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THE REPRESENTATION PROBLEM OF PAIRS OF SYMMETRIC SECOND-ORDER TENSORS IN THE CONTEXT OF SOLID MECHANICS

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The representation problem of pairs of symmetric second-order tensors in the context of Solid Mechanics

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1 Introduction

This paper deals with the problem of determine a basis of invariants for a pair of symmetric second-order tensors in the context of solid mechanics, sometimes referred as representation theorems. We have restricted ourselves to the particular case of pairs of 3×3 real symmetric matrices, although some of the results here could be thought in a more general setting.

Much is known and written in this area (see, for instance, [2], [3], [8], [9] and the references therein). Our aim here is to approach in a slight different way by focussing our attention on the set of similarity classes of pairs of 3×3 real symmetric matrices, what we have called $\mathcal{SP}(3)$ (see Section 2 for the precise definitions). In this sense, our first result, Theorem 3.1, states that anisotropic second-order valued functions are the functions defined on $\mathcal{SP}(3)$ and gives a new proof that the three plus three direct invariants $\mathrm{tr}(A)$, $\mathrm{tr}(A^2)$, $\mathrm{tr}(A^3)$ and $\mathrm{tr}(B)$, $\mathrm{tr}(B^2)$, $\mathrm{tr}(B^3)$ and the four mixed invariants $\mathrm{tr}(AB)$, $\mathrm{tr}(A^2B)$, $\mathrm{tr}(AB^2)$, $\mathrm{tr}(A^2B^2)$ completely determine the similarity class of the pair of second-order tensors (A, B). Moreover, we provide an example, that not any single trace of the ten is superfluous. Once we have seen the relevance of $\mathcal{SP}(3)$ in this context, we give a parametrization of it, closely connected to the eigenvalues of the tensors and the Euler Angles relating the principal axes of them. This is Theorem 4.4. Using this parametrization we suggest a physical meaning to the mixed invariants and give a constructive way to build a pair of symmetric second-order tensors from its standard ten traces. Later we prove what we consider the third main result of the paper, Theorem 6.5: any set of nine anisotropic second-order valued functions, six of them independent of the relative position of the principal axes, do not determine their similarity class. This would explain why the six direct traces plus three mixed are not a basis of invariants.

The paper is organized as follows. In Section 2, we introduce most of the notations and definitions that will be needed on the paper. This is clearly tedious but has the advantage of facilitating the subsequent reading. The main subject of the paper is presented in Section 3. Section 4 is devoted to provide a parametrization of $\mathcal{SP}(3)$ in terms of orthostocastic matrices and next one puts its emphasis on the properties of doubly stochastic and orthostocastic matrices. Section 6 exploits all the properties and results presented in the former sections. Finally, Section 7 centers its attention on relationships among invariants.

2 Notations and preliminaries

To allow a further easier reading, we concentrate some notations in this section (see, for instance, [4]) so the reader may consult them if necessary.

All matrices we consider will be $n \times n$ real matrices unless specified to the contrary. The subset of all non-singular matrices is called the linear group and will be denoted by $\mathrm{GL}_n(\mathbb{R})$. Two matrices A and B connected by the relation $B = S^{-1}AS$, where S is a non-singular matrix, are called similar. Two pairs of matrices (A, B) and (C, D) are called similar if there exists a non-singular matrix S such that $C = S^{-1}AS$ and $D = S^{-1}BS$. Given a pair (A, B), its similarity class will be denoted by $[(A, B)]$.

The subgroup of $\mathrm{GL}_n(\mathbb{R})$ of matrices whose inverse coincide with their transpose is called the orthogonal group and will be denoted by $\mathcal{O}(n)$. A real matrix that coincides with its transpose is called a symmetric matrix. The set of all $n \times n$ symmetric matrices will be denoted by $\mathrm{Sym}(n)$. The set of all similarity classes of pairs of $n \times n$ real symmetric matrices will be denoted by $\mathcal{SP}(n)$. Its study is the main goal of this paper.

If $\mathcal{S}_n$ stands for the n-th symmetric group, that is, the group of the permutations on n elements, let $\rho : \mathcal{S}_n \to \mathrm{GL}_n(\mathbb{R})$ be the linear representation sending σ to the $n \times n$ matrix obtained from the identity matrix by acting the permutation σ^{-1} on its columns. Let $\mathrm{P}(n) = \rho(\mathcal{S}_n)$ be the image, which is clearly a subgroup of $\mathcal{O}(n)$. Its elements will be called permutation matrices.

A diagonal matrix with ordered eigenvalues $\lambda = (\lambda_1, \ldots, \lambda_n)$ will be denoted by $D(\lambda)$ and the set of all $n \times n$ real diagonal matrices will be denoted by $\mathrm{Diag}(n)$. If $D(\lambda)$ and $D(\alpha)$ are similar, it is well known that $\{\lambda_1, \ldots, \lambda_n\} = \{\alpha_1, \ldots, \alpha_n\}$. So there exists $\sigma \in \mathcal{S}_n$ such that $\lambda_i = \alpha_{\sigma(i)}$ ($\lambda = \sigma(\alpha)$ for short). In particular, $D(\lambda) = D(\sigma(\alpha)) = p(\sigma)D(\alpha)p(\sigma)^{-1}$. Diagonal matrices with eigenvalues ± 1 are called diagonal square root matrices of the identity. The set of all $n \times n$ diagonal square root matrices of the identity will be denoted by $\mathcal{OD}(n)$. In fact, $\mathcal{OD}(n) = \mathcal{O}(n) \cap \mathrm{Diag}(n)$.

A nonnegative matrix is a real matrix with nonnegative entries. A nonnegative matrix is called stochastic if the sum of the elements on each row is equal to 1. It is said double stochastic if itself and its transpose are both stochastic. The set of $n \times n$ double stochastic matrices will be denoted by $\Omega(n)$. The Hadamard product of two $n \times n$ matrices $A = (a_{i,j})$, $B = (b_{i,j})$ is defined as the $n \times n$ matrix $A * B = (a_{i,j} b_{i,j})$. $A^{[2]}$ will stand for the Hadamard product of A by itself and called the Hadamard square of the matrix A. The Hadamard square of an orthogonal matrix is called an orthostochastic matrix. The set of $n \times n$ orthostochastic matrices will be denoted by $\mathrm{Ort}(n)$. Clearly $\mathrm{Ort}(n) \subset \Omega(n)$.

3 Similarity pairs of symmetric tensors

In this section we give a new proof that ten traces uniquely determine a pair of symmetric tensors. In particular, $\mathcal{SP}(3)$ can be seen as a subset of $\mathbb{R}^{10}$. Concretely, given a pair of $n \times n$ real matrices (A, B), let $t_{i,j}$ stand for the trace of the product $A^i B^j$. Since AB and BA have the same trace, $t(A, B)$ defined as $t(A, B) = (t_{1,0}, t_{0,1}, t_{2,0}, t_{1,1}, t_{0,2}, t_{3,0}, t_{2,1}, t_{1,2}, t_{0,3}, t_{2,2})$ is an invariant of the similarity class of the pair (A, B).

