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Salutació

Ara fa una any saludava amb satisfacció la presentació escrita dels primers Grups d'Estudi de Matemàtica i Tecnologia. La celebració dels segons GEMT, i la publicació de les seves conclusions que avui disposem, corroboren l'interès i l'oportunitat d'aquella iniciativa, configurant-la com un fòrum molt adient per afavorir les relacions entre els matemàtics i el món de la tecnologia. Per a la Facultat de Matemàtiques i Estadística, que té un compromís actiu amb l'entorn tecnològic que l'envolta, és un plaer contribuir a la realització d'aquestes iniciatives.
Un cop més hem de celebrar l'èxit organitzatiu i de participació aconseguit pels organitzadors, els professors Enric Fossas i Joan Solà-Morales, que van crear les condicions idònies pel desenvolupament de les jornades, i la generositat i dedicació de tots els participants.

Barcelona, Octubre de 2002

Pere Pascual
Degà FME

Presentació

Aquest llibret que teniu a les mans recull els problemes que es varen proposar als Grups d'Estudi de Matemàtiques i Tecnologia, el Juliol del 2002. Així mateix conté les respostes que vàrem ser capaços de donar. Els problemes pertanyen a àmbits ben diferents, com són: els procediments de puntuar en un concurs, la resistència de materials i el processat (costures) d'imatges. Els resultats obtinguts són, com a mínim, satisfactoris, doncs, si bé no es proporciona una resposta definitiva a tots els problemes plantejats, s'apunten els procediments que han de portar a completar-la. De manera que acabar-ho, ara només seria una qüestió de temps.

L'assistència a la trobada, una trentena de participants, és suficient per a pensar en tornar-la a organitzar entre juny i juliol de 2003. El format -primer la presentació dels problemes a tots els participants, després, els debats per grups en quatre sessions de tres hores, en funció del tema d'interès de cadascú, que no necessàriament de l'expertesa, i finalment la presentació conjunta de les aportacions de cada grup- també ens sembla apropiat. És evident que el temps que es dedica als debats no és suficient per a resoldre gairebé cap dels problemes plantejats. De fet, tampoc és l'objectiu d'aquests grups d'estudi, ja que el que es pretén és desbrossar el problema apuntant camins per a la seva resolució. En cas que qui ha proposat el problema hi estigui interessat, ja demanarà si es poden continuar els treballs. També és cert que una característica comú dels participants en els grups d'estudi és ser molt gelosos del propi temps, i una cosa és dedicar tres dies a desentumir els reflexes i una altra, comprometre's en un projecte de recerca o transferència de tecnologia, més o menys allunyat de la recerca de cadascú, per a un període mínim de mig any.

La participació en els grups d'estudi ha tingut un cost nul per als participants. Agraïm la col·laboració de la Facultat de Matemàtiques i Estadística de la Universitat Politècnica de Catalunya, per l'ajuda en l'organització i en l'edició i publicació d'aquest llibret. També han col·laborat el Vicerrectorat

de Recerca de la UPC i el Departament d'Universitats, Recerca i Societat de la Informació.

Pel que fa als grups d'estudi celebrats l'any anterior, els GEMT'01, podem dir que han donat lloc a alguna realització: s'ha continuat treballant en el problema *Codificació espacial i temporal*, en Xavier Muñoz dirigeix un projecte fi de carrera a l'Escola Tècnica Superior d'Enginyeria de Telecomunicació de Barcelona. També ens consta que els que varen proposar el problema *Estudi de la variabilitat del ritme cardíac* estaven interessats en que s'hi continués treballant.

Per acabar, només assenyalar que l'èxit d'aquestes trobades és degut a la implicació dels participants. Moltes gràcies, doncs, als que hi heu assistit. Els organitzadors també volem deixar constància del nostre agraïment als qui van plantejar-nos els problemes; ens agradaria que aquesta relació no acabés aquí.

