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A FORMULATION OF ISOTROPIC AND ANISOTROPIC DAMAGE WITH EVOLUTION LAWS IN PSEUDO-LOG SPACE

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Abstract. *Anisotropic damage modeling still poses a number of open challenges. One of the most important is how to formulate the evolution laws in a way that is reasonably simple and makes physical sense. The theory tells us that loading function and damage rule are better defined in the space of forces conjugate to the primary damage variable. Choosing the 2nd-order integrity tensor (or any of the related tensors) as such variable, the resulting conjugate force tensor lacks physical meaning, and proposing physically meaningful evolution laws becomes a difficult task. A 2nd-order pseudo-log rate of damage has been recently proposed which remedies this problem and exhibits a number of additional advantages. To develop this idea, a 'basic' secant formulation has been considered in which the degraded stiffness depends on only 5 independent parameters. A first anisotropic model which distinguishes tension and compression has been developed based on these ideas, and has been verified in a complex loading case involving rotation of principal axes. All these recent developments are summarized in this paper. In addition, the extension of the secant stiffness from 5 to 6 independent parameters is finally outlined. This extension encompasses purely volumetric and von Mises isotropic damage, and is formulated in a way that preserves most of the convenient features of the basic formulation.*

1 INTRODUCTION

Anisotropic degradation and damage entails considerably complexity, with a number of aspects which are still not resolved at present [1]. In a sequence of papers, the authors have contributed with the proposal of a unified theoretical framework for elastic degradation and damage [2], the analysis of spurious energy dissipation of stiffness recovery schemes [3], and the study of the constitutive localization properties of scalar damage models, based on the spectral analysis of the acoustic tensor [4, 5]. The most recent contributions aim at the representation of anisotropic degradation and at the formulation of the corresponding evolution laws [6, 7, 8]. These latest developments are summarized in this paper.

2 THEORETICAL FRAMEWORK FOR ELASTIC DEGRADATION AND DAMAGE

In the simplest setting of purely elastic degradation, it is assumed that unloading always leads to the origin with some secant stiffness/compliance, and reloading follows the same path until the envelope is reached again and nonlinear behavior resumes (Fig. 1).

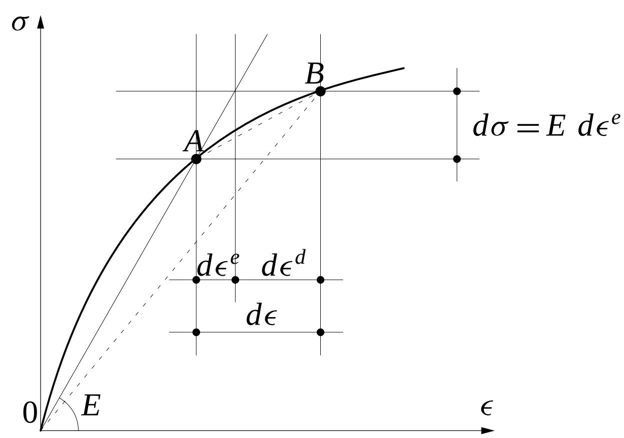

Fig. 1. Elastic-Degrading behavior and decomposition of the strain increments

Total values of stresses and strains at any time are related by the secant expressions

$$\boldsymbol{\sigma} = \mathbf{E} : \boldsymbol{\epsilon} \qquad ; \qquad \boldsymbol{\epsilon} = \mathbf{C} : \boldsymbol{\sigma} \tag{1a,b}$$

where $\mathbf{E}$ and $\mathbf{C}$ are the fourth-order stiffness and compliance tensors, endowed with major and minor symmetries, which are inverse to each other, i.e. $\mathbf{E} : \mathbf{C} = \mathbf{C} : \mathbf{E} = \mathbf{I}_4^{\text{sym}}$ (fourth-order symmetric identity tensor, defined as $\mathbf{I}_4^{\text{sym}} = (\mathbf{I} \overline{\otimes} \mathbf{I} + \mathbf{I} \underline{\otimes} \mathbf{I})/2$, where $\mathbf{I}$=second-order identity tensor with Cartesian components $I_{ij} = \delta_{ij}$ Kronecker delta, and products $\mathbf{A} = \mathbf{b} \overline{\otimes} \mathbf{c}$ and $\mathbf{A} = \mathbf{b} \underline{\otimes} \mathbf{c}$ correspond to the Cartesian component expressions $A_{ijkl} = b_{ik} c_{jl}$ and $A_{ijkl} = b_{il} c_{jk}$ respectively; see [8] for a full derivation without recourse to index notation).

It is also assumed that the stiffness and compliance are functions of a damage variable $\boldsymbol{\mathcal{D}}$, which may be scalar, vector- or tensor-valued. Thereby, the elastic energy per unit volume at any stage of the damage process, u, may be expressed as

$$u = \frac{1}{2}\,\boldsymbol{\epsilon} : \mathbf{E}(\boldsymbol{\mathcal{D}}) : \boldsymbol{\epsilon} = \frac{1}{2}\,\boldsymbol{\sigma} : \mathbf{C}(\boldsymbol{\mathcal{D}}) : \boldsymbol{\sigma} \tag{2a,b}$$

For isothermal conditions, one may differentiate to obtain the equations of incremental energy balance, dissipation $\dot{d}$, and conjugate forces $-\boldsymbol{\mathcal{Y}}$:

$$\dot{u} = \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} - \dot{d} \quad , \quad \dot{d} = (-\boldsymbol{\mathcal{Y}}) \star \dot{\boldsymbol{\mathcal{D}}} \quad , \quad -\boldsymbol{\mathcal{Y}} = -\frac{\partial u}{\partial \boldsymbol{\mathcal{D}}}\Big|_{\epsilon} = \frac{\partial u}{\partial \boldsymbol{\mathcal{D}}}\Big|_{\sigma} = \frac{1}{2}[\boldsymbol{\sigma} \otimes \boldsymbol{\sigma}] :: \frac{\partial \mathbf{C}}{\partial \boldsymbol{\mathcal{D}}} \tag{3a,b,c,d}$$

where the symbol $\star$ means full contraction of all indices of the damage variable.

The conjugate forces $-\boldsymbol{\mathcal{Y}}$ constitute the space in which the loading surface $F(-\boldsymbol{\mathcal{Y}}, \mathbf{p}) = 0$ is defined, and where the normal to the surface $\boldsymbol{\mathcal{N}}$ and 'flow rule' for damage (or *damage rule*) $\boldsymbol{\mathcal{M}}$ are established:

$$\boldsymbol{\mathcal{N}} = \frac{\partial F}{\partial(-\boldsymbol{\mathcal{Y}})}\Big|_{\lambda} \quad , \quad \dot{\boldsymbol{\mathcal{D}}} = \dot{\lambda}\,\boldsymbol{\mathcal{M}} \tag{4a,b}$$

These thermodynamic concepts are perfectly compatible with plasticity-like concepts and expressions in stress (or strain) space, which make the formulation more intuitive [2]. A first step is to consider the (fourth-order) space of forces $-\mathbf{Y}$ conjugate to the compliance rate $\dot{\mathbf{C}}$, in which we can rephrase the dissipation, loading function $F(-\mathbf{Y}, \mathbf{p})$, and define fourth-order *compliance rule* $\mathbf{M}$ and normal to the surface $\mathbf{N}$:

$$\begin{gathered} \dot{d} = \frac{1}{2}\,\boldsymbol{\sigma} : \dot{\mathbf{C}} : \boldsymbol{\sigma} = (-\mathbf{Y}) :: \dot{\mathbf{C}} \quad ; \quad -\mathbf{Y} = \frac{\partial u}{\partial \mathbf{C}}\Big|_{\sigma} = \frac{1}{2}\,\boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \quad ; \quad -\boldsymbol{\mathcal{Y}} = (-\mathbf{Y}) :: \frac{\partial \mathbf{C}}{\partial \boldsymbol{\mathcal{D}}} \\ \mathbf{N} = \frac{\partial F}{\partial(-\mathbf{Y})}\Big|_{\lambda} = \boldsymbol{\mathcal{N}} \star \frac{\partial \mathbf{C}}{\partial \boldsymbol{\mathcal{D}}} \quad ; \quad \dot{\mathbf{C}} = \dot{\lambda}\,\mathbf{M} \quad ; \quad \mathbf{M} = \boldsymbol{\mathcal{M}} \star \frac{\partial \mathbf{C}}{\partial \boldsymbol{\mathcal{D}}} \end{gathered} \tag{5a-i}$$

