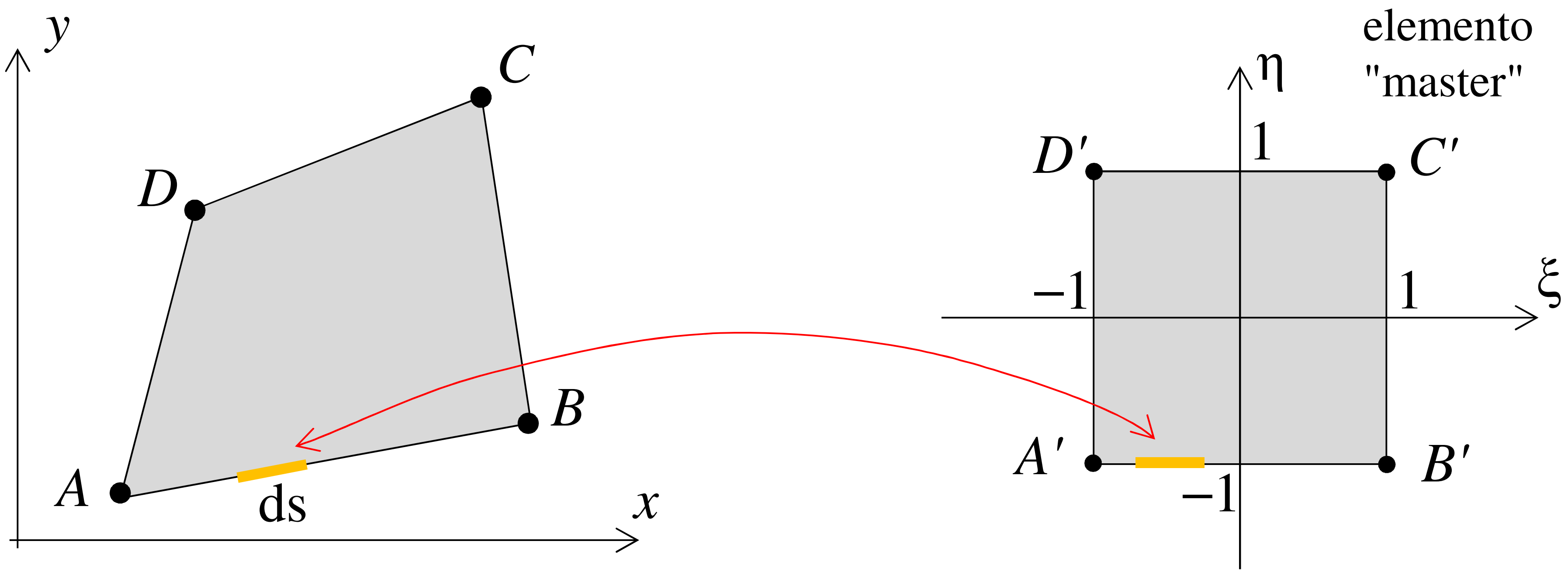
Calcolo della matrice $\underline{\underline{\mathbf{K}}}$



$$\underline{\underline{\mathbf{K}}}_e = \int_{V_e} \left(\underline{\nabla}\,\underline{\Psi}_e^T(\underline{\mathbf{x}}) \right)^T \underline{\underline{\mathbf{k}}}\, \underline{\nabla}\,\underline{\Psi}_e^T(\underline{\mathbf{x}})\, dV = \int_{A_e} t_e \left(\underline{\nabla}\,\underline{\Psi}_e^T(\underline{\mathbf{x}}) \right)^T \underline{\underline{\mathbf{k}}}\, \underline{\nabla}\,\underline{\Psi}_e^T(\underline{\mathbf{x}})\, dA =$$

$$= \int_{-1}^{1}\int_{-1}^{1} t_e\, \underline{\underline{\mathbf{B}}}_e^T(\xi)\, \underline{\underline{\mathbf{k}}}\, \underline{\underline{\mathbf{B}}}_e(\xi) \left| \det\left(\underline{\underline{\mathbf{j}}}_e \right) \right| d\xi\, d\eta$$