Barcelona, octubre de 2002

Enric Fossas i Joan Solà-Morales. Universitat Politècnica de Catalunya

Some problems of tensor calculus in the context of solid mechanics

Problema proposat per Ignacio Carol, ETSECCPB, UPC, Dept. of Geotechnical Engineering and Geo-Sciences i Egidio Rizzi, University of Bergamo, Faculty of Engineering at Dalmine, Engineering Dept. Italy

3.1 Introduction

We list below a series of problems of tensor calculus arising in our on-going research works in the area of solid mechanics. The typical framework under consideration is that of the so-called constitutive laws, i.e. the formulation of appropriate stress/strain relations apt to describe specific material behaviors. These material responses typically involve: non-linear behavior, elastic stiffness degradation, plasticity, large deformations (large spatial displacement gradients), brittle/quasi-brittle/ductile behavior. Characteristic materials under concern are: concrete-type materials and geomaterials, rocks, metals, composites, ceramics. Fields of applications regard: aeronautical, civil, geotechnical, materials, mechanical and structural engineering. Samples of the mechanical contexts in which the questions presented below arise are available in [**3**], [**5**] and refer primarily to the modeling of anisotropic (orthotropic) damage of the elastic properties (progressive stiffness degradation and loss of strength).

Notation. Compact or index tensor notation is adopted. Vectors and second-order tensors are identified by boldface characters (e.g. $\mathbf{n}$, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, $\boldsymbol{\phi}$, $\mathbf{A}$, respectively), whereas fourth-order tensors are denoted by blackboard-bold fonts (e.g. $\mathbb{A}, \mathbb{C}, \mathbb{E}$). Symbols '$\cdot$' and ':' denote the inner products with single and double contraction, respectively. Superscript $^{\mathrm{T}}$ indicates the transpose operation applied either to second-order tensors, i.e. $\mathbf{v}\cdot\mathbf{A}^{\mathrm{T}}\cdot\mathbf{u}=\mathbf{u}\cdot\mathbf{A}\cdot\mathbf{v}$, for any second-order tensor $\mathbf{A}$ and vectors $\mathbf{u}, \mathbf{v}$, or to fourth-order tensors, i.e. $\mathbf{B}{:}\mathbb{A}^{\mathrm{T}}{:}\mathbf{C}{=}\mathbf{C}{:}\mathbb{A}{:}\mathbf{B}$, for any fourth-order tensor $\mathbb{A}$ and second-order tensors $\mathbf{B}, \mathbf{C}$; componentwise $(A^{\mathrm{T}})_{ij}{=}A_{ji}$ and $(A^{\mathrm{T}})_{ijkl}{=}A_{klij}$. The dyadic product of either vectors or second-order tensors is indicated with '$\otimes$' and defined respectively as $(\mathbf{u}{\otimes}\mathbf{v}){\cdot}\mathbf{z}{=}(\mathbf{v}{\cdot}\mathbf{z})\,\mathbf{u}$, for any vectors $\mathbf{u}, \mathbf{v}, \mathbf{z}$, and $(\mathbf{A}{\otimes}\mathbf{B}){:}\mathbf{C}{=}(\mathbf{B}{:}\mathbf{C})\,\mathbf{A}$, for any second-order tensors $\mathbf{A}, \mathbf{B}, \mathbf{C}$, whereas '$\overline{\underline{\otimes}}$' denotes the symmetrized dyadic product of second-order tensors defined as $(\mathbf{A}\,\overline{\underline{\otimes}}\,\mathbf{B}){:}\mathbf{C}{=}\mathbf{A}{\cdot}\mathbf{C}^s{\cdot}\mathbf{B}^{\mathrm{T}}$, for any second-order tensors $\mathbf{A}, \mathbf{B}, \mathbf{C}$, where $\mathbf{C}^s{=}(\mathbf{C}{+}\mathbf{C}^{\mathrm{T}})/2$ is the symmetric part of $\mathbf{C}$; componentwise $(\mathbf{u}{\otimes}\mathbf{v})_{ij}{=}u_i\,v_j$, $(\mathbf{A}{\otimes}\mathbf{B})_{ijkl}{=}A_{ij}\,B_{kl}$ and $(\mathbf{A}\,\overline{\underline{\otimes}}\,\mathbf{B})_{ijkl}{=}(A_{ik}\,B_{jl}{+}A_{il}\,B_{jk})/2$. $\mathbf{I}$ and $\mathbb{I}^s{=}\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I}$ are respectively the second-order and symmetric (major and minor symmetries) fourth-order identity tensors; componentwise $I_{ij}{=}\delta_{ij}$ and $I^s_{ijkl}{=}(\delta_{ik}\delta_{jl}{+}\delta_{il}\delta_{jk})/2$, where δ_{ij} is the Kronecker delta ($\delta_{ij}{=}1$ if $i{=}j$, $\delta_{ij}{=}0$ if $i{\neq}j$). $\mathbb{I}^s$ maps any second-order tensor $\mathbf{A}$ into its symmetric part $\mathbf{A}^s$, i.e. $\mathbb{I}^s{:}\mathbf{A}{=}\mathbf{A}^s$ and any symmetric second-order tensor $\mathbf{B}{=}\mathbf{B}^{\mathrm{T}}$ into itself, i.e. $\mathbb{I}^s{:}\mathbf{B}{=}\mathbf{B}$. Symbol 'tr' denotes the trace operator applied to second-order tensors, i.e. $\mathrm{tr}\,\mathbf{A}{=}\mathbf{I}{:}\mathbf{A}{=}A_{ii}$. For more detailed definitions see e.g. [**5**] , Appendix A.