These quantities may be finally related to the usual stress-space in which loading function is given as $F(\boldsymbol{\sigma}, \mathbf{p})$, with normal to the surface $\mathbf{n}$, degrading strain rate $\dot{\boldsymbol{\epsilon}}^d$ (Fig. 1) and 'flow' rule $\mathbf{m}$:

$$\mathbf{n} = \frac{\partial F}{\partial \boldsymbol{\sigma}}\Big|_{\lambda} = \mathbf{N} : \boldsymbol{\sigma} \quad ; \quad \dot{\boldsymbol{\epsilon}}^d = \dot{\lambda}\,\mathbf{m} \quad ; \quad \mathbf{m} = \mathbf{M} : \boldsymbol{\sigma} \tag{6a,b,c,d}$$

Using these concepts, one may write rate equations which are identical to the ones in elasto-plasticity, and obtain the well known expressions of the inelastic multiplier $\dot{\lambda}$ and tangential stiffness, in which the only difference is the secant stiffness $\mathbf{E}$ instead of the initial:

$$\dot{\lambda} = \frac{1}{H}\,\mathbf{n} : \mathbf{E} : \dot{\boldsymbol{\epsilon}} \quad ; \quad \dot{\boldsymbol{\sigma}} = \mathbf{E}^{\tan} : \dot{\boldsymbol{\epsilon}} \quad ; \quad \mathbf{E}^{\tan} = \mathbf{E} - \frac{1}{H}\,\mathbf{E} : \mathbf{m} \otimes \mathbf{n} : \mathbf{E} \tag{7a,b}$$

The hardening parameters in strain and stress space are defined in the usual way:

$$\bar{H} = -\frac{\partial F}{\partial \lambda}\Big|_{\epsilon} = H + \mathbf{n} : \mathbf{E} : \mathbf{m} \quad , \quad H = -\frac{\partial F}{\partial \lambda}\Big|_{\sigma} \tag{8a,b}$$

Similar to plasticity, H and $\mathbf{m}$ are restricted in such a way that the denominator $\bar{H} = H + \mathbf{n} : \mathbf{E} : \mathbf{m}$ remains always positive. The model is called associated *in the stress space* (traditional definition) when $\mathbf{m}$ is proportional to $\mathbf{n}$ and consequently the tangent stiffness exhibits major symmetry. If $\mathbf{m}$ is derived from a potential Q, associativity may be alternatively stated as $Q = F$. Other definitions of associativity may be established at the compliance level if $\mathbf{M}$ is parallel to $\mathbf{N}$, which implies the former, or in the damage space if $\boldsymbol{\mathcal{M}}$ is parallel to $\boldsymbol{\mathcal{N}}$, which implies all of them. The latter may be termed full associativity [2].

3 BASIC ISOTROPIC DAMAGE

Using the theoretical framework described in Sec. 2, it is possible to formulate a variety of damage models depending mainly on the nature and choice of damage variables $\boldsymbol{\mathcal{D}}$, and the dependency of stiffness or compliance on those variables, $\mathbf{E} = \mathbf{E}(\boldsymbol{\mathcal{D}})$ or $\mathbf{C} = \mathbf{C}(\boldsymbol{\mathcal{D}})$. The simplest models are those in which the initial stiffness (and therefore also the compliance) is isotropic, and its degraded counterpart also maintains isotropy. In particular, the traditional "$(1-D)$" scalar damage model is one in which all the components of the stiffness tensor are reduced with the same coefficient $(1-D)$, where D is a damage variable varying from 0 to 1. In this section, this formulation is rephrased to introduce the concept of logarithmic scalar damage.

First, consider the general form of the isotropic stiffness and compliance tensors:

$$\mathbf{E} = \Lambda\, \mathbf{I} \otimes \mathbf{I} + G(\mathbf{I} \overline{\otimes} \mathbf{I} + \mathbf{I} \underline{\otimes} \mathbf{I}) \quad , \quad \mathbf{C} = \frac{-\nu}{E} \mathbf{I} \otimes \mathbf{I} + \frac{1+\nu}{2E}(\mathbf{I} \overline{\otimes} \mathbf{I} + \mathbf{I} \underline{\otimes} \mathbf{I}) \tag{9a,b}$$

where Λ and G are the Lamé constants, linked to the Young's modulus E and Poisson's ratio ν by the classical relations

$$\Lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad , \qquad G = \frac{E}{2(1+\nu)} \tag{10a,b}$$

In the "$(1-D)$" scalar damage model, the following well know expressions are assumed for the secant stiffness and its inverse compliance:

$$\mathbf{E} = (1-D)\, \mathbf{E}^0 \qquad ; \qquad \mathbf{C} = \frac{1}{1-D}\, \mathbf{C}^0 \tag{11a,b}$$

where D varies between 0 and 1 as damage progresses, and $\mathbf{E}^0$ and $\mathbf{C}^0$ are the initial stiffness and compliance tensors given by (9a,b) with initial values of elastic constants Λ^0, G^0 or E^0, ν^0. A new logarithmic scalar damage variable L, which varies between 0 and ∞, is introduced and considered as the primary damage variable $\boldsymbol{\mathcal{D}}$:

$$\boldsymbol{\mathcal{D}} = \text{scalar} = L = \ln\frac{1}{1-D} \quad ; \quad D = 1 - e^{-L} \quad , \quad \mathbf{E} = e^{-L}\, \mathbf{E}^0 \quad , \quad \mathbf{C} = e^{L}\, \mathbf{C}^0 \tag{12a,b,c,d,e}$$

The partial derivative $\partial\mathbf{C}/\partial\boldsymbol{\mathcal{D}}$ may be immediately calculated, $\mathbf{C}$ may be differentiated, and $\dot{L}$ itself may be chosen as the inelastic multiplier of the formulation:

$$\frac{\partial \mathbf{C}}{\partial \boldsymbol{\mathcal{D}}} = \frac{\partial \mathbf{C}}{\partial L} = \mathbf{C} \quad , \quad \dot{\mathbf{C}} = \dot{L}\, e^{L} \mathbf{C}^0 = \dot{L}\, \mathbf{C} \quad , \quad \dot{\lambda} = \dot{L} = \frac{\dot{D}}{1-D} \tag{13a,b,c,d,e}$$

This leads to the identification of the "m" terms of the general theory, which take the convenient simple form of the current value of compliance and strain:

$$\boldsymbol{\mathcal{M}} = scalar = 1 \quad ; \quad \mathbf{M} = \mathbf{C} \quad ; \quad \mathbf{m} = \mathbf{C} : \boldsymbol{\sigma} = \boldsymbol{\epsilon} \tag{14a-d}$$

The dissipation equation leads to the force $-\boldsymbol{\mathcal{Y}}$=scalar=$-\mathcal{Y}$, conjugate to the logarithmic damage L, which turns out to be equal to the current (secant) elastic energy:

$$\dot{d} = \frac{1}{2}\boldsymbol{\sigma} : \mathbf{C} : \boldsymbol{\sigma}\ \dot{L} = (-\mathcal{Y})\ \dot{L} \quad ; \quad -\mathcal{Y} = \frac{1}{2}\boldsymbol{\sigma} : \mathbf{C} : \boldsymbol{\sigma} = u \tag{15a,b,c,d}$$

In order to achieve an associated formulation, the loading surface is written in terms of the conjugate force $-\mathcal{Y} = u$ and the damage state L (equivalent to D), in the format

$$F = u - r(L) = 0 \tag{16}$$

From F, the various gradients of the loading function at constant damage may be obtained:

$$\boldsymbol{\mathcal{N}} = scalar = \frac{\partial F}{\partial(-\mathcal{Y})} = 1 \quad ; \quad \mathbf{N} = \frac{\partial F}{\partial(-\mathbf{Y})} = \mathbf{C} \quad ; \quad \mathbf{n} = \frac{\partial F}{\partial \boldsymbol{\sigma}} = \mathbf{C} : \boldsymbol{\sigma} = \boldsymbol{\epsilon} \tag{17a,b,c}$$

Note that the three gradients $\mathcal{N}$, $\mathbf{N}$ and $\mathbf{n}$ are equal to the corresponding rules $\mathcal{M}$, $\mathbf{M}$ and $\mathbf{m}$ in the theory, which means associativity at all levels.