Calcolo della matrice $\underline{\underline{\mathbf{H}}}$

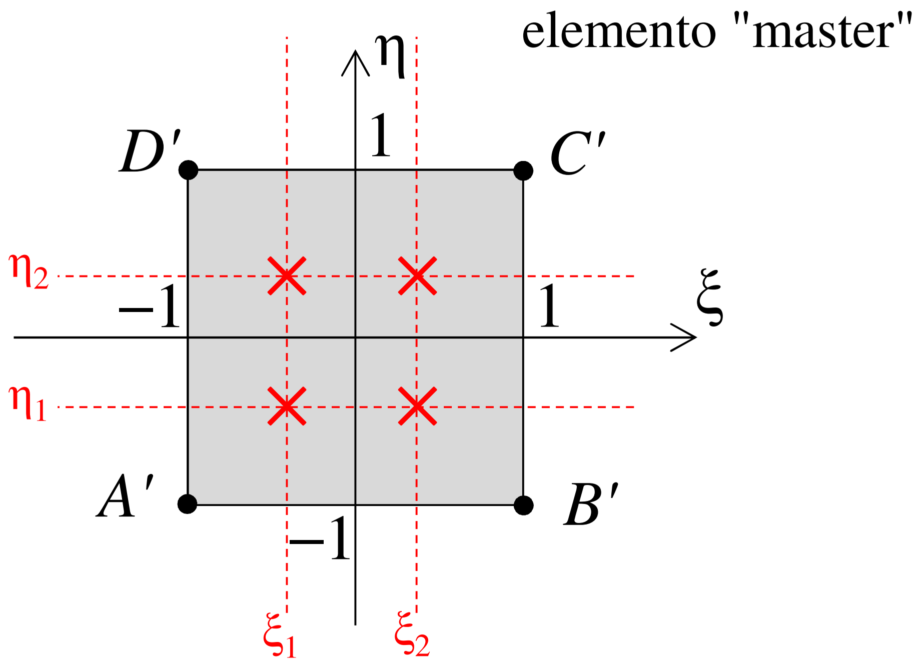


$$\underline{\underline{\mathbf{H}}}_e = \int_{S_h^e} h\,\underline{\boldsymbol{\Psi}}_e(\underline{\mathbf{x}})\,\underline{\boldsymbol{\Psi}}_e^T(\underline{\mathbf{x}})\,dS = \int_{\Gamma_h^e} t_e\, h\,\underline{\boldsymbol{\Psi}}_e(\underline{\mathbf{x}})\,\underline{\boldsymbol{\Psi}}_e^T(\underline{\mathbf{x}})\,ds =$$

$$= \int_{-1}^{1} t_e\, h\,\underline{\boldsymbol{\varphi}}_e(\xi,-1)\,\underline{\boldsymbol{\varphi}}_e^T(\xi,-1)\,\overbrace{j_h d\xi}^{\mathrm{ds}}, \qquad j_h = \sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2}$$

Integrazione numerica

Le matrici e, in modo analogo, i vettori possono essere calcolati mediante l'integrazione numerica, ad es. col metodo di Gauss.