Theorem 3.1 *Let (A, B) and (C, D) be two pairs of 3×3 real symmetric matrices. Then, the following conditions are equivalent:*

(i) (A, B) and (C, D) are similar.

(ii) There exists an orthogonal matrix U such that $C = U^{-1}AU$ and $D = U^{-1}BU$.

(iii) $t(A, B) = t(C, D)$.

Before proving the theorem, we need a couple of lemmas.

Lemma 3.2 *Let A, B be two 3×3 real symmetric matrices. Then, the following are equivalent:*

(i) *A and B are similar.*

(ii) *There exists an orthogonal matrix U such that $B = U^{-1}AU$.*

(iii) $(\mathrm{tr}(A), \mathrm{tr}(A^2), \mathrm{tr}(A^3)) = (\mathrm{tr}(B), \mathrm{tr}(B^2), \mathrm{tr}(B^3))$.

Proof. $(i) \Leftrightarrow (ii)$ is well-known and $(i) \Leftrightarrow (iii)$ follows from Newton's formulas (see, for instance, [4], Vol.1, pages 274, 87). ∎

Remark 3.3 If $A = D(\lambda)$ and $B = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$, then it is straightforward that:

$$\begin{pmatrix} \mathrm{tr}(B) & \mathrm{tr}(B^2) \\ \mathrm{tr}(AB) & \mathrm{tr}(AB^2) \\ \mathrm{tr}(A^2B) & \mathrm{tr}(A^2B^2) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix} \begin{pmatrix} a & a^2+b^2+c^2 \\ d & b^2+d^2+e^2 \\ f & c^2+e^2+f^2 \end{pmatrix} \text{ and}$$

$$\mathrm{tr}(B^3) = 3\begin{pmatrix} a & d & f \end{pmatrix} \begin{pmatrix} a^2+b^2+c^2 \\ b^2+d^2+e^2 \\ c^2+e^2+f^2 \end{pmatrix} - 2\left(a^3+d^3+f^3\right) + 6bce.$$

Lemma 3.4 *Let (A, B) and (A, C) be two pairs of 3×3 real symmetric matrices, $A = D(\lambda)$. If $t(A, B) = t(A, C)$, there exists an orthogonal matrix U such that $A = U^{-1}AU$ and $C = U^{-1}BU$.*

Proof. Writing $B = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$, $C = \begin{pmatrix} g & h & i \\ h & j & k \\ i & k & l \end{pmatrix}$ and using Remark 3.3, we have

$$\begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix} \left[\begin{pmatrix} a & a^2+b^2+c^2 \\ d & b^2+d^2+e^2 \\ f & c^2+e^2+f^2 \end{pmatrix} - \begin{pmatrix} g & g^2+h^2+i^2 \\ j & h^2+j^2+k^2 \\ l & i^2+k^2+l^2 \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

If $\lambda_1, \lambda_2, \lambda_3$ are distinct, we deduce $a = g$, $d = j$, $f = l$, $b^2 = h^2$, $c^2 = i^2$ and $e^2 = k^2$. In particular, by Remark 3.3 again, $bce = hik$. Thus C must be one of the following four matrices:

$$\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}, \begin{pmatrix} a & -b & -c \\ -b & d & e \\ -c & e & f \end{pmatrix}, \begin{pmatrix} a & -b & c \\ -b & d & -e \\ c & -e & f \end{pmatrix}, \begin{pmatrix} a & b & -c \\ b & d & -e \\ -c & -e & f \end{pmatrix}.$$

And one sees $A = U^{-1}AU$, $C = U^{-1}BU$ taking $U = D(1,1,1)$ in the first case, $U = D(-1,1,1)$ in the second case, $U = D(1,-1,1)$ in the third case and $U = D(1,1,-1)$ in the fourth case. If $\lambda_1 = \lambda_2 \neq \lambda_3$, we deduce $f = l$, $a + d = g + j$, $c^2 + e^2 = i^2 + k^2$ and $a^2 + 2b^2 + d^2 = g^2 + 2h^2 + j^2$. In particular, $ad - b^2 = gj - h^2$ and $a + d = g + j$. Thus, $B_1 = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$ and $C_1 = \begin{pmatrix} g & h \\ h & j \end{pmatrix}$ are similar. Let $S_1, T_1 \in \mathcal{O}(2)$ be such that

$S_1^{-1}B_1S_1 = D_1 = T_1^{-1}C_1T_1$, $D_1 = D(\alpha,\beta)$. Take $S = \begin{pmatrix} S_1 & 0 \\ 0 & 1 \end{pmatrix}$, $T = \begin{pmatrix} T_1 & 0 \\ 0 & 1 \end{pmatrix}$, $M = S^{-1}BS = \begin{pmatrix} D_1 & P \\ P^\top & f \end{pmatrix}$, $P^\top = \begin{pmatrix} \gamma & \delta \end{pmatrix}$, $N = T^{-1}CT = \begin{pmatrix} D_1 & Q \\ Q^\top & f \end{pmatrix}$, $Q^\top = \begin{pmatrix} \varepsilon & \varphi \end{pmatrix}$. Clearly $t(A,M) = t(A,B) = t(A,C) = t(A,N)$. Remark that, in this very particular case, this is equivalent to $(\mathrm{tr}(M^2), \mathrm{tr}(M^3)) = (\mathrm{tr}(N^2), \mathrm{tr}(N^3))$. But $\mathrm{tr}(M^2) = \mathrm{tr}(N^2)$ means $\gamma^2 + \delta^2 = \varepsilon^2 + \varphi^2$ and $\mathrm{tr}(M^3) = \mathrm{tr}(N^3)$ implies $\alpha\gamma^2 + \beta\delta^2 = \alpha\varepsilon^2 + \beta\varphi^2$. In particular, there exists $V_1 \in \mathcal{O}(2)$ such that $V_1Q = P$ (if $\alpha = \beta$, it is obvious and if $\alpha \neq \beta$, then $\gamma = \pm\varepsilon$, $\delta = \pm\varphi$ and $V_1 = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$). For $V = \begin{pmatrix} V_1 & 0 \\ 0 & 1 \end{pmatrix}$, $N = V^{-1}MV$ and taking the orthogonal matrix $U = SVT^{-1}$, we have $A = U^{-1}AU$ and $C = U^{-1}BU$. Finally, if $\lambda_1 = \lambda_2 = \lambda_3$, since $(\mathrm{tr}(B), \mathrm{tr}(B^2), \mathrm{tr}(B^3)) = (\mathrm{tr}(C), \mathrm{tr}(C^2), \mathrm{tr}(C^3))$, any orthogonal matrix such that $C = U^{-1}BU$ verifies $A = U^{-1}AU$. ∎