The hardening/softening modulus $H = -\partial F/\partial\lambda$ at constant stress, may be calculated from (16), and with $\mathbf{n}$ and $\mathbf{m}$ the tangential stiffness is obtained:

$$\mathbf{E}^{\tan} = e^{-L}\ \mathbf{E}^0 - \frac{1}{H}\boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \quad ; \quad \bar{H} = \frac{\partial r}{\partial L} + u \tag{18a,b}$$

As described above, it only remains to define the hardening/softening function $r(L)$ (or, equivalently, $r(D)$). This function may be identified from a single stress-strain curve from experiments, for instance from a uniaxial test. Once it has been chosen, however, all other features of the model are automatically fixed.

4 ANISOTROPIC SECANT STIFFNESS USING 2nd-ORDER DAMAGE TENSORS

4.1 Damage variables

Disregarding vectors due to theoretical and practical shortcomings [9, 10], a second-order symmetric tensor seems to be the simplest way to represent anisotropic damage with reasonable generality. Several authors have proposed either the direct generalization of D to a *second-order damage tensor* $\mathbf{D}$ which varies between $\mathbf{0}$ and $\mathbf{I}$ as damage progresses [11, 12, 13], or the use of an *integrity tensor* $\bar{\boldsymbol{\phi}} = \mathbf{I} - \mathbf{D}$ which has exactly the opposite variation [14, 15]. Actually, one can think of a number of second-order tensors to characterize damage, all with the same principal axes and simple relations between their principal values. Additionally to the integrity tensor $\bar{\boldsymbol{\phi}}$, here we introduce its square root $\bar{\mathbf{w}}$ (which also varies from $\mathbf{I}$ to $\mathbf{0}$) and their inverses $\boldsymbol{\phi}$ and $\mathbf{w}$ (which vary from $\mathbf{I}$ to $\boldsymbol{\infty}$). These tensors and their principal values satisfy the following relations:

$$\bar{\boldsymbol{\phi}} = \bar{\mathbf{w}} \cdot \bar{\mathbf{w}} \quad , \quad \boldsymbol{\phi} = \mathbf{w} \cdot \mathbf{w} \quad , \quad \bar{\boldsymbol{\phi}} \cdot \boldsymbol{\phi} = \boldsymbol{\phi} \cdot \bar{\boldsymbol{\phi}} = \mathbf{I} \quad , \quad \bar{\mathbf{w}} \cdot \mathbf{w} = \mathbf{w} \cdot \bar{\mathbf{w}} = \mathbf{I} \qquad \text{(19a,b,c,d)}$$

$$\bar{\phi}_{(i)} = \bar{w}^2_{(i)} \quad , \quad \phi_{(i)} = w^2_{(i)} \quad , \quad \bar{\phi}_{(i)} = \frac{1}{\phi_{(i)}} \quad , \quad \bar{w}_{(i)} = \frac{1}{w_{(i)}} \qquad \text{(20a,b,c,d)}$$

In the case of isotropic degradation, all these tensors reduce to the volumetric form:

$$\bar{\boldsymbol{\phi}} = \bar{\phi}\, \mathbf{I} \quad , \quad \bar{\mathbf{w}} = \bar{w}\, \mathbf{I} \quad , \quad \boldsymbol{\phi} = \phi\, \mathbf{I} \quad , \quad \mathbf{w} = w\, \mathbf{I} \qquad \text{(21a,b,c,d)}$$

Due to the *energy equivalence approach* which will be introduced next, equivalence of these variables to the scalar D used in the previous section will involve a square root

$$\bar{\phi} = \bar{w}^2 = \frac{1}{\phi} = \frac{1}{w^2} = \sqrt{1-D} \qquad \text{(22a,b,c,d)}$$

In the case of anisotropic degradation, tensors $\boldsymbol{\phi}$, $\bar{\boldsymbol{\phi}}$, $\mathbf{w}$ and $\bar{\mathbf{w}}$ may be decomposed as the product of 1/3 of their determinant times an isochoric (unit determinant) tensor:

$$\bar{\boldsymbol{\phi}} = \bar{\phi}\, \bar{\boldsymbol{\psi}} \quad , \quad \bar{\mathbf{w}} = \bar{w}\, \bar{\mathbf{v}} \quad , \quad \boldsymbol{\phi} = \phi\, \boldsymbol{\psi} \quad , \quad \mathbf{w} = w\, \mathbf{v} \qquad \text{(23a,b,c,d)}$$

$$\bar{\phi} = (det\, \bar{\boldsymbol{\phi}})^{1/3} \quad , \quad \bar{w} = (det\, \bar{\mathbf{w}})^{1/3} \quad , \quad \phi = (det\, \boldsymbol{\phi})^{1/3} \quad , \quad w = (det\, \mathbf{w})^{1/3} \qquad \text{(24a,b,c,d)}$$

Tensors $\boldsymbol{\psi}$, $\bar{\boldsymbol{\psi}}$, $\mathbf{v}$ and $\bar{\mathbf{v}}$ satisfy relations similar to (19), while their scalar counterparts ϕ, $\bar{\phi}$, w and $\bar{w}$ satisfy relations similar to (20). Note that, in all these product-type decompositions of the damage tensors, the “product-volumetric” part (determinant to power 1/3) may be interpreted as representing the isotropic part of the damage, while the isochoric part (with unit determinant) would represent its anisotropic part.

4.2 Effective stress and strain, energy equivalence

Degradation may be understood as the average effect of distributed microcracks. *Effective stress* $\boldsymbol{\sigma}^{\text{eff}}$ and *effective strain* $\boldsymbol{\epsilon}^{\text{eff}}$ may be introduced as the stress and strain to which the material between microcracks is subjected. In this context, the relation between effective stress and effective strain describes the behavior of the undamaged material skeleton, which in this case is assumed to be linear elastic, i.e.

$$\boldsymbol{\sigma}^{\text{eff}} = \mathbf{E}^0 : \boldsymbol{\epsilon}^{\text{eff}} \quad ; \quad \boldsymbol{\epsilon}^{\text{eff}} = \mathbf{C}^0 : \boldsymbol{\sigma}^{\text{eff}} \qquad \text{(25a,b)}$$

Damage variables must relate the effective quantities to their *nominal* or apparent counterparts, which are the ones measured externally and must satisfy equilibrium and compatibility at structural level. In the literature, the relation between nominal and effective quantities has been established mainly in three ways: strain equivalence, stress equivalence and energy equivalence. The most widely used, strain equivalence [16], assumes that effective and nominal strains are equal and stresses differ, while stress equivalence is the opposite. These approaches, however, lead to non-symmetric secant stiffness tensors, which do not ensure energy conservation in unloading-reloading situations.

In contrast, *energy equivalence* produces symmetric secant stiffness and compliance tensors. It is assumed [12] that the elastic energy stored in terms of effective quantities with undamaged stiffness and in terms of nominal quantities with secant stiffness must be the same. As a result, neither effective strain nor effective stress coincide with their nominal counterparts. Rather, assuming that the relations are linear, they must be given by the same fourth-order "damage-effect" tensor $\bar{\boldsymbol{\alpha}}$, or its inverse $\boldsymbol{\alpha}$ (i.e. $\boldsymbol{\alpha}:\bar{\boldsymbol{\alpha}}=\bar{\boldsymbol{\alpha}}:\boldsymbol{\alpha}=\mathbf{I}_4^{\mathrm{sym}}$) in the following reciprocal form:

$$\boldsymbol{\sigma} = \bar{\boldsymbol{\alpha}} : \boldsymbol{\sigma}^{\mathrm{eff}} \quad , \quad \boldsymbol{\sigma}^{\mathrm{eff}} = \boldsymbol{\alpha} : \boldsymbol{\sigma} \quad , \quad \boldsymbol{\epsilon} = \boldsymbol{\alpha}^T : \boldsymbol{\epsilon}^{\mathrm{eff}} \quad , \quad \boldsymbol{\epsilon}^{\mathrm{eff}} = \bar{\boldsymbol{\alpha}}^T : \boldsymbol{\epsilon} \qquad \text{(26a,b,c,d)}$$

where superscript $()^T$ stands for transposed in the major sense, (i.e. $\alpha^T_{ijkl} = \alpha_{klij}$). Combining equations (26) with (25), one recovers the secant relations (1a,b) where

$$\mathbf{E} = \bar{\boldsymbol{\alpha}} : \mathbf{E}^0 : \bar{\boldsymbol{\alpha}}^T \quad , \quad \mathbf{C} = \boldsymbol{\alpha}^T : \mathbf{C}^0 : \boldsymbol{\alpha} \qquad \text{(27a,b)}$$