$$\xi_1 = \eta_1 = -\frac{1}{\sqrt{3}}$$

$$\xi_2 = \eta_2 = +\frac{1}{\sqrt{3}}$$

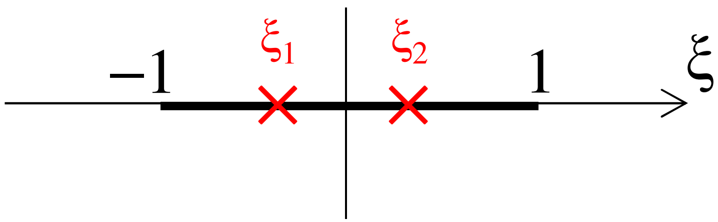
$$w_1 = w_2 = 1$$

$$\underline{\underline{\mathbf{K}}}_e = \int_{-1}^{1}\int_{-1}^{1} \underbrace{t_e\, \underline{\underline{\mathbf{B}}}_e^T \underline{\underline{\mathbf{k}}}\, \underline{\underline{\mathbf{B}}}_e \left|\det\left(\underline{\underline{\mathbf{j}}}_e\right)\right|}_{\underline{\underline{\mathbf{g}}}(\xi,\eta)} d\xi\, d\eta \cong \sum_{s=1}^{2}\sum_{r=1}^{2} w_s\, w_r\, \underline{\underline{\mathbf{g}}}\left(\xi_r, \eta_s\right)$$

Integrazione numerica

Le matrici e, in modo analogo, i vettori possono essere calcolati mediante l'integrazione numerica, ad es. col metodo di Gauss.

elemento "master"



$$\xi_1 = -\frac{1}{\sqrt{3}}, \qquad \xi_2 = +\frac{1}{\sqrt{3}}$$

$$w_1 = w_2 = 1$$

$$\underline{\underline{\mathbf{H}}}_e = \int_{-1}^{1} t_e\, h\, \underline{\varphi}_e(\xi,-1)\, \underline{\varphi}_e^T(\xi,-1)\, \overbrace{j_h d\xi}^{\mathrm{ds}} \cong$$

$$\cong \sum_{s=1}^{2} w_s\, t_e\, h\, \underline{\varphi}_e(\xi_s,-1)\, \underline{\varphi}_e^T(\xi_s,-1)\; j_h(\xi_s,-1)$$

Sistema risolvente

Da ultimo, utilizzando la classica regola di assemblaggio (in questo caso limitata ai numeri dei nodi, dato che per ogni nodo è definito un solo valore di temperatura) si pone:

$$\underline{\mathbf{T}}_e = \underline{\underline{\mathbf{A}}}_e \underline{\mathbf{T}}, \qquad \underline{\mathbf{F}}_e = \underline{\underline{\mathbf{A}}}_e \underline{\mathbf{F}}$$

e si ottiene il sistema algebrico governante:

$$\underline{\mathbf{F}} \left(\sum_{e=1}^{N_{EF}} \underline{\underline{\mathbf{A}}}_e^T \underline{\underline{\mathbf{C}}}_e \underline{\underline{\mathbf{A}}}_e \dot{\underline{\mathbf{T}}} + \sum_{e=1}^{N_{EF}} \underline{\underline{\mathbf{A}}}_e^T \underline{\underline{\mathbf{K}}}_e \underline{\underline{\mathbf{A}}}_e \underline{\mathbf{T}} + \sum_{e=1}^{N_{EF}} \underline{\underline{\mathbf{A}}}_e^T \underline{\underline{\mathbf{H}}}_e \underline{\underline{\mathbf{A}}}_e \underline{\mathbf{T}} \right) =$$

$$= \underline{\mathbf{F}} \sum_{e=1}^{N_{EF}} \underline{\underline{\mathbf{A}}}_e^T \left(\dot{\underline{\mathbf{Q}}}_e - \overline{\underline{\mathbf{Q}}}_e + \underline{\mathbf{Q}}_\infty^e - \underline{\mathbf{Q}}_n^e \right) \qquad \forall \underline{\mathbf{F}}$$

cioè:

$$\boxed{\underline{\underline{\mathbf{C}}} \dot{\underline{\mathbf{T}}} + \left(\underline{\underline{\mathbf{K}}} + \underline{\underline{\mathbf{H}}} \right) \underline{\mathbf{T}} = \dot{\underline{\mathbf{Q}}} - \overline{\underline{\mathbf{Q}}} + \underline{\mathbf{Q}}_\infty - \underline{\mathbf{Q}}_n}$$