Proof of Theorem 3.1. $(ii) \Rightarrow (i)$ and $(i) \Rightarrow (iii)$ are clear. Let us see $(iii) \Rightarrow (ii)$. By Lemma 3.2, there exist orthogonal matrices S, T such that $E = S^{-1}AS$ and $E = T^{-1}CT$, E diagonal. Then $t(E, S^{-1}BS) = t(A,B) = t(C,D) = t(E, T^{-1}DT)$. By Lemma 3.4, there exists an orthogonal matrix V such that $E = V^{-1}EV$ and $T^{-1}DT = V^{-1}S^{-1}BSV$. Taking the orthogonal matrix $U = SVT^{-1}$, we have $C = U^{-1}AU$ and $D = U^{-1}BU$. ∎

Remark 3.5 The equivalence $(i) \Leftrightarrow (ii)$ in Theorem 3.1 shows that any anisotropic function on $\mathrm{Sym}(3) \times \mathrm{Sym}(3)$ factorizes through $\mathcal{SP}(3)$ (see, for instance, [9]). On the other hand, the equivalence $(i) \Leftrightarrow (iii)$ in Theorem 3.1 shows that $\mathcal{SP}(3)$ can be seen as a subset of $\mathbb{R}^{10}$ through the well-defined injective map $t : \mathcal{SP}(3) \to \mathbb{R}^{10}$, $t([(A,B)]) = t(A,B)$. Next example shows that not a single trace of the ten can be determined in terms of the other nine.

Example 3.6 Let (A,B) be the pair $A = D(\lambda)$ and $B = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$. For the following values of $\lambda_1, \lambda_2, \lambda_3$ and a, b, c, d, e, f:

λ_1	λ_2	λ_3	a	b	c	d	e	f
1	2	3	0	0	1	0	1	0
1	2	3	0	$\sqrt{1/2}$	0	0	$\sqrt{3/2}$	0
1	2	3	0	$\sqrt{3/8}$	0	0	$\sqrt{13/8}$	0
1	2	3	0	$\sqrt{1/7}$	0	0	$\sqrt{12/7}$	0
1	2	3	0	1	1	0	1	0
1	2	3	0	1	1	0	-1	0
1	1	0	0	1	1	0	1	0
1	1	0	2/3	$\sqrt{19/18}$	1	-2/3	0	1
1	-1	0	1	0	1	-1	1	0
1	-1	0	-1	0	1	1	1	0

then $t_{i,j} = \mathrm{tr}(A^iB^j)$ are equal to:

$t_{1,0}$	$t_{0,1}$	$t_{2,0}$	$t_{1,1}$	$t_{0,2}$	$t_{3,0}$	$t_{2,1}$	$t_{1,2}$	$t_{0,3}$	$t_{2,2}$
6	0	14	0	4	36	0	9	0	23
6	0	14	0	4	36	0	9	0	22
6	0	14	0	4	36	0	37/4	0	23
6	0	14	0	26/7	36	0	9	0	23
6	0	14	0	6	36	0	12	6	28
6	0	14	0	6	36	0	12	-6	28
2	0	2	0	6	2	0	4	6	4
2	1	2	0	6	2	0	4	6	4
0	0	2	2	6	0	0	0	0	4
0	0	2	-2	6	0	0	0	0	4

4 Parametrizing pairs of symmetric tensors

In the last section we have seen that $\mathcal{SP}(3)$ is a subset of $\mathbb{R}^{10}$ and a relevant object in the context of anisotropic second-order valued functions. On the other hand, intuitively a pair of symmetric tensors is determined by the three eigenvalues of each matrix and the three Euler Angles relating the principal axes of each tensor. Nevertheless, permutations of the eigenvalues, choice of the Euler Angles and changes in the axes orientation provide the same class of a similarity pair of symmetric tensors. The purpose of this section is to deepen in the study of this natural parametrization.

Since the problem of the permutation of the eigenvalues can be solved by considering ordered ones or substituting them by the three principal traces, we focus our atention on the relation among the principal axes.

Consider the map $p : \mathbb{R}^6 \times \mathcal{O}(3) \to \mathcal{SP}(3)$, where if $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$, $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3$ and $S \in \mathcal{O}(3)$, then $p(\alpha, \beta, S) = [(D(\alpha), SD(\beta)S^{-1})]$. It is clear that p is surjective. Indeed, given any $[(A, B)] \in \mathcal{SP}(3)$, let α be the three eigenvalues of A, β the three eigenvalues of B and $U, V \in \mathcal{O}(3)$ such that $U^{-1}AU = D(\alpha)$, $V^{-1}BV = D(\beta)$. If $S = U^{-1}V$, then

$$\begin{aligned} &p(\alpha,\beta,S) = [(D(\alpha), SD(\beta)S^{-1})] = [(D(\alpha), U^{-1}VD(\beta)V^{-1}U)] = \\ &[(D(\alpha), U^{-1}BU)] = [(UD(\alpha)U^{-1}, B)] = [(A, B)], \end{aligned}$$

as desired. Therefore, $p : \mathbb{R}^6 \times \mathcal{O}(3) \to \mathcal{SP}(3)$ defined by $p(\alpha, \beta, S) = [(D(\alpha), SD(\beta)S^{-1})]$ is a parametrization of $\mathcal{SP}(3)$. In other words, a similarity class of a pair (A, B) is determined by a three-component vector (α, β, S), where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ are the three eigenvalues of A, $\beta = (\beta_1, \beta_2, \beta_3)$ are the three eigenvalues of B and $S = U^{-1}V$, where the columns of U are an orthonormal basis of eigenvectors of A of respective eigenvalues α and the columns of V are an orthonormal basis of eigenvectors of B of respective eigenvalues β. Remark that this parametrization avoids the choice of the Euler Angles. The following lemma, whose proof is omitted, solves the problem of the changes in the orientation of the principal axes and is a key point for the next proposition.

Lemma 4.1 *Let $S, T \in \mathcal{O}(3)$. Then $S^{[2]} = T^{[2]}$ if and only if there exist $R_1, R_2 \in \mathcal{OD}(3)$ such that $T = R_1SR_2$.*

Now we are able to formally summarize all the previous discussion when the three eigenvalues of A are distinct and the three eigenvalues of B are distinct too.