4.3 Symmetrized nominal-effective relations and resulting secant stiffness/compliance

Trying to establish relation (26a) in terms of a second-order damage tensor as a direct generalization of the one-dimensional relation $\sigma = \bar{\phi}\sigma^{\mathrm{eff}}$ where $\bar{\phi}$ is an effective area reduction, one has $\boldsymbol{\sigma} = \bar{\boldsymbol{\phi}} \cdot \boldsymbol{\sigma}^{\mathrm{eff}}$, where symmetry cannot be ensured for $\boldsymbol{\sigma}$ even if $\boldsymbol{\sigma}^{\mathrm{eff}}$ and $\bar{\boldsymbol{\phi}}$ are symmetric. This suggests that some form of symmetrization should be applied. Due to its advantages [6], a product-type symmetrization is adopted, which leads to the following nominal-effective relations and subsequent damage-effect tensors:

$$\boldsymbol{\sigma}^{\mathbf{eff}} = \mathbf{w} \cdot \boldsymbol{\sigma} \cdot \mathbf{w} \quad , \quad \boldsymbol{\epsilon}^{\mathbf{eff}} = \bar{\mathbf{w}} \cdot \boldsymbol{\epsilon} \cdot \bar{\mathbf{w}} \quad , \quad \boldsymbol{\sigma} = \bar{\mathbf{w}} \cdot \boldsymbol{\sigma}^{\mathbf{eff}} \cdot \bar{\mathbf{w}} \quad , \quad \boldsymbol{\epsilon} = \mathbf{w} \cdot \boldsymbol{\epsilon}^{\mathbf{eff}} \cdot \mathbf{w} \qquad \text{(28a,b,c,d)}$$

$$\bar{\boldsymbol{\alpha}} = \frac{1}{2} \left(\bar{\mathbf{w}} \,\overline{\otimes}\, \bar{\mathbf{w}} + \bar{\mathbf{w}} \,\underline{\otimes}\, \bar{\mathbf{w}} \right) \quad ; \quad \boldsymbol{\alpha} = \frac{1}{2} \left(\mathbf{w} \,\overline{\otimes}\, \mathbf{w} + \mathbf{w} \,\underline{\otimes}\, \mathbf{w} \right) \qquad \text{(29a,b)}$$

In a 6×6 matrix representation in the principal axes of damage, tensor $\bar{\boldsymbol{\alpha}}$ exhibits the following diagonal form [12, 17, 18, 19]:

$$\bar{\boldsymbol{\alpha}} = \begin{bmatrix} \bar{\phi}_{(1)} & & & & & \\ & \bar{\phi}_{(2)} & & & & \\ & & \bar{\phi}_{(3)} & & & \\ & & & \bar{w}_{(1)}\bar{w}_{(2)} & & \\ & & & & \bar{w}_{(2)}\bar{w}_{(3)} & \\ & & & & & \bar{w}_{(3)}\bar{w}_{(1)} \end{bmatrix} \qquad \text{(30)}$$

Replacing expressions (28) into (25), one obtains the following equations (better expressed in Cartesian component form):

$$E_{ijkl} = \bar{w}_{ip}\bar{w}_{jq}\bar{w}_{kr}\bar{w}_{ls}E^0_{pqrs} \quad ; \quad C_{ijkl} = w_{ip}w_{jq}w_{kr}w_{ls}C^0_{pqrs} \tag{31a,b}$$

Further replacing the isotropic elastic stiffness and compliance tensors and making the appropriate products and substitutions, one finally obtains

$$\mathbf{E} = \Lambda^0\, \bar{\boldsymbol{\phi}} \otimes \bar{\boldsymbol{\phi}} + G^0\, (\bar{\boldsymbol{\phi}}\ \overline{\otimes}\ \bar{\boldsymbol{\phi}} + \bar{\boldsymbol{\phi}}\ \underline{\otimes}\ \bar{\boldsymbol{\phi}}) \ ; \quad \mathbf{C} = -\frac{\nu^0}{E^0}\boldsymbol{\phi}\otimes\boldsymbol{\phi} + \frac{1+\nu^0}{2E^0}\,(\boldsymbol{\phi}\ \overline{\otimes}\ \boldsymbol{\phi} + \boldsymbol{\phi}\ \underline{\otimes}\ \boldsymbol{\phi}) \tag{32a,b}$$

which can be equally rewritten in terms of any other pair of elastic constants, to obtain expressions analogous to the isotropic ones (9), in which all second-order unit tensors **I** have been replaced by $\bar{\boldsymbol{\phi}}$ (stiffness) or $\boldsymbol{\phi}$ (compliance). Note that the expression for **E** (32a) actually corresponds to the model proposed by Valanis (1990), although in that case it was derived directly from a specific form of the elastic potential, rather than using effective stress and effective strain concepts.

It is also useful to represent the secant compliance **C**, in a 6×6 matrix form, selecting as the reference system the principal axes of damage. This matrix may be compared to the traditional compliance matrix for orthotropic elasticity:

$$\mathbf{C} = \begin{bmatrix} \phi_1^2\frac{1}{E} & \phi_1\phi_2\frac{-\nu}{E} & \phi_1\phi_3\frac{-\nu}{E} & & & \\ \phi_1\phi_2\frac{-\nu}{E} & \phi_2^2\frac{1}{E} & \phi_2\phi_3\frac{-\nu}{E} & & & \\ \phi_1\phi_3\frac{-\nu}{E} & \phi_2\phi_3\frac{-\nu}{E} & \phi_3^2\frac{1}{E} & & & \\ & & & \phi_1\phi_2\frac{1+\nu}{E} & & \\ & & & & \phi_2\phi_3\frac{1+\nu}{E} & \\ & & & & & \phi_1\phi_3\frac{1+\nu}{E} \end{bmatrix} , \quad \mathbf{C}^{\mathrm{orth}} = \begin{bmatrix} \frac{1}{E_1} & \frac{-\nu_{12}}{E_2} & \frac{-\nu_{13}}{E_3} & & & \\ \frac{-\nu_{21}}{E_1} & \frac{1}{E_2} & \frac{-\nu_{23}}{E_3} & & & \\ \frac{-\nu_{31}}{E_1} & \frac{-\nu_{32}}{E_2} & \frac{1}{E_3} & & & \\ & & & \frac{1}{G_{12}} & & \\ & & & & \frac{1}{G_{23}} & \\ & & & & & \frac{1}{G_{31}} \end{bmatrix} \tag{33a,b}$$

This leads to the following equivalences:

$$E_1 = \bar{\phi}_1^2 E\ , \quad E_2 = \bar{\phi}_2^2 E\ , \quad E_3 = \bar{\phi}_3^2 E\ , \quad G_{12} = \frac{\bar{\phi}_1\bar{\phi}_2 E}{2(1+\nu)}\ , \quad G_{23} = \frac{\bar{\phi}_2\bar{\phi}_3 E}{2(1+\nu)}\ , \quad G_{31} = \frac{\bar{\phi}_3\bar{\phi}_1 E}{2(1+\nu)}$$

$$\nu_{12} = \frac{\bar{\phi}_2}{\phi_1}\nu\ , \quad \nu_{13} = \frac{\bar{\phi}_3}{\phi_1}\nu\ , \quad \nu_{21} = \frac{\bar{\phi}_1}{\phi_2}\nu\ , \quad \nu_{23} = \frac{\bar{\phi}_3}{\phi_2}\nu\ , \quad \nu_{31} = \frac{\bar{\phi}_1}{\phi_3}\nu\ , \quad \nu_{32} = \frac{\bar{\phi}_2}{\phi_3}\nu \tag{34a-l}$$

In these relations, the 9 independent orthotropic elastic constants (the 12 in previous equations are subject to the three symmetry constraints $\nu_{12}/E_2 = \nu_{21}/E_1$, etc.), are generated from 5 independent parameters: E, ν plus the three principal values of damage ϕ_i. Therefore, this secant stiffness corresponds only to a restricted form of orthotropy, which will lead to what we call *basic formulation* of anisotropic damage. For the particular case of isotropic damage, the 'basic' formulation collapses into the "$1-D$" model, which is also known to be a restricted form of isotropic damage in which only E degrades while ν remains constant. In spite of its limited character, the 'basic' secant formulation seems the most appropriate framework to introduce the concepts of pseudo-log rate of damage and related evolution laws.

