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Ignacio Carol, Egidio Rizzi and Kaspar Willam

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An 'extended' volumetric/deviatoric formulation of anisotropic damage based on a pseudo-log rate

Ignacio Carol[1], Egidio Rizzi[2] and Kaspar Willam[3]

Abstract

Following a framework of elastic degradation and damage previously proposed by the authors, an 'extended' formulation of orthotropic damage in initially-isotropic materials, based on volumetric/deviatoric decomposition, is presented. The formulation is founded on the concept of energy equivalence and makes use of second-order symmetric tensor damage variables. It is characterized by fourth-order damage-effect tensors (relating nominal to effective stresses and strains) built from the underlying second-order damage tensors and decomposed in product-form in isotropic and anisotropic parts. The formulation is developed in two steps. First, secant relations are established. In the isotropic case, the model embeds a path-parameter allowing to range between pure volumetric to pure deviatoric damage. With the two undamaged material constants this makes a total of three constant parameters plus an evolving scalar damage variable, giving rise to a four-parameter model with two varying isotropic material coefficients. In the anisotropic case, the model is still characterized by the same three material constants plus three evolving variables which are the principal values of a second-order damage tensor. This leads to a six-parameter restricted form of orthotropic damage. In the second step, damage evolution rules are formulated in terms of a pseudo-logarithmic rate of damage. This allows to define meaningful conjugate forces that constitute a feasible space in which loading functions and damage evolution rules can be defined. The present 'extended' formulation is closed by the derivation of the tangent stiffness.

Keywords: Continuum Damage Mechanics, elastic damage, anisotropic damage, damage tensor, damage-effect tensor, pseudo-logarithmic damage rate, tangent stiffness.

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[1]Assoc. Prof. of Civil Eng., ETSECCPB, Technical Univ. of Catalonia, Dept. of Geotechnical Eng. and Geo-Sciences, E–08034 Barcelona, Spain. Adjunct Assoc. Prof., Dept. CEAE, University of Colorado, Boulder, CO 80309, USA.

Corresponding author, email: Ignacio.Carol@upc.es, fax: ++34.93.401.6504.

[2]Assoc. Prof. of Civil Eng., Politecnico di Bari, Facoltà di Ingegneria di Taranto, Dip. di Ingegneria Civile e Ambientale, Via Orabona 4, I-70125 Bari, Italia.

[3]Prof. of Civil Eng., Dept. CEAE, University of Colorado, Boulder, CO 80309, USA.

1 Introduction

The pioneering works of Kachanov (1958) and Rabotnov (1969) laid the foundations of the branch of Continuum Mechanics normally referred to as Continuum Damage Mechanics (CDM). After the first generation of formulations of elastic degradation and damage was proposed (Dougill 1976; Lemaitre and Chaboche, 1978; Bažant and Kim, 1979, Dragon and Mróz, 1979; Maier and Hueckel, 1979; Cordebois and Sidoroff, 1982; Ortiz, 1985; Murakami, 1988; to cite only a few), the number of papers dealing with this subject has increased rapidly (see e.g. the extensive reference list provided in the monograph by Krajcinovic, 1995). Initially, most of the literature was centered on modelling isotropic stiffness degradation based on scalar damage variables (e.g. Mazars and Lemaitre, 1984; Simo and Ju, 1987; Ju 1989; Neilsen and Schreyer, 1992), while in recent years the focus has gradually shifted to anisotropic stiffness degradation based on damage tensors of various orders (e.g. Cordebois and Sidoroff, 1982; Ladevèze, 1983; Chow and Wang, 1987; Yazdani and Schreyer, 1988; Ju, 1990; see also the references listed in Zheng and Betten, 1996 and Carol et al., 2001a). Indeed, anisotropic degradation is crucial both for damage-induced anisotropic processes in initially-isotropic materials (as e.g. quasi-brittle materials such as concrete, geomaterials, ceramics, etc.) and for damage development in anisotropic materials already embedding anisotropic textures at virgin state (e.g. rocks, composites, etc.). However, the problem of modelling anisotropic degradation still remains a challenging research issue. This includes the selection of appropriate tensor damage variables, the derivation of secant stiffness and compliance, the definition of appropriate loading functions, the postulation of evolution laws based on experimental observation, the description of coupling to plasticity, unilateral effects, stiffness recovery, and so on.