3.2 A typical mechanical context: secant stress/strain relations in elastic damage

To give some mechanical flavor of the tensorial context relevant to the last issue mentioned in the Introduction (with reference to [**1**], [**3**], [**5**]), consider the definition of the elastic response of a material that has undergone some damage process. At any damage state the nominal (small) strain tensor $\boldsymbol{\epsilon}$ (symmetric part of the displacement gradient) and stress tensor $\boldsymbol{\sigma}$ can be related by the following secant elastic constitutive law (Fig. 1):

$$\boldsymbol{\epsilon} = \mathbb{C}(\mathbb{C}_0, \boldsymbol{\mathcal{D}}) : \boldsymbol{\sigma}\,; \qquad \boldsymbol{\sigma} = \mathbb{E}(\mathbb{E}_0, \bar{\boldsymbol{\mathcal{D}}}) : \boldsymbol{\epsilon}\,, \tag{3.2.1}$$

where $\mathbb{C}$ and $\mathbb{E}$ are the current positive-definite fourth-order compliance and stiffness tensors, inverse of each other (i.e. $\mathbb{C}{:}\mathbb{E}{=}\mathbb{E}{:}\mathbb{C}{=}\mathbb{I}^s$) and endowed with both major and minor symmetries. The current compliance $\mathbb{C}(\mathbb{C}_0, \boldsymbol{\mathcal{D}})$ and stiffness $\mathbb{E}(\mathbb{E}_0, \bar{\boldsymbol{\mathcal{D}}})$ start from their initial values $\mathbb{C}_0$, $\mathbb{E}_0$ in the undamaged state and evolve as functions of generally-defined damage tensor variables $\boldsymbol{\mathcal{D}}$, or of a dual damage tensor variables $\bar{\boldsymbol{\mathcal{D}}}$.

The underlying damage variables $\boldsymbol{\mathcal{D}}$ and $\bar{\boldsymbol{\mathcal{D}}}$ can be taken for example as positive-definite symmetric second-order tensors: the so-called integrity tensor $\bar{\boldsymbol{\phi}}$, varying between $\mathbf{I}$ and $\mathbf{0}$, or its inverse $\boldsymbol{\phi}{=}\bar{\boldsymbol{\phi}}^{-1}$, with complementary

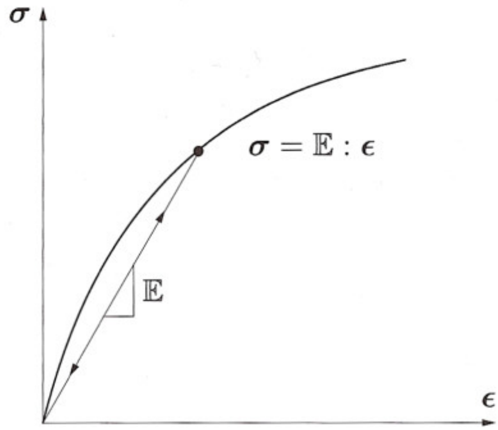

Figura 3.1: *Symbolic stress/strain curve and current secant stiffness* $\mathbb{E}$.