5 PSEUDO-LOG RATE OF DAMAGE AND CONJUGATE FORCE

Next step is to select the primary damage variable which will play the role of $\boldsymbol{\mathcal{D}}$ in the general theory of Sect. 2, and to calculate the corresponding conjugate force $-\boldsymbol{\mathcal{Y}}$. First, $\mathbf{C}$ given in (32b) is differentiated and the result is substituted into (5a). The resulting dissipation rate $\dot{d}$ is:

$$\dot{d} = \left[\frac{-\nu^0}{E^0} \left(\boldsymbol{\sigma} : \boldsymbol{\phi}\right) \boldsymbol{\sigma} + \frac{1+\nu^0}{E^0} \boldsymbol{\sigma} \cdot \boldsymbol{\phi} \cdot \boldsymbol{\sigma} \right] : \dot{\boldsymbol{\phi}} \tag{35}$$

If the inverse integrity tensor $\boldsymbol{\phi}$ itself is taken as the primary damage variable, the term between brackets may be immediately identified as the corresponding conjugate force. This force, analogous to what was obtained in [20] in terms of stiffness and strains, has no clear physical meaning, which makes it difficult to propose and interpret loading functions and damage rules, and the same happens by selecting as primary damage variable any other of the damage tensors defined, or any other simple function of them.

A solution to this apparent dead-end street is to introduce the *pseudo-logarithmic rate* of damage, $\dot{\mathbf{L}}$, defined as:

$$\dot{\mathbf{L}} = -2\, \mathbf{w} \cdot \dot{\bar{\boldsymbol{\phi}}} \cdot \mathbf{w} = 2\, \bar{\mathbf{w}} \cdot \dot{\boldsymbol{\phi}} \cdot \bar{\mathbf{w}} \qquad \text{or} \qquad \dot{\boldsymbol{\phi}} = \frac{1}{2}\, \mathbf{w} \cdot \dot{\mathbf{L}} \cdot \mathbf{w} \quad , \quad \dot{\bar{\boldsymbol{\phi}}} = -\frac{1}{2}\, \bar{\mathbf{w}} \cdot \dot{\mathbf{L}} \cdot \bar{\mathbf{w}} \tag{36a,b,c}$$

If (36b) is substituted into (35), the two factors $\mathbf{w}$ enter the brackets and we obtain

$$\dot{d} = \left[\frac{-\nu^0}{E^0} (tr\, \boldsymbol{\sigma}^{\text{eff}}) \boldsymbol{\sigma}^{\text{eff}} + \frac{1+\nu^0}{E^0} \boldsymbol{\sigma}^{\text{eff}} \cdot \boldsymbol{\sigma}^{\text{eff}} \right] : \dot{\mathbf{L}} \tag{37}$$

The term between brackets may be indentified with the new force conjugate to $\dot{\mathbf{L}}$, which, using linear isotropic elastic relations between effective stress and strain, may be rewritten in the simple form:

$$-\boldsymbol{\mathcal{Y}} = \frac{1}{2}\, \boldsymbol{\sigma}^{\mathbf{eff}} \cdot \boldsymbol{\epsilon}^{\mathbf{eff}} \qquad \text{or, in principal values, } -\mathcal{Y}_{(i)} = \frac{1}{2} \sigma^{\text{eff}}_{(i)} \epsilon^{\text{eff}}_{(i)}, \quad i = 1, 2, 3 \tag{38a,b}$$

The second equation holds because $\boldsymbol{\sigma}^{\mathbf{eff}}$ and $\boldsymbol{\epsilon}^{\mathbf{eff}}$ remain coaxial due to the underlying assumption of isotropic elasticity in effective space, and therefore the conjugate force $-\boldsymbol{\mathcal{Y}}$ also shares the same principal axes. Note also the meaning of the first invariant $tr(-\boldsymbol{\mathcal{Y}}) = \boldsymbol{\sigma}^{\mathbf{eff}} : \boldsymbol{\epsilon}^{\mathbf{eff}}/2 = u$, current elastic energy.

If the principal axes of damage remain constant (no rotation), the new pseudo-log damage rate $\dot{\mathbf{L}}$ coincides with the rate of the logarithm of the square inverse integrity tensor. In these particular conditions, the total value of the pseudo-log damage tensor is defined as $\mathbf{L} = \ln \boldsymbol{\phi}^2$ (logarithm of a tensor defined as a tensor function, i.e. with same principal axes and logarithm of the principal values). In the general case of rotating principal axes, $\dot{\mathbf{L}}$ is not an exact differential, and the total pseudo-logarithmic damage tensor is simply not defined. However, the previous relations hold for the volumetric part $\dot{L} = tr(\dot{\mathbf{L}})/3$, the total value of which always exists and is $L = 2\ \ln(\phi_{(1)}\phi_{(2)}\phi_{(3)})/3$.

The lack of a general relation between total values of $\mathbf{L}$ and $\boldsymbol{\phi}$ does not actually represent a practical difficulty because the pseudo-log damage is only used in rate form due to its properties of exhibiting a convenient conjugate force. Once the damage rule in terms of $\dot{\mathbf{L}}$ is established, the rate $\dot{\boldsymbol{\phi}}$ may be always evaluated with (36b) and the integration process needed in the numerical implementation of the model can be carried out directly in terms of $\boldsymbol{\phi}$, which is the variable that enters directly the expressions of the secant compliance and stiffness.

Another important property is that the product-type decomposition of $\dot{\boldsymbol{\phi}}$ becomes a sum-type decomposition for $\dot{\mathbf{L}}$ [6]. Due to that, *the volumetric part of the damage rule, $\boldsymbol{\mathcal{M}}^V$, only generates increments of isotropic damage*, while *its deviatoric part, $\boldsymbol{\mathcal{M}}^D$, is solely responsible for anisotropic degradation*. In Fig. 2, this is represented in the space of the principal values of the conjugate force, $-\mathcal{Y}_{(1)}$,$-\mathcal{Y}_{(2)}$,$-\mathcal{Y}_{(3)}$. With this representation, the isotropic damage model is recovered if the damage rule is always aligned with the p-axis of this space. If the model is associated, this corresponds to a loading surface given by a π-plane.

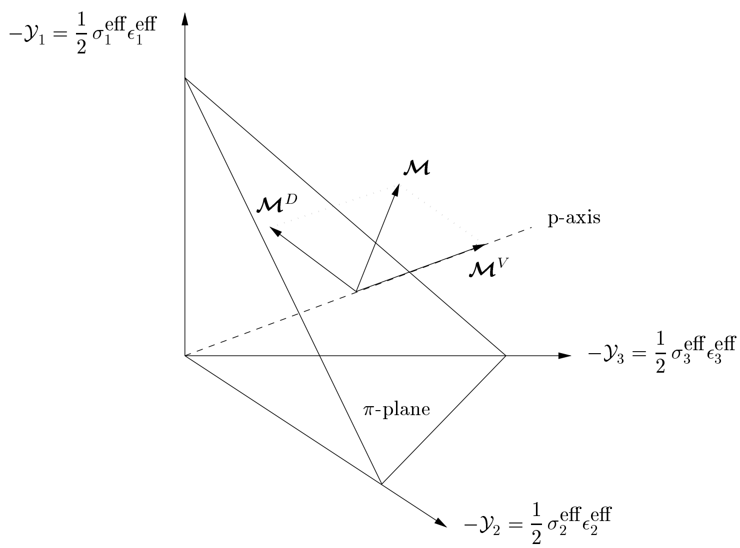

Fig. 2. Space of principal values of the conjugate force $-\mathcal{Y}_1$, $-\mathcal{Y}_2$, $-\mathcal{Y}_3$, and volumetric-deviatoric decomposition of the damage rule