Considering second-order damage tensors as one of the most convenient options to represent anisotropic (orthotropic) degradation without excessive complexity, one problem is to find the most convenient forms of secant stiffness and compliance in terms of such second-order tensors. Such expressions should lead to sufficiently comprehensive forms of anisotropy (desirably approaching general orthotropy), but at the same time should keep sufficiently manageable for the practical implementation. The secant stiffness of the Valanis' damage model based on a second-order integrity variable (Valanis, 1990) provides an elegant and practical form of restricted orthotropy based on 5 parameters (two initial material constants and three evolving principal values of damage). At the same time, the model turns out to be compatible with traditional concepts of CDM such as effective stress and strain, energy equivalence, and exhibits a number of convenient properties (see Section 2.4). This may be considered as a good departure point for the development of more general anisotropic damage models. From a more rigorous viewpoint, some authors have applied representation theorems to obtain general expressions of orthotropic stiffness and compliance in fabric elasticity in terms of second-order fabric tensors (Cowin 1985, Zysset and Cournier, 1995), or have started from micromechanical considerations to represent damaged elastic properties in terms of series expansion of appropriate orientation distribution functions (Ladevèze, 1983; Lubarda and Krajcinovic, 1993; He and Curnier, 1995). However, these general expressions are either quite complex and, at least in their original form, privilege only the characteristics of the elastic properties at a given state (not the evolution), or they do not automatically preserve (or actually even refuse in methodological sense) the convenient features provided by the CDM framework (e.g. notion of effective quantities, modularity of the constitutive formulation, convenient structure to combine eventually damage with plasticity, viscoelasticity, or other rheological models for the undamaged material, including anisotropic behavior). Other developments within the framework of CDM (Lam and Zhang, 1995; Zheng and Betten, 1996) are confined instead to specific aspects of a damage model, as e.g. the definition of the fourth-order damage-effect tensor defining the linear map between nominal and effective stress and strain quantities, and the arising secant moduli.

In a sequence of papers, the authors have contributed to the proposal of a unified theoretical framework of elastic degradation and damage (Carol et al., 1994), the analysis of spurious energy dissipation of stiffness recovery schemes (Carol and Willam, 1996), and the study of the constitutive localization properties of scalar damage models, based on the spectral analysis of the acoustic tensor (Rizzi et al., 1995; Rizzi et al., 1996; Carol and Willam, 1997). Most recent contributions aimed at the representation of anisotropic elastic degradation through a second-order symmetric damage tensor and at the formulation of convenient damage evolution laws (Carol et al., 2001a,b; Rizzi and Carol, 2001). Such formulation, here referred-to as *'basic'*, describes a Valanis-type restricted form of orthotropic damage combined with convenient evolution laws defined in terms of a pseudo-logarithmic damage rate. The classical scalar damage of the '$(1-D)$'-type, which we will call 'basic' isotropic damage, is recovered as a particular case of the formulation when the damage tensor becomes spherical.

In the present paper, the 'basic' formulation is further extended to provide a more general description of orthotropic damage in initially-isotropic materials (*'extended'* formulation). The extension proposed is based on the volumetric/deviatoric decomposition of the underlying isotropic undamaged stiffness and compliance. The new developments focus on the following constitutive aspects: 'extended' secant relations of pure elastic damage based on damage-effect tensors decomposed in product form in isotropic and anisotropic parts, definition of the pseudo-logarithmic rate of damage and of the relevant conjugate force in the 'extended' framework, general damage evolution laws in pseudo-logarithmic space and resulting tangent stiffness. The interested reader is referred to Carol et al. (1994) and Carol et al. (2001a,b) for the details of the previous formulations and for comparison to the present extension. Indeed, to avoid undue repetitions, the presentation is restricted here to the new developments of the 'extended' formulation. However, a necessary brief account of the earlier framework of elastic degradation and damage is presented first in Section 2 in order to introduce the basic terminology and definition of the quantities involved in the 'extended' model. The main developments of the paper are then provided in Section 3, where 'extended' isotropic degradation is concerned, and in Section 4, which deals with 'extended' anisotropic degradation. A few closing remarks and perspectives of the present study are outlined in the closing section.

Notation. Compact or index tensor notation is used throughout. Vectors and second-order tensors are identified by boldface characters, whereas fourth-order tensors are denoted by blackboard-bold fonts (e.g. $\mathbb{A}, \mathbb{C}, \mathbb{E}$). Superscript $^{\mathrm{T}}$ indicates the transpose operation (on first and second couple of indices for fourth-order tensors, i.e. componentwise $(A^{\mathrm{T}})_{ijkl}=A_{klij}$), while 'tr' is the trace operator. Symbols '$\cdot$' and ':' denote the inner products with single and double contraction. The dyadic product is indicated with '$\otimes$', whereas '$\underline{\overline{\otimes}}$' denotes the symmetrized outer product defined as $(\mathbf{A}\,\underline{\overline{\otimes}}\,\mathbf{B}){:}\mathbf{C}{=}\mathbf{A}{\cdot}\mathbf{C}_s{\cdot}\mathbf{B}^{\mathrm{T}}$, for any arbitrary second-order tensors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, where $\mathbf{C}_s{=}(\mathbf{C}{+}\mathbf{C}^{\mathrm{T}})/2$ is the symmetric part of $\mathbf{C}$; componentwise $(\mathbf{A}\,\underline{\overline{\otimes}}\,\mathbf{B})_{ijkl}{=}(A_{ik}B_{jl}{+}A_{il}B_{jk})/2$. $\mathbf{I}$ and $\mathbb{I}_s{=}\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}$ are respectively the second-order and symmetric fourth-order identity tensors.

2 Theoretical framework of elastic degradation and damage

In the present section, the main ingredients of the theory of elastic degradation and damage presented in Carol et al. (1994) are briefly summarized. Reference is made