Proposition 4.2 *If $(\alpha, \beta, S), (\lambda, \mu, T) \in \mathbb{R}^6 \times \mathcal{O}(3)$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ are distinct and $\beta = (\beta_1, \beta_2, \beta_3)$ are distinct, then the following conditions are equivalent:*

(i) $p(\alpha, \beta, S) = p(\lambda, \mu, T)$,

(ii) *There exist $P_1, P_2 \in \mathrm{P}(3)$, $R_1, R_2 \in \mathcal{OD}(3)$, such that $D(\lambda) = P_1^{-1} D(\alpha) P_1$, $D(\mu) = P_2^{-1} D(\beta) P_2$ and $T = P_1^{-1} R_1 S R_2 P_2$.*

(iii) *There exist $P_1, P_2 \in \mathrm{P}(3)$ such that $D(\lambda) = P_1^{-1} D(\alpha) P_1$, $D(\mu) = P_2^{-1} D(\beta) P_2$ and $T^{[2]} = P_1^{-1} S^{[2]} P_2$.*

Proof. Let us prove $(i) \Rightarrow (ii)$. If $p(\alpha, \beta, S) = p(\lambda, \mu, T)$, there exists $U \in \mathcal{O}(3)$ such that $D(\lambda) = U^{-1} D(\alpha) U$, $T D(\mu) T^{-1} = U^{-1} S D(\beta) S^{-1} U$. In particular, $D(\lambda)$ and $D(\alpha)$ are similar, and $D(\mu)$ and $D(\beta)$ are similar too. Thus, there exist $P_1, P_2 \in \mathrm{P}(3)$ such that $D(\lambda) = P_1^{-1} D(\alpha) P_1$ and $D(\mu) = P_2^{-1} D(\beta) P_2$. Then,

$$P_1^{-1} D(\alpha) P_1 = D(\lambda) = U^{-1} D(\alpha) U,$$

which implies $D(\alpha) P_1 U^{-1} = P_1 U^{-1} D(\alpha)$. Since the α are distinct, $P_1 U^{-1}$ must be a diagonal matrix, and since $P_1, U \in \mathcal{O}(3)$, then $P_1 U^{-1} = R_1 \in \mathcal{OD}(3)$ and $U = R_1 P_1$. On the other hand,

$$T P_2^{-1} D(\beta) P_2 T^{-1} = T D(\mu) T^{-1} = U^{-1} S D(\beta) S^{-1} U,$$

which implies $D(\beta) P_2 T^{-1} U^{-1} S = P_2 T^{-1} U^{-1} S D(\beta)$. Since the β are distinct, $P_2 T^{-1} U^{-1} S$ must be a diagonal matrix, and since $P_2, T^{-1}, U^{-1}, S \in \mathcal{O}(3)$, then $P_2 T^{-1} U^{-1} S = R_2 \in \mathcal{OD}(3)$ and $T = U^{-1} S R_2 P_2 = P_1^{-1} R_1 S R_2 P_2$, where $P_1, P_2 \in \mathrm{P}(3)$ and $R_1, R_2 \in \mathcal{OD}(3)$.

Let us prove $(ii) \Rightarrow (iii)$. By hypothesis $P_1 T P_2^{-1} = R_1 S R_2$. Take their Hadarmard squares. Since P_1, P_2 are permutation matrices, then $(P_1 T P_2^{-1})^{[2]} = P_1 T^{[2]} P_2$. Since R_1, R_2 are diagonal square roots of the identity matrix, then $(R_1 S R_2)^{[2]} = S^{[2]}$. Therefore $P_1 T^{[2]} P_2^{-1} = S^{[2]}$ and $T^{[2]} = P_1^{-1} S^{[2]} P_2$.

Suppose (iii): $D(\lambda) = P_1^{-1} D(\alpha) P_1$, $D(\mu) = P_2^{-1} D(\beta) P_2$ and $T^{[2]} = P_1^{-1} S^{[2]} P_2$. Then $(P_1 T P_2^{-1})^{[2]} = P_1 T^{[2]} P_2^{-1} = S^{[2]}$. Since $P_1 T P_2^{-1}$ and $S \in \mathcal{O}(3)$ and their Hadamard square coincide, by Lemma 4.1 there must exist $R_1, R_2 \in \mathcal{OD}(3)$ such that $P_1 T P_2^{-1} = R_1 S R_2$. Then,

$$\begin{aligned}
&p(\lambda, \mu, T) = [(D(\lambda), T D(\mu) T^{-1})] = \\
&[(P_1^{-1} D(\alpha) P_1, P_1^{-1} R_1 S R_2 P_2 P_2^{-1} D(\beta) P_2 P_2^{-1} R_2 S^{-1} R_1 P_1)] = \\
&[(D(\alpha), R_1 S R_2 D(\beta) R_2 S^{-1} R_1)] = \\
&[(R_1 D(\alpha) R_1, S R_2 D(\beta) R_2 S^{-1})] = [(D(\alpha), S D(\beta) S^{-1})] = p(\alpha, \beta, S)
\end{aligned}$$

since $R_1 D(\alpha) R_1 = D(\alpha)$ and $R_2 D(\beta) R_2 = D(\beta)$. ∎

In particular, once the eigenvalues are already ordered, we have:

Corollary 4.3 *If $(\alpha, \beta, S) \in \mathbb{R}^6 \times \mathcal{O}(3)$, $T \in \mathcal{O}(3)$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ are distinct and $\beta = (\beta_1, \beta_2, \beta_3)$ are distinct, then the following conditions are equivalent:*

(i) $p(\alpha, \beta, S) = p(\alpha, \beta, T)$.

(ii) $T^{[2]} = S^{[2]}$.

Proof. In this case, P_1 and P_2 of the proposition must be the identity matrix. ∎

We summarize the section in the next theorem.

Theorem 4.4 *There exist a well-defined surjective map* $p : \mathbb{R}^6 \times \mathrm{Ort}(3) \to \mathcal{SP}(3)$ *sending* $(\alpha, \beta, M) \in \mathbb{R}^6 \times \mathrm{Ort}(3)$ *to* $[(D(\alpha), SD(\beta)S^{-1})] \in \mathcal{SP}(3)$, *where* $M = S^{[2]}$, $S \in \mathcal{O}(3)$. *Moreover, if* $(\alpha, \beta, M), (\lambda, \mu, N) \in \mathbb{R}^6 \times \mathrm{Ort}(3)$, *are such that* $\alpha_1 < \alpha_2 < \alpha_3$, $\beta_1 < \beta_2 < \beta_3$, $\lambda_1 < \lambda_2 < \lambda_3$ *and* $\mu_2 < \mu_2 < \mu_3$, *then* $p(\alpha, \beta, M) = p(\lambda, \mu, N)$ *if and only if* $(\alpha, \beta, M) = (\lambda, \mu, N)$.