For anisotropic damage models, the surface is in general convex and includes the origin of the $-\boldsymbol{\mathcal{Y}}$ space. Additional constraints to the pseudo-log damage rule (and to the loading surface as well, if the model is associated) may be derived from the condition that damage on any direction $\mathbf{s}$ (unit vector $\mathbf{s}\cdot\mathbf{s} = 1$), as given by the projection $\mathbf{s}\cdot\boldsymbol{\phi}\cdot\mathbf{s}$, must always increase [6]. Developing that condition, one obtains that the pseudo-log damage rule $\boldsymbol{\mathcal{M}}$ must be positive semi-definite, which means that all its three principal components be positive or zero, but not be negative. In Fig. 2, this means that the vector representing the damage rule should be part of the positive-positive-positive octant, which is a severe restriction if compared with traditional flow rules in stress space. For instance, associated models with surfaces similar to von Mises or Drucker-Prager (in which the normal has negative components on one or more of the axes) are not allowed in this case. On the other hand, a surface similar to Rankine in the $-\mathcal{Y}_{(1)}$,$-\mathcal{Y}_{(2)}$,$-\mathcal{Y}_{(3)}$ space would sit in the limit of the stated restriction, with only one positive principal value of $\boldsymbol{\mathcal{N}}$ at a time, while the other two are zero. This surface is the basis for the model developed in the next section.

6 DISTINGUISHING TENSION AND COMPRESSION: MODIFIED CONJUGATE FORCES

From Eqns (37), (38) and (4b), and assuming that the damage rule shares principal axes with the conjugate forces (i.e. it can be represented in the principal force space as shown in Fig. 2), dissipation may be written as:

$$\dot{d} = (-\mathcal{Y}_{ij})\dot{L}_{ij} = (-\mathcal{Y}_{ij})\mathcal{M}_{ij}\dot{\lambda} = \dot{\lambda}\sum_{k=1}^{3}(-\mathcal{Y}_{(k)})\mathcal{M}_{(k)} \qquad \text{(39a,b,c)}$$

As given by (38a,b), the conjugate force $-\boldsymbol{\mathcal{Y}}$ does not distinguish between tension and compression, which is a fundamental aspect of the behavior of quasi-brittle materials such as concrete. This would require that $\mathcal{M}_{(k)} > 0$ only if the corresponding principal direction is subject to tension, and otherwise $\mathcal{M}_{(k)} = 0$. Another difficulty with that definition of conjugate forces, is that negative values $-\mathcal{Y}_{(k)} < 0$ could be obtained when $\sigma^{\text{eff}}_{(k)} > 0$ and $\epsilon^{\text{eff}}_{(k)} < 0$, or *vice versa*. This could correspond for instance to cases with principal stresses of the same sign but values much higher in one direction than the other, and could lead to undesirable negative dissipation values. To avoid that, the restriction is imposed that $\mathcal{M}_{(k)} = 0$ whenever $-\mathcal{Y}_{(k)} < 0$.

Taking advantage of previous requirements and restrictions, it is possible to consider the following modified conjugate forces without changing the resulting dissipation [7]:

$$\begin{aligned} -\hat{\mathcal{Y}}_{(1)} &= \frac{1}{2}\,\langle\sigma^{\text{eff}}_{(1)}\rangle\,\langle\epsilon^{\text{eff}}_{(1)}\rangle \\ -\hat{\mathcal{Y}}_{(2)} &= \frac{1}{2}\,\langle\sigma^{\text{eff}}_{(2)}\rangle\,\langle\epsilon^{\text{eff}}_{(2)}\rangle \\ -\hat{\mathcal{Y}}_{(3)} &= \frac{1}{2}\,\langle\sigma^{\text{eff}}_{(3)}\rangle\,\langle\epsilon^{\text{eff}}_{(3)}\rangle \end{aligned} \qquad \text{(40a,b,c)}$$

where the angle brackets are McAuley brackets with usual meaning (i.e. $\langle x\rangle = x$ if $x > 0$ and $\langle x\rangle = 0$ otherwise). These new forces can directly replace the old ones for the definition of the loading surface, with the advantadge that tension and compression cases will be automatically distinguished, and damage will only be activated in tension.

7 GENERALIZED PSEUDO-RANKINE MODEL FOR TENSILE DAMAGE

F is defined according to the general expression:

$$F = f(-\hat{\boldsymbol{\mathcal{Y}}}) - r(history) \tag{41}$$

where the algebraic expression of f is taken from the literature [21], and r is assumed an exponential decay function of the volumetric component of the logarithmic damage tensor, L:

$$f = \left((-\hat{\mathcal{Y}}_{(1)})^{b+1} + (-\hat{\mathcal{Y}}_{(2)})^{b+1} + (-\hat{\mathcal{Y}}_{(3)})^{b+1}\right)^{\frac{1}{b+1}} \quad , \quad r = \frac{\sigma^2_{\text{peak}}}{2E^0}\ \exp\left(-\frac{3\sigma^2_{\text{peak}}}{2E^0 g_f} L\right) \tag{42a,b}$$

The surface $F = 0$ takes different shapes in $-\hat{\mathcal{Y}}_{(1)}, -\hat{\mathcal{Y}}_{(2)}, -\hat{\mathcal{Y}}_{(3)}$ space depending on parameter b. For $b = 0$, the surface is a π-plane and the model collapses into isotropic damage. For $b \rightarrow \infty$, the surface approaches a Rankine-type criterion and the model exhibits maximum anisotropic character. The cross-section of such a surface with the $-\hat{\mathcal{Y}}_{(1)}$,$-\hat{\mathcal{Y}}_{(2)}$ plane, is represented in Fig. 3 for different values of b.

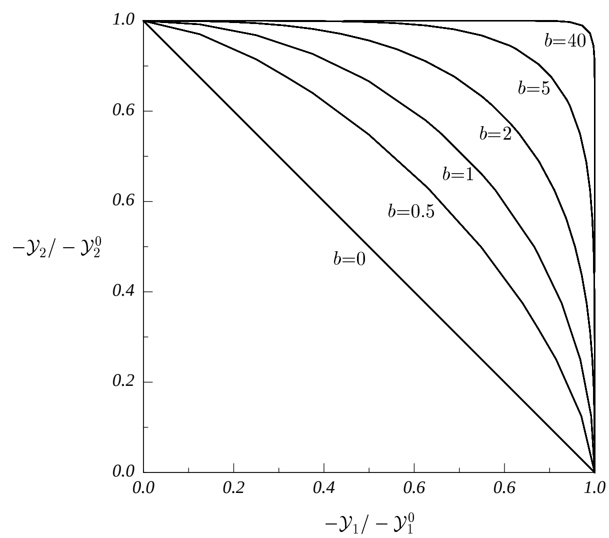

Fig. 3. 2-D cross-section of the loading surface with coordinate plane $-\mathcal{Y}_3 = 0$, for various values of b.

The coefficients in the exponential resistance function (42b) are simple expressions of the tensile strength σ^{peak} and the fracture energy per unit volume g_f (area enclosed under the uniaxial stress-strain diagram). These expressions are obtained by first assuming a generic exponential function $r = r_0 \exp(-kL)$ and then identifying r_0 and k from the analytical solution of the pure tension case, which is available for any value of $b \geq 0$. Other analytical solutions of the model for simple loading cases such as pure shear, pure distortion, and loading-unloading-reloading in a perpendicular direction, are available as well [7].

The loading surface may also be represented in principal stress space. In general, this representation is only possible in terms of the effective stresses. However, for initial conditions with no damage, effective stresses coincide with nominal stresses. Considering again the 2-D case with $\sigma^{\text{eff}}_{(3)} = 0$, the principal effective strains in (40) may be replaced in terms of principal effective stresses via linear elasticity, and the resulting forces may be substituted into the loading function (42). The McAuley brackets determine three regions with different algebraic expressions, and the resulting loading surface $F = 0$ is represented in Fig. 4, with a general view in Fig. 4a and close-up of the tension-tension sector in Fig. 4b. Overall, the shape of the surface agrees well with the tensile-dominated parts of the standard biaxial failure diagram for concrete [22]. In Regions 2 and 3, the surface is not affected by parameter b, which in contrast has significant influence in the tension-tension corner of Region 1.