variation between **I** and ∞. The square-root tensors $\mathbf{w}=\boldsymbol{\phi}^{1/2}$, $\bar{\mathbf{w}}=\bar{\boldsymbol{\phi}}^{1/2}$ may be adopted as well where more convenient in notation.
The damage-state relations $\mathbb{C}=\mathbb{C}(\mathbb{C}_0,\boldsymbol{\mathcal{D}})$ and $\mathbb{E}=\mathbb{E}(\mathbb{E}_0,\bar{\boldsymbol{\mathcal{D}}})$ are derived by following typical steps in Continuum Damage Mechanics. Often they are expressed in terms of the so-called fourth-order damage-effect tensors $\mathbb{A}, \bar{\mathbb{A}}$:

$$\mathbb{C}(\mathbb{C}_0,\boldsymbol{\mathcal{D}}) = \mathbb{A}^{\mathrm{T}}(\boldsymbol{\mathcal{D}}) : \mathbb{C}_0 : \mathbb{A}(\boldsymbol{\mathcal{D}}) \; ; \qquad \mathbb{E}(\mathbb{C}_0,\bar{\boldsymbol{\mathcal{D}}}) = \bar{\mathbb{A}}(\bar{\boldsymbol{\mathcal{D}}}) : \mathbb{E}_0 : \bar{\mathbb{A}}^{\mathrm{T}}(\bar{\boldsymbol{\mathcal{D}}}) \; ,$$

where $\mathbb{A}(\boldsymbol{\mathcal{D}})=\bar{\mathbb{A}}^{-1}(\bar{\boldsymbol{\mathcal{D}}})$ and $\bar{\mathbb{A}}(\bar{\boldsymbol{\mathcal{D}}})=\mathbb{A}^{-1}(\boldsymbol{\mathcal{D}})$ are dual non-singular fourth-order damage-effect tensors, inverse of each other (i.e. $\mathbb{A}{:}\bar{\mathbb{A}}=\bar{\mathbb{A}}{:}\mathbb{A}=\mathbb{I}^{s}$) and endowed with minor symmetries (not necessarily major symmetry).
In this context the need of managing tools of tensor calculus that are useful to handle a number of operations involving the second- and fourth-order tensor quantities introduced above spontaneously arises.

3.3 List of tensorial problems

1. Given symmetric second-order tensors $\mathbf{A}, \mathbf{U}$ and scalar c explore the eigenspectrum of second-order tensor $\mathbf{B}$ obtained by the transformation

$$\mathbf{B} = c\ \mathbf{U} \cdot \mathbf{A} \cdot \mathbf{U} \, . \tag{3.3.1}$$

   In particular express the rotation of the principal directions of $\mathbf{A}$ to get those of $\mathbf{B}$. This aspect is relevant to the context of defining appropriate strain measures in large deformation analysis and of modeling anisotropic damage (see e.g. the definition of the pseudo-logarithmic rate of damage [**3**], [**5**] recalled below in Item 5).

2. So-called representation theorems (e.g. [**2**], [**8**] are often adopted to express scalar or tensor functions of tensorial arguments. A typical context is that of expressing the elastic properties of a material endowed with fabric structure (fabric elasticity, e.g. [**9**], or undergoing anisotropic damage [**1**], [**3**], [**5**] (e.g. characterized by a second-order tensor as in Section 2). Given two second-order tensors $\mathbf{A}, \mathbf{B}$, consider the three plus three direct invariants $\operatorname{tr}\mathbf{A}, \operatorname{tr}\mathbf{A}^2, \operatorname{tr}\mathbf{A}^3$ and $\operatorname{tr}\mathbf{B}, \operatorname{tr}\mathbf{B}^2, \operatorname{tr}\mathbf{B}^3$, and the four mixed invariants

$$\operatorname{tr}(\mathbf{A}\cdot\mathbf{B}),\quad \operatorname{tr}(\mathbf{A}^2\cdot\mathbf{B}),\quad \operatorname{tr}(\mathbf{A}\cdot\mathbf{B}^2),\quad \operatorname{tr}(\mathbf{A}\cdot\mathbf{B})^2\ \text{(or } \operatorname{tr}(\mathbf{A}^2\cdot\mathbf{B}^2)\text{)}. \tag{3.3.2}$$

Determine the irreducible basis of the invariants of the system $(\mathbf{A}, \mathbf{B})$. Attach a clear physical meaning to the mixed invariants and link them if possible to the relationship between the relative angles of the principal axes of $\mathbf{A}$ and $\mathbf{B}$. In this respect, the question arises whether or not four vs. three mixed invariants are providing redundant information. Furthermore, due to repeated application of the Cayley-Hamilton theorem the use of $\operatorname{tr}(\mathbf{A}\cdot\mathbf{B})^2$ rather than $\operatorname{tr}(\mathbf{A}^2\cdot\mathbf{B}^2)$ seems rather arbitrary (see e.g. [**2**], [**9**]). Then, one wonders if there is any reason to prefer one of the two (e.g. $\operatorname{tr}(\mathbf{A}\cdot\mathbf{B})^2$) referring also to the physical meaning mentioned above.