5 Doubly stochastic matrices

In the previous section we have shown that the set $\mathcal{SP}(3)$ is closely related to $\mathbb{R}^6 \times \mathrm{Ort}(3)$, concretely, a dense open set of $\mathcal{SP}(3)$ is made of copies of $\mathrm{Ort}(3)$. This fact guides us to one of the main results of the paper: $\mathcal{SP}(3)$ can not be described as a subset of $\mathbb{R}^9$. But before proving this we need to study some properties of $\mathrm{Ort}(3)$ and $\Omega(3)$. To begin with, we recall the following theorem due to Birkhoff (see [5], page 117).

Theorem 5.1 *The set* $\Omega(n)$ *of the* $n \times n$ *doubly stochastic matrices is the convex hull (in* $\mathrm{M}_n(\mathbb{R})$*) of the set* $\mathrm{P}(n)$ *of the* $n \times n$ *permutation matrices. That is, if* $M \in \mathrm{M}_n(\mathbb{R})$, *then* $M \in \Omega(n)$ *if and only if there exist* $a \in \mathbb{R}^{n!}$ *with* $a_i \geq 0$ *and* $\sum_{i=1}^{n!} a_i = 1$ *such that* $M = \sum_{i=1}^{n!} a_i P_i$, *where* $P_i \in \mathrm{P}(n)$, $i = 1 \ldots, n!$, *are the permutation matrices.*

Remark 5.2 The expression of a doubly stochastic matrix as a convex combination of permutation matrices is far from being unique. The reason is that, for $n \geq 3$, permutation matrices are not linearly independent. Concretely, for $n = 3$, and if we call

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \; P_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$P_4 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; P_5 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \; P_6 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

one can easily check that $\sum_{i=1}^{6} a_i P_i = 0$ if and only if $(a_1, \ldots, a_6) = \lambda(1, -1, -1, -1, 1, 1)$. In other words, if $\Delta = \{(a_1, \ldots, a_6) \in \mathbb{R}^6 \mid a_i \geq 0, \sum_{i=1}^{6} a_i = 1\}$ is the convex hull of the natural basis of $\mathbb{R}^6$ and $\Phi : \mathbb{R}^6 \to \mathrm{M}_3(\mathbb{R})$ is the linear map defined by $\Phi(a_1, \ldots, a_6) = a_1 P_1 + \ldots + a_6 P_6$, then its image is $\Phi(\Delta) = \Omega(3)$ and its kernel is generated by the vector $v = (1, -1, -1, -1, 1, 1)$. So $\Omega(3)$ might be seen as the image of the projection of the standard simplex Δ in the direction of the vector v.

Notation 5.3 Let $\Delta_{i_1, \ldots, i_r}$ be stand for the r-codimensional face of Δ determined by the equations $a_{i_1} = \ldots = a_{i_r} = 0$. For instance, there are 6 1-codimensional faces, namely $\Delta_1, \ldots, \Delta_6$, and 15 2-codimensional faces, $\Delta_{1,2}, \Delta_{1,3}, \ldots, \Delta_{5,6}$. Let $C_{i_1, \ldots, i_r} = \Phi(\Delta_{i_1, \ldots, i_r})$ and let $\partial\Omega(n)$ stand for the boundary of $\Omega(n)$ as a topological subspace of $\mathrm{M}_n(\mathbb{R}) \subset \mathbb{R}^{n \times n}$.

Next proposition characterizes when the decomposition of a doubly stochastic matrix as a convex combination of permutation matrices is unique.

Proposition 5.4 *The boundary of* $\Omega(3)$ *is* $\partial(\Omega(3)) = C_{1,2} \cup C_{1,3} \cup C_{1,4} \cup C_{2,5} \cup C_{3,5} \cup C_{4,5} \cup C_{2,6} \cup C_{3,6} \cup C_{4,6}$. *Moreover, if* $M \in \Omega(3)$, *then* $M \in \partial\Omega(3)$ *if and only if* $M = \Phi(a)$ *for a unique* $a = (a_1, \dots, a_6) \in \Delta$.

Proof. The map Φ can be seen as a projection from $\mathbb{R}^6$ onto the dimension 5 subspace generated by the 6 permutation matrices, which is an open map. Hence, any point inside Δ goes to a point inside $\Omega(3)$. One can prove that any codimension 1 face of Δ does not contain the direction of projection. Hence codimension 1 faces of Δ map injectively through Φ to $\Omega(3)$. So any point in the interior of a codimension 1 face goes to the interior of $\Omega(3)$. The statement follows by studying the images of the codimension 2 faces. Now, let $M \in \partial\Omega(3)$ and suppose $M \in C_{1,2}$. If $M = \Phi(a)$, then $a = (0, 0, a_3, \dots, a_6)$. Since the kernel is generated by $v = (1, -1, -1, -1, 1, 1)$, all the preimages of M are of the form $a + tv = (t, -t, a_3 - t, a_4 - t, a_5 + t, a_6 + t)$. But $a + tv \in \Delta$ if and only if $t = 0$. So a is the only preimage of M in Δ. The rest of the cases would be shown analogously. ∎

Remark 5.5 Let $M \in \mathrm{Ort}(3)$, $M = S^{[2]} = \begin{pmatrix} a^2 & b^2 & c^2 \\ d^2 & e^2 & f^2 \\ g^2 & h^2 & i^2 \end{pmatrix}$, $S = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in \mathcal{O}(3)$.
Since $\mathrm{Ort}(3) \subset \Omega(3)$ and $\Omega(3)$ is the convex hull of the set of permutations matrices, there exists $a_i \geq 0$, $i = 1, \dots, 6$, with $\sum_{i=1}^{6} a_i = 1$ such that $M = \sum_{i=1}^{6} a_i P_i$. Equating both expressions of M, one deduces the existence of a $t \in \mathbb{R}$ such that

$$(a_1, a_2, a_3, a_4, a_5, a_6) = (t - (g^2 - e^2), -t + f^2, -t + g^2, -t + b^2, t - (b^2 - d^2), t).$$

Since $a_i \geq 0$, then $\max\{0, (g^2 - e^2), (b^2 - d^2)\} \leq t \leq \min\{f^2, b^2, g^2\}$. In particular, there are 9 possible equalities (3 for the possible values of the maximum and 3 for the minimum) which should correspond to the 9 components of the boundary of $\Omega(3)$.

Next result follows from [7].

Proposition 5.6 *Let* $N(x, y, z, t) \in \mathbb{R}[x, y, z, t]$ *be the following polynomial:*

$$N(x,y,z,t) = x^2t^2 + y^2z^2 - 2xyzt - 2xt(x+t) - 2yz(y+z) - 2(xyz + xyt + xzt + yzt)$$
$$+x^2 + y^2 + z^2 + t^2 + 2(xy + xz + yt + zt + 2xt + 2yz) - 2(x + y + z + t) + 1.$$

Then,

(*a*) $\mathrm{Ort}(3) = \{A = (a_{i,j}) \in \Omega(3) \mid N(a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}) = 0\}$.