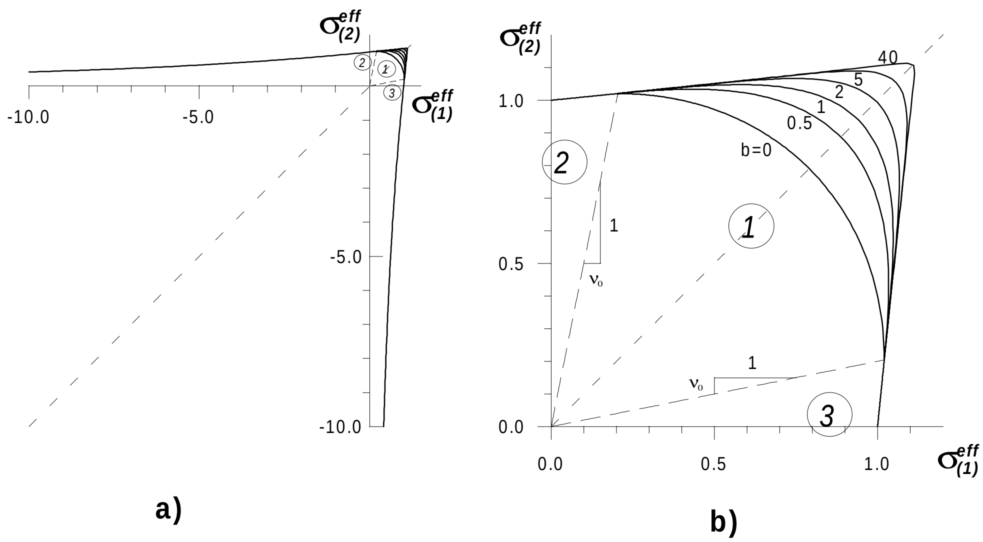

Fig. 4. 2D representation of the loading surface in effective stress space.

8 NUMERICAL RESULTS FOR WILLAM'S TEST

To verify its capabilities under complex loading, the model has been implemented and used to solve Willam's test [23], which is becoming a typical benchmark for anisotropic cracking and damage formulations. This test consists of two load steps in plane stress. First, uniaxial loading is applied until peak stress is reached. Second, strain increments are applied to all in-plane degrees of freedom in the proportion $[\epsilon_{xx}, \epsilon_{yy}, \epsilon_{xy}] = [1, 1.5, 1]$. This represents increments of tensile strain in all directions, accompanied by a rotation of the principal axes which slows down progressively with a final asymptotic value of 52.02°. Fixed parameter values are $E^0 = 10^7$ kPa, $\nu^0 = 0.2$, $\sigma_{\text{peak}} = 10^4$ kPa and $g_f = 15$ kPa (i.e. three times the elastic energy at peak). The analysis is repeated for various values of parameter b. Some of the salient results for the extreme cases of $b = 0$ (isotropic damage) and $b = 40$ (very close to pure pseudo-Rankine, maximum anisotropy), are shown in Fig. 5. The extensive representation of the evolution of all variables for all parameter values, may be found in [7].

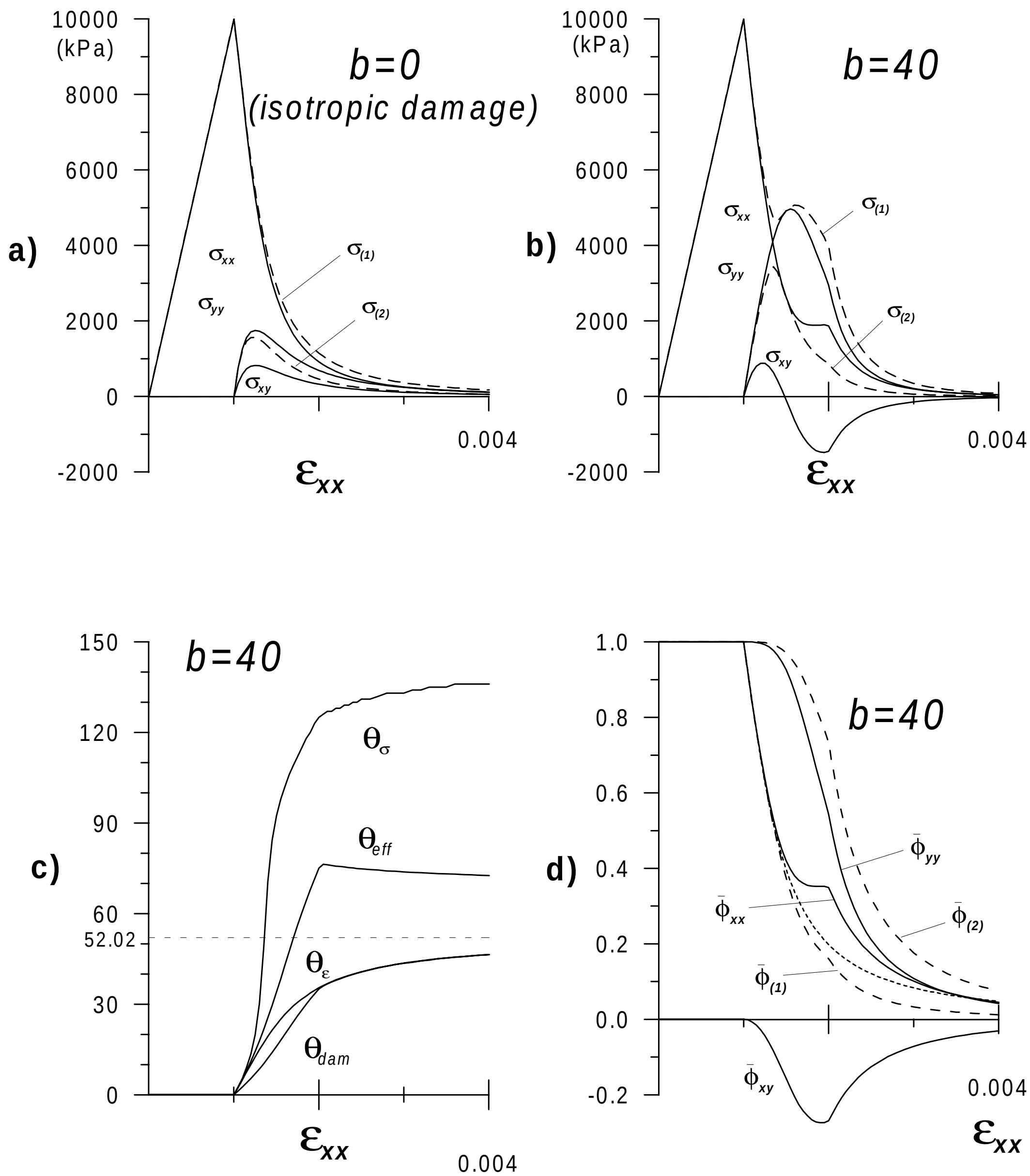

Fig. 5. Results of Willam's test. Evolution of: a) stress components for isotropic damage (b=0); b) same for highly anisotropic damage (b=40); c) angles of the first principal directions of prescribed strain, stress, damage and effective stress with x axis, for b=40; and d) damage components for b=40.

9 'EXTENDED' SECANT FORMULATION

As already mentioned, the 'basic' secant formulation of Sect. 4 represents only a restricted form of anisotropy, with 5 independent parameters in contrast for instance to 9 in general orthotropy. Also, the traditional "$(1-D)$" scalar damage model, which appears as the particular case of the formulation when the various damage tensors remain spherical (Sec. 3, Eqns. (21)), is a restricted form of isotropic degradation equivalent to only reducing the Young's modulus $E=(1-D)E^0$, while Poisson's ratio remains constant $\nu \equiv \nu^0$ [24]. Alternative more general options of single scalar damage variables affecting alternative elastic constants (e.g. K, G, Λ) are reviewed in [25]. An interesting case is that of purely deviatoric or 'von Mises' damage, in which only the shear modulus G is decreased while the bulk modulus K remains unaffected [20].