3. An isotropic second-order tensor valued function $\mathbf{f}$ of second-order tensor $\mathbf{A}$ can be expressed as (e.g. [**6**]):

$$\mathbf{f}(\mathbf{A}) = f_0\,\mathbf{I} + f_1\,\mathbf{A} + f_2\,\mathbf{A}^2\,, \tag{3.3.3}$$

where f_0, f_1, f_2 are functions of the three principal invariants of $\mathbf{A}$ (which can be classically defined as ${}^{\mathbf{A}}I_1 = \operatorname{tr}\mathbf{A}$, ${}^{\mathbf{A}}I_2 = (\operatorname{tr}^2\mathbf{A} - \operatorname{tr}\mathbf{A}^2)/2$, ${}^{\mathbf{A}}I_3 = \det\mathbf{A} = \operatorname{tr}\mathbf{A}^3/3 + \operatorname{tr}^3\mathbf{A}/6 - \operatorname{tr}\mathbf{A}\,\operatorname{tr}\mathbf{A}^2/2$ and enter the Cayley-Hamilton theorem applied to $\mathbf{A}$, i.e. $\mathbf{A}^3 - {}^{\mathbf{A}}I_1\,\mathbf{A}^2 + {}^{\mathbf{A}}I_2\,\mathbf{A} - {}^{\mathbf{A}}I_3\,\mathbf{I} = \mathbf{0}$). Consider typical sample functions, e.g. $\mathbf{f}(\mathbf{A}) = \ln\mathbf{A}, \exp\mathbf{A}, \mathbf{A}^n, \mathbf{A}^{1/n}$, etc., and give the coefficients f_0, f_1, f_2 in full invariant form. Mechanical contexts referring to this problem are for example: the definition of logarithmic strain measures, the adoption of power or logarithmic damage variables.

4. Given the specific functions selected above, evaluate the tensor derivatives

$$\frac{\partial\mathbf{f}(\mathbf{A})}{\partial\mathbf{A}} = \ ? \tag{3.3.4}$$

expressed as fourth-order tensors in terms of $\mathbf{A}$ and of its invariants (see e.g. the derivative of the square-root function derived in [**4**] and used in [**5**]. The task should be that of performing all the operations in compact form (i.e. without resorting to spectral representations).

5. A more specific task within the previous context is considered next. Let:

$$\begin{aligned} -\boldsymbol{\mathcal{Y}} &= \frac{1}{2}\Big(-\frac{\nu_0}{E_0}\, tr\,\boldsymbol{\sigma}_{\text{eff}}\ \boldsymbol{\sigma}_{\text{eff}} + \frac{1+\nu_0}{E_0}\,\boldsymbol{\sigma}_{\text{eff}}{}^2 \Big) \,, \\ \boldsymbol{\sigma}_{\text{eff}} &= \mathbf{w}\cdot\boldsymbol{\sigma}\cdot\mathbf{w} \,, \\ \mathbf{w} &= \boldsymbol{\phi}^{1/2} \,, \quad \boldsymbol{\phi} = \mathbf{w}^2 \,, \\ \dot{\boldsymbol{\phi}} &= \frac{1}{2}\,\mathbf{w}\cdot\overset{\circ}{\mathbf{L}}\cdot\mathbf{w} \,, \end{aligned} \tag{3.3.5}$$

where scalars E_0, ν_0 are classical undamaged isotropic elastic constants (Young's modulus and Poisson's ratio) and all other tensor quantities are second-order tensors (physical meaning is explained in [**5**]). Symbol $\dot{\boldsymbol{\phi}}$ marks the time rate of $\boldsymbol{\phi}$ while $\overset{\circ}{\mathbf{L}}$ denotes a non-holonomic rate ($\overset{\circ}{\mathbf{L}}$ is defined only as a rate, i.e. it is not the rate of a finite quantity $\mathbf{L}$). Equation (3.3.5) gives an example of transformation (3.3.1) referred-to earlier. Evaluate the partial derivative:

$$\left.\frac{\partial(-\boldsymbol{\mathcal{Y}})}{\partial \mathbf{L}}\right|_{\boldsymbol{\sigma}} = \ ? \tag{3.3.6}$$

in compact form in terms of $\boldsymbol{\sigma}_{\text{eff}}, \mathbf{w}$ and the relevant invariants.