(*b*) $\mathrm{Ort}(3)$ *is homeomorphic to a 3-dimensional sphere.*

Proof. Part (a) directly follows from proposition 2.2 in [7]. There, it is shown too that $K = \{A = (a_{i,j}) \in \Omega(3) \mid N(a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}) \leq 0\}$ is a star-shaped set whose center is the matrix C_0 with entries 1/3 and whose boundary is $\mathrm{Ort}(3)$. Moreover, lemma 2.1 of [7] proves that the ray joining any $M \in \mathrm{Ort}(3)$ with C_0 cuts $\mathrm{Ort}(3)$ just in M. Take now a basis of the four dimensional linear subspace parallel to the the linear variety defined by the permutation matrices. For a given $M \in \mathrm{Ort}(3)$, let $v_M \in \mathbb{R}^4 - \{0\}$ the components in this basis of the vector $M - C_0$. So we have the continuos injective map $\mu : \mathrm{Ort}(3) \longrightarrow \mathcal{S}^3$ defined by $\mu(M) = \frac{v_M}{\|v_M\|}$. Since K is compact and $\mathrm{Ort}(3)$ is its boundary, μ is sujective. Since $\mathrm{Ort}(3)$ is compact, μ is closed. Therefore, μ is an homeomorphism. ∎

6 Building pairs of symmetric tensors from traces

Keeping in mind the parametrization given in Theorem 4.4, we begin this section by describing the traces of a pair of 3×3 symmetric matrices (A, B) parametrized by $(\alpha, \beta, M) \in \mathbb{R}^6 \times \mathrm{Ort}(3)$ as polynomials in their eigenvalues linearly weighted by the components of the matrix M. Since $\mathrm{Ort}(3) \subset \Omega(3)$ and $\Omega(3)$ is the convex hull of the set of permutations matrices, one gets a kind of "physical" interpretation (closely related to the energy of the system) for the mixed traces as a convex combination of permuted products of powers of their eigenvalues.

Proposition 6.1 *Let (A, B) be a pair of 3×3 real symmetric matrices parametrized by $(\alpha, \beta, M) \in \mathbb{R}^6 \times \mathrm{Ort}(3)$. If $t_{i,j} = \mathrm{tr}(A^i B^j)$, then*

$$t_{i,j} = \left(\alpha_1^i, \alpha_2^i, \alpha_3^i\right) M \begin{pmatrix} \beta_1^j \\ \beta_2^j \\ \beta_3^j \end{pmatrix}. \qquad (1)$$

In particular, $t_{i,j}$ are linear functions on the components of M. Moreover, there exists $a \in \mathbb{R}^6$, with $a_i \geq 0$ and $\sum_{i=1}^6 a_i = 1$, such that

$$t_{i,j} = a_1(\alpha_1^i\beta_1^j + \alpha_2^i\beta_2^j + \alpha_3^i\beta_3^j) + a_2(\alpha_1^i\beta_1^j + \alpha_2^i\beta_3^j + \alpha_3^i\beta_2^j) + a_3(\alpha_1^i\beta_3^j + \alpha_2^i\beta_2^j + \alpha_3^i\beta_1^j) +$$
$$a_4(\alpha_1^i\beta_2^j + \alpha_2^i\beta_1^j + \alpha_3^i\beta_3^j) + a_5(\alpha_1^i\beta_3^j + \alpha_2^i\beta_1^j + \alpha_3^i\beta_2^j) + a_6(\alpha_1^i\beta_2^j + \alpha_2^i\beta_3^j + \alpha_3^i\beta_1^j).$$

Remark 6.2 Developing Equality (1) writing $M = S^{[2]} = \begin{pmatrix} a^2 & b^2 & c^2 \\ d^2 & e^2 & f^2 \\ g^2 & h^2 & i^2 \end{pmatrix}$, one deduces:

$$\begin{pmatrix}
\alpha_1 & \alpha_1 & \alpha_1 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_3 & \alpha_3 & \alpha_3 \\
\alpha_1^2 & \alpha_1^2 & \alpha_1^2 & \alpha_2^2 & \alpha_2^2 & \alpha_2^2 & \alpha_3^2 & \alpha_3^2 & \alpha_3^2 \\
\alpha_1^3 & \alpha_1^3 & \alpha_1^3 & \alpha_2^3 & \alpha_2^3 & \alpha_2^3 & \alpha_3^3 & \alpha_3^3 & \alpha_3^3 \\
\beta_1 & \beta_2 & \beta_3 & \beta_1 & \beta_2 & \beta_3 & \beta_1 & \beta_2 & \beta_3 \\
\beta_1^2 & \beta_2^2 & \beta_3^2 & \beta_1^2 & \beta_2^2 & \beta_3^2 & \beta_1^2 & \beta_2^2 & \beta_3^2 \\
\beta_1^3 & \beta_2^3 & \beta_3^3 & \beta_1^3 & \beta_2^3 & \beta_3^3 & \beta_1^3 & \beta_2^3 & \beta_3^3 \\
\alpha_1\beta_1 & \alpha_1\beta_2 & \alpha_1\beta_3 & \alpha_2\beta_1 & \alpha_2\beta_2 & \alpha_2\beta_3 & \alpha_3\beta_1 & \alpha_3\beta_2 & \alpha_3\beta_3 \\
\alpha_1^2\beta_1 & \alpha_1^2\beta_2 & \alpha_1^2\beta_3 & \alpha_2^2\beta_1 & \alpha_2^2\beta_2 & \alpha_2^2\beta_3 & \alpha_3^2\beta_1 & \alpha_3^2\beta_2 & \alpha_3^2\beta_3 \\
\alpha_1\beta_1^2 & \alpha_1\beta_2^2 & \alpha_1\beta_3^2 & \alpha_2\beta_1^2 & \alpha_2\beta_2^2 & \alpha_2\beta_3^2 & \alpha_3\beta_1^2 & \alpha_3\beta_2^2 & \alpha_3\beta_3^2 \\
\alpha_1^2\beta_1^2 & \alpha_1^2\beta_2^2 & \alpha_1^2\beta_3^2 & \alpha_2^2\beta_1^2 & \alpha_2^2\beta_2^2 & \alpha_2^2\beta_3^2 & \alpha_3^2\beta_1^2 & \alpha_3^2\beta_2^2 & \alpha_3^2\beta_3^2
\end{pmatrix}
\begin{pmatrix} a^2 \\ b^2 \\ c^2 \\ d^2 \\ e^2 \\ f^2 \\ g^2 \\ h^2 \\ i^2 \end{pmatrix}
=
\begin{pmatrix} t_{1,0} \\ t_{2,0} \\ t_{3,0} \\ t_{0,1} \\ t_{0,2} \\ t_{0,3} \\ t_{1,1} \\ t_{2,1} \\ t_{1,2} \\ t_{2,2} \end{pmatrix} \qquad (2)$$