Attempts have have been made to derive the most general form of orthotropic secant stiffness using representation theorems [26, 27]. In that way, however, some of the nice properties satisfied by the 'basic' formulation (concepts such as effective stress, effective strain and energy equivalence, perfect duality and equivalence of the stress- and strain-based formulations, linearity and simplicity in most of the tensor relations, etc.) may be lost. What is outlined in this section, is a generalization of the basic secant formulation, which extends the number of independent parameters from 5 to 6, while still preserving the convenient properties of the basic formulation. Even if this is still far from general orthotropy, the 'extended' formulation exhibits interesting new features, such as encompassing a more general type of isotropic degradation [28] in which the two elastic constants may have different variations. This includes the purely deviatoric or von Mises damage as a particular case.

We consider the following expression for the secant compliance:

$$\mathbf{C} = \frac{1}{3K^0}\left[\frac{1}{3}\boldsymbol{\phi}^K \otimes \boldsymbol{\phi}^K\right] + \frac{1}{2G^0}\left[\frac{1}{2}(\boldsymbol{\phi}^G \overline{\otimes}\, \boldsymbol{\phi}^G + \boldsymbol{\phi}^G \underline{\otimes}\, \boldsymbol{\phi}^G) - \frac{1}{3}\boldsymbol{\phi}^G \otimes \boldsymbol{\phi}^G\right] \tag{43}$$

This corresponds to the standard initial elastic compliance written in terms of bulk and shear moduli, in which Kronecker deltas have been replaced by inverse integrity tensors $\boldsymbol{\phi}^K$ and $\boldsymbol{\phi}^G$, *which are different* for the volumetric and deviatoric stiffnesses.

The assumption is made that the two inverse integrity tensors are proportional, i.e. they have a common isochoric part $\boldsymbol{\psi}$ ($\det\boldsymbol{\psi}=1$), and only differ in their product-volumetric parts ϕ^K and ϕ^G:

$$\boldsymbol{\phi}^K = \phi^K \boldsymbol{\psi} \qquad , \qquad \boldsymbol{\phi}^G = \phi^G \boldsymbol{\psi} \tag{44a,b}$$

The compliance tensor may be then rewritten in the following ways:

$$\begin{aligned}\mathbf{C} =& \frac{1}{3K}\left[\frac{1}{3}\boldsymbol{\psi} \otimes \boldsymbol{\psi}\right] + \frac{1}{2G}\left[\frac{1}{2}(\boldsymbol{\psi} \overline{\otimes}\, \boldsymbol{\psi} + \boldsymbol{\psi} \underline{\otimes}\, \boldsymbol{\psi}) - \frac{1}{3}\boldsymbol{\psi} \otimes \boldsymbol{\psi}\right] = \\ =& -\frac{\nu}{E}\, \boldsymbol{\psi} \otimes \boldsymbol{\psi} + \frac{1+\nu}{2E}\, (\boldsymbol{\psi} \overline{\otimes}\, \boldsymbol{\psi} + \boldsymbol{\psi} \underline{\otimes}\, \boldsymbol{\psi})\end{aligned} \tag{45a,b}$$

where the new 'secant' elastic cofficients have expressions:

$$K = \frac{K^0}{(\phi^K)^2} \quad , \quad G = \frac{G^0}{(\phi^G)^2} \quad \text{and} \quad E = \frac{9KG}{3K+G} \quad , \quad \nu = \frac{3K-2G}{2(3K+G)} \qquad \text{(46a,b,c,d)}$$

One more assumption is made: that the variation of ϕ^K and ϕ^G is not completely independent, but is given by one single free variable ϕ and a fixed parameter η ($-1 \leq \eta \leq 1$), in the following fashion:

$$\phi^K = (\phi)^{1-\eta} \qquad , \qquad \phi^G = (\phi)^{1+\eta} \qquad \text{(47a,b)}$$

By reintroducing the single inverse integrity tensor $\boldsymbol{\phi} = \phi\boldsymbol{\psi}$, this leads to the following final form of the compliance tensor:

$$\mathbf{C} = -\frac{\hat{\nu}}{\hat{E}}\,\boldsymbol{\phi} \otimes \boldsymbol{\phi} + \frac{1+\hat{\nu}}{2\hat{E}}\,(\boldsymbol{\phi}\ \overline{\otimes}\ \boldsymbol{\phi} + \boldsymbol{\phi}\ \underline{\otimes}\ \boldsymbol{\phi}) \qquad \text{(48)}$$

where

$$\hat{K} = (\phi)^{2\eta} K^0 \quad , \quad \hat{G} = (\phi)^{-2\eta} G^0 \quad , \quad \hat{E} = \frac{9\hat{K}\hat{G}}{3\hat{K}+\hat{G}} \quad , \quad \hat{\nu} = \frac{3\hat{K}-2\hat{G}}{2(3\hat{K}+\hat{G})} \quad , \qquad \text{(49a,b,c,d)}$$

Expressing this compliance tensor as a 6×6 matrix in the principal axes of damage, and comparing to orthotropic elasticity (33), one obtains the following new set of equivalences in terms of the principal values of the integrity tensor $\overline{\boldsymbol{\phi}} = \boldsymbol{\phi}^{-1}$:

$$E_1 = \overline{\phi}_1^2 \hat{E} \ , \quad E_2 = \overline{\phi}_2^2 \hat{E} \ , \quad E_3 = \overline{\phi}_3^2 \hat{E} \ , \quad G_{12} = \frac{\overline{\phi}_1 \overline{\phi}_2 \hat{E}}{2(1+\hat{\nu})} \ , \quad G_{23} = \frac{\overline{\phi}_2 \overline{\phi}_3 \hat{E}}{2(1+\hat{\nu})} \ , \quad G_{31} = \frac{\overline{\phi}_3 \overline{\phi}_1 \hat{E}}{2(1+\hat{\nu})}$$
$$\nu_{12} = \frac{\overline{\phi}_2}{\overline{\phi}_1}\hat{\nu} \ , \quad \nu_{13} = \frac{\overline{\phi}_3}{\overline{\phi}_1}\hat{\nu} \ , \quad \nu_{21} = \frac{\overline{\phi}_1}{\overline{\phi}_2}\hat{\nu} \ , \quad \nu_{23} = \frac{\overline{\phi}_3}{\overline{\phi}_2}\hat{\nu} \ , \quad \nu_{31} = \frac{\overline{\phi}_1}{\overline{\phi}_3}\hat{\nu} \ , \quad \nu_{32} = \frac{\overline{\phi}_2}{\overline{\phi}_3}\hat{\nu} \qquad \text{(50a-l)}$$

i.e. the same as in Eqns (34), except for $\hat{E}$, $\hat{\nu}$ replacing the initial E^0, ν^0. The nine orthotropic elastic constants are now functions of *six* independent parameters: the same five of the basic formulation, E^0, ν^0, ϕ_1, ϕ_2, ϕ_3, plus the new one η, which is hidden in $\hat{E}$ and $\hat{\nu}$. Note that, by setting parameter $\eta = 0$, we obtain $\hat{E} = E^0$ and $\hat{\nu} = \nu^0$, and the basic formulation is recovered. On the other hand, for $\eta < 0$, damage progresses faster for the volumtric than the deviatoric part, and the opposite for $\eta > 0$. In the limit cases of $\eta = -1$ or $\eta = 1$, the model will only degrade respectively the volumetric part or deviatoric part of stiffness.

Note finally the convenient feature that, in the format introduced, the extended formulation is compatible with most concepts and developments described in previous sections for the basic formulation, such as pseudo-logarithmic damage rate and its properties, physically meaningful conjugate forces, etc. Taking advantadge of that, an extended version of the Generalized pseudo-Rankine model of Sect. 8 is currently under development to investigate the practical effect of parameter η in the model response.

10 CONCLUDING REMARKS

The theoretical framework for elastic degradation and damage proposed by the authors some years ago is shown to be a powerful base for development of models dealing in an elegant way with complex material behavior. The concept of pseudo-logarithmic damage rate leads to convenient properties which open the door to rational treatment of anisotropic damage and its evolution laws. The simple format of a 'basic' anisotropic secant formulation, even if representing only a restricted form of orthotropy, allows us to implement all those features in the 'Generalized pseudo-Rankine' model. With only five material parameters with clear physical meaning, the model exhibits simple analytical solutions for basic loading scenarios, and consistent behavior under complex loading with rotation of principal directions. An 'extended' version of the anisotropic formulation has also been outlined. All this sets a very promising scene for further developments and practical applications of the new formulations proposed.

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