6. Let $\mathbf{n}$ be a unit direction in space. Examine the properties of

$$\mathrm{f}\,(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n}) \quad \text{vs.} \quad \mathbf{n}\cdot\mathbf{f}(\mathbf{A})\cdot\mathbf{n} \,, \tag{3.3.7}$$

i.e. of the function of the projection of $\mathbf{A}$ in the direction $\mathbf{n}$ vs. of the projection of the function. Focus on the cases of the functional dependencies considered earlier, e.g. $\mathbf{f}(\mathbf{A}) = \mathbf{A}^n$, $\mathbf{f}(\mathbf{A}) = \ln\mathbf{A}$. Relevant contexts are: so-called microplane models apt to represent effective properties of microcracked continua, determination of orientation properties in fabric and damaged materials, eigenanalysis of the elastic acoustic tensor in wave propagation (where similarly $\mathbf{f}(\mathbf{n}\cdot\mathbb{E}\cdot\mathbf{n})$ vs. $\mathbf{n}\cdot\mathbb{f}(\mathbb{E})\cdot\mathbf{n}$ should be considered).

7. Consider the orthotropic representation of a symmetric fourth-order tensor $\mathbb{A}$ [**9**]:

$$\begin{aligned} \mathbb{A} = {} & a_1\,\mathbf{I}\otimes\mathbf{I} + a_2\,\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I} + a_3\,\mathbf{w}\otimes\mathbf{w} + a_4\,(\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{I} + \mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{w}) \\ & + a_5\,\mathbf{w}^2\otimes\mathbf{w}^2 + a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w} + a_7\,(\mathbf{w}\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{w}) \\ & + a_8\,(\mathbf{w}^2\otimes\mathbf{w} + \mathbf{w}\otimes\mathbf{w}^2) + a_9\,(\mathbf{w}^2\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{w}^2) \,, \end{aligned} \tag{3.3.8}$$

where the nine scalar coefficients a_i, i=1-9, are any arbitrary polynomial functions of the three principal invariants of second-order tensor $\mathbf{w}$. Imagine to seek the inverse of $\bar{\mathbb{A}} = \mathbb{A}^{-1}$ in terms of $\bar{\mathbf{w}} = \mathbf{w}^{-1}$ as represented in a form dual to (3.3.8) with (unknown) dual coefficients $\bar{a}_i$, i=1-9. Determine all the specific subsets of coefficients that span the same sets of tensor terms after inversion and give the explicit expressions of the coefficients $\bar{a}_i$ in full invariant form. Specific examples of the sought correspondence are the two-coefficients isotropic case:

$$\mathbb{A} = a_2\,\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I} + a_1\,\mathbf{I}\otimes\mathbf{I}\,; \quad \bar{\mathbb{A}} = \bar{a}_2\,\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I} + \bar{a}_1\,\mathbf{I}\otimes\mathbf{I}\,, \tag{3.3.9}$$

with

$$\bar{a}_2 = \frac{1}{a_2}\,; \qquad \bar{a}_1 = -\frac{a_1}{a_2\,(3a_1 + a_2)}\,, \tag{3.3.10}$$

and the so-called two-coefficients Valanis-type structure:

$$\mathbb{A} = a_6\;\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w} + a_3\;\mathbf{w}\otimes\mathbf{w}\,; \quad \bar{\mathbb{A}} = \bar{a}_6\;\bar{\mathbf{w}}\,\overline{\underline{\otimes}}\,\bar{\mathbf{w}} + \bar{a}_3\;\bar{\mathbf{w}}\otimes\bar{\mathbf{w}}\,, \tag{3.3.11}$$

with

$$\bar{a}_6 = \frac{1}{a_6}\,; \qquad \bar{a}_3 = -\frac{a_3}{a_6\,(3a_3 + a_6)}\,. \tag{3.3.12}$$

8. Examine the possibility to address/solve (part of) the problems above by using the theory of quaternions and biquaternions (see e.g. [**7**] in the context of the algebra decomposition of fourth-order tensors). Since the equivalence of quaternions with no real component to 3D rotations has been already established with very simple representation of composed rotations, etc., one wonders if and how these tools could be conveniently used to represent symmetric second-order tensors (e.g. biquaternions with no real or imaginary component), or to handle in symbolic form representation/manipulation of fourth-order tensors in the sense delineated above (e.g. elasticity tensors, damage tensors, etc.).

3.4 Referències

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