Write system (2) as $HX = T$ for short. Given a vector $t = (t_1, \ldots, t_{10}) \in \mathbb{R}^{10}$, one can ask whether there exists a pair of 3×3 real symmetric matrices (A, B) such that $t = t(A, B)$, i.e., $(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}) = (t_{1,0}, t_{0,1}, t_{2,0}, t_{1,1}, t_{0,2}, t_{3,0}, t_{2,1}, t_{1,2}, t_{0,3}, t_{2,2})$, and if this is the case, how to find it. Given a pair of 3×3 real symmetric matrices (A, B), it is clear that $\mathrm{tr}(A^2) \geq 0$ and $\mathrm{tr}(B^2) \geq 0$. Moreover, by Remark 3.3 or Proposition 6.1, $\mathrm{tr}(A^2B^2) \geq 0$. Therefore, the first three necessary conditions to $t = t(A, B)$ are $t_3, t_5, t_{10} \geq 0$. Given a 3×3 real matrix A, by Newton's Formula (see, for instance, [4]), its characteristic polynomial in terms of its traces is

$$Q_A(\lambda) = \lambda^3 - \mathrm{tr}(A)\lambda^2 + \frac{1}{2}\left(-\mathrm{tr}(A^2) + \mathrm{tr}^2(A)\right)\lambda - \left(\frac{1}{3}\mathrm{tr}(A^3) - \frac{1}{2}\mathrm{tr}(A)\mathrm{tr}(A^2) + \frac{1}{6}\mathrm{tr}^3(A)\right).$$

If A is real symmetric, $Q_A(\lambda)$ must have three real roots. By Cardano's Formula, this is equivalent to $C(\mathrm{tr}(A), \mathrm{tr}(A^2), \mathrm{tr}(A^3)) \leq 0$, where $C(x, y, z) \in \mathbb{R}[x, y, z]$ is the polynomial

$C(x,y,z) = x^6 - 9x^4y + 8x^3z + 21x^2y^2 - 36xyz - 3y^3 + 18z^2$. If $C(\mathrm{tr}(A), \mathrm{tr}(A^2), \mathrm{tr}(A^3)) < 0$, then the three roots are distinct. Therefore, the second two necessary conditions to $t = t(A,B)$ are $C(t_1,t_3,t_6) \leq 0$ and $C(t_2,t_5,t_9) \leq 0$.

Suppose now one has a vector $t = (t_1,\dots,t_{10}) \in \mathbb{R}^{10}$ verifying $t_3,t_5,t_{10} \geq 0$ and $C(t_1,t_3,t_6) \leq 0$ and $C(t_2,t_5,t_9) \leq 0$. From t_1,t_3,t_6 one can construct the characteristic polynomial of what will be A and find its three eigenvalues $\alpha = \alpha_1,\alpha_2,\alpha_3$ which will be real since $C(t_1,t_3,t_6) \leq 0$. Analogously, from (t_2,t_5,t_9) one can construct the characteristic polynomial of what will be B and find its three roots $\beta = \beta_1,\beta_2,\beta_3$ which will be real since $C(t_2,t_5,t_9) \leq 0$. Once one has α and β, then one can construct the former system $HX = T$, where T is defined from t by reordering adequately its components. If α are non zero and pairwise different, i.e., $C(t_1,t_3,t_6) < 0$, and if β are non zero and pairwise different too, i.e., $C(t_2,t_5,t_9) < 0$, then it is not difficult to see that the rank of H is nine. Therefore, the system $HX = T$ is compatible if and only if $\det(H \mid T) = 0$. In such a case, it has a unique solution X. It remains to check that the correspondant matrix M is doubly stochastic, which is not difficult. Finally, to see $M \in \mathrm{Ort}(3)$ one can use the equation given in Proposition 5.6.

Remark 6.3 Consider the six unmixed traces $t_{1,0}, t_{2,0}, t_{3,0}, t_{0,1}, t_{0,2}, t_{0,3}$, which completely determine the eigenvalues of A and B. If a mixed trace of the set $\{t_{1,1}, t_{2,1}, t_{1,2}, t_{2,2}\}$ is skipped, then the correspondant row in the system (2) must be removed. It can be shown that this remaining matrix has rank eight. Thus, the correspondant set of solutions $L_{i,j}$ is a line. From the first six equations, one can check that $L_{i,j}$ must be contained into the linear variety defined by the permutation matrices. By Theorem 5.1, $\Omega(3)$ is the convex hull of the set of permutation matrices and, by Proposition 5.6, $\mathrm{Ort}(3)$ is the boundary of a star-shaped set contained in $\Omega(3)$ whose rays joining any point of the boundary with the origin cut the boundary just in that point. Therefore, $L_{i,j} \bigcap \mathrm{Ort}(3)$ contains at least two points unless it is tangent to $\mathrm{Ort}(3)$.

Notation 6.4 Recall that an anisotropic second-order valued function might be thought as a function $f : \mathcal{SP}(3) \to \mathbb{R}$ (see Remark 3.5). We will say that f is *position-free* if there exists $g : \mathbb{R}^6 \to \mathbb{R}$ such that the composition $f \circ p : \mathbb{R}^6 \times \mathrm{Ort}(3) \to \mathbb{R}$ verifies $(f \circ p)(\alpha,\beta,M) = g(\alpha,\beta)$ for all $(\alpha,\beta,M) \in \mathbb{R}^6 \times \mathrm{Ort}(3)$, i.e., the expresion of f in terms of the parametrization does not depend on M. For instance, the unmixed traces $t_{1,0}, t_{2,0}, t_{3,0}, t_{0,1}, t_{0,2}, t_{0,3}$ are position-free anisotropic second-order valued functions. Next theorem is the main result of this section. Endowing $\mathcal{SP}(3)$ with the natural topology and using topological reasonings, we generalize the fact that 9 traces do not determine pairs of symmetric tensors.

Theorem 6.5 *Any set of nine anisotropic continuous second-order valued functions, six of them position-free, do not determine the similarity class of a pair of symmetric tensors.*

Proof. Suppose there exist $f_i : \mathcal{SP}(3) \to \mathbb{R}$, $i = 1,\dots,9$, six of them position-free, defining an injective continuous map $f : \mathcal{SP}(3) \to \mathbb{R}^9$. Fix $\alpha_1 < \alpha_2 < \alpha_3$ and $\beta_1 < \beta_2 < \beta_3$, the map $\Phi : \mathrm{Ort}(3) \to \mathbb{R}^9$, defined by $\Phi(M) = (f \circ p)(\alpha,\beta,M)$ is an injective continuous map. Since six of the nine f_i are position-free, then six of the components of Φ are constant. Thus, Φ induces an injective continuous map $\Psi : \mathrm{Ort}(3) \to \mathbb{R}^3$. Since $\mathrm{Ort}(3)$ is homeomorphic to a 3-dimensional sphere (see Proposition 5.6), one has an injective continuos map from a 3-dimensional sphere to $\mathbb{R}^3$. But this is not possible by Borsuk-Ulam's Theorem (see, for instance, [6]). ∎

7 Equations relating traces

By remark 3.5, $\mathcal{SP}(3)$ is a subset of $\mathbb{R}^{10}$ through the injective map defined by $t : \mathcal{SP}(3) \to \mathbb{R}^{10}$, $t([(A,B)]) = t(A,B)$. In other words, similarity classes of pairs of 3×3 real symmetric matrices are globally described by ten traces. On the other hand, the similarity class of any (A,B) is locally described by nine parameters: its six eigenvalues and three Euler Angles relating the principal axes. Therefore, there should exist a relationship among the ten traces which could be computed explicitly following Remark 6.2. This relation is a new equation relating the 10 classical traces. By elimination theory, it should be polynomial (since all the functions involved are polynomials) and its degree in each mixed trace would be at least two (by Example 3.6). This function must vanish on an open dense set (pairs of symmetric matrices with non zero and pairwise distinct eigenvalues) but, being continuous, it would vanish too on all the set $\mathcal{SP}(3)$. Similarly, one could find a relationship among any set of anisotropic (linear on the components of M) second-order valued functions as in 2.

Question 7.1 For a given pair of 3×3 real symmetric matrices (A,B), and for all integers $n_1, m_1, \ldots, n_r, m_r \geq 0$, can $\mathrm{tr}(A^{n_1}B^{m_1} \ldots A^{n_r}B^{m_r})$ be expressed as a polynomial in the coordinates of $t(A,B)$? Some well-known identities satisfied by 3×3 matrices ([8]) might be useful to give and answer.

For the simplest case we have the following well-known example.

Example 7.2 Let (A,B) be a pair of 3×3 real symmetric matrices. Using the Theorem of Cayley Hamilton or the Theory of Invariants (see, for instance, [1]), it is easy to prove:

$$\begin{aligned}&2\mathrm{tr}(ABAB) = (\mathrm{tr}^2(A) - \mathrm{tr}(A^2))(\mathrm{tr}^2(B) - \mathrm{tr}(B^2)) + 4\mathrm{tr}(A)\mathrm{tr}(AB^2) + 4\mathrm{tr}(B)\mathrm{tr}(A^2B)\\ &-4\mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(AB) + 2\mathrm{tr}^2(AB) - 4\mathrm{tr}(A^2B^2)\,.\end{aligned}$$

We finish by giving the formula which relates the ten traces for the particular case of M being the identity.

Example 7.3 Let (A,B) be a pair of 3×3 real symmetric matrices and $t(A,B) \in \mathbb{R}^{10}$. If A and B are diagonal and x stands for $x = \frac{1}{6}t_{1,0}^4 - t_{1,0}^2 t_{2,0} + \frac{4}{3}t_{1,0}t_{3,0} + \frac{1}{2}t_{2,0}^2$, using Remark 3.3

$$t_{0,3} = \begin{pmatrix} t_{0,1} & t_{1,1} & t_{2,1} \end{pmatrix} \begin{pmatrix} 1 & t_{1,0} & t_{2,0} \\ t_{1,0} & t_{2,0} & t_{3,0} \\ t_{2,0} & t_{3,0} & x \end{pmatrix}^{-1} \begin{pmatrix} t_{0,2} \\ t_{1,2} \\ t_{2,2} \end{pmatrix},$$

which implies

$$\begin{aligned}&\left(\frac{1}{6}t_{1,0}^6 - \frac{7}{6}t_{1,0}^4 t_{2,0} + \frac{4}{3}t_{1,0}^3 t_{3,0} + \frac{3}{2}t_{1,0}^2 t_{2,0}^2 - \frac{10}{3}t_{1,0}t_{2,0}t_{3,0} + \frac{1}{2}t_{2,0}^3 + t_{3,0}^2\right) t_{0,3} +\\ &\left(\frac{1}{6}t_{1,0}^4 t_{2,0} - t_{1,0}^2 t_{2,0}^2 + \frac{4}{3}t_{1,0}t_{2,0}t_{3,0} + \frac{1}{2}t_{2,0}^3 - t_{3,0}^2\right) t_{0,1}t_{0,2} +\\ &\left(t_{2,0}t_{3,0} - \frac{1}{6}t_{1,0}^5 + t_{1,0}^3 t_{2,0} - \frac{4}{3}t_{1,0}^2 t_{3,0} - \frac{1}{2}t_{1,0}t_{2,0}^2\right)(t_{0,1}t_{1,2} + t_{1,1}t_{0,2}) +\\ &(t_{1,0}t_{3,0} - t_{2,0}^2)(t_{0,1}t_{2,2} + t_{0,2}t_{2,1}) +\\ &\left(\frac{1}{6}t_{1,0}^4 - t_{1,0}^2 t_{2,0} + \frac{4}{3}t_{1,0}t_{3,0} - \frac{1}{2}t_{2,0}^2\right) t_{1,1}t_{1,2} +\\ &(t_{1,0}t_{2,0} - t_{3,0})(t_{1,1}t_{2,2} + t_{2,1}t_{1,2}) + (t_{2,0} - t_{1,0}^2)t_{2,1}t_{2,2} = 0\,.\end{aligned}$$

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References

[1] M. Bocher: Introduction to Higher Algebra. The Macmillan Company, New York, 1907.

[2] J.P. Boehler. Applications of Tensor Functions in Solid Mechanics. CISM Adv. School, Springer-Verlag, 1987.

[3] I. Carol, E. Rizzi: *Some problems of tensor calculus in the context of solid mechanics.* Grups d'Estudi de Matemàtica i Tecnologia. Juliol 2002.

[4] F.R. Gantmacher: The Theory of Matrices. Chelsea Publishing Company, New York, N.Y., 1959.

[5] H. Minc: Nonnegative matrices. Wiley-Interscience, 1988.

[6] J. R. Munkres: *Topology, a first course.* Prentice-Hall Inc., 1975.

[7] H. Nakazato: *Set of 3x3 orthostochastic matrices*, Nihonkai Mathematical Journal vol. 7 (1996), No. 2, pp. 83-100.

[8] A.J.M. Spencer, R.S. Rivlin: *The Theory of Matrix Polynomials and its Application to the Mechanics of Isotropic Continua.* Arch. Rational Mech. Anal., Vol 2, 1958/1959, 309-336.

[9] Q.S. Zheng: *Theory of representations for tensor functions. A unified invariant approach to constitutive equations.* Appl. Mech. Rev. vol 47, no 11, November 1994, 545-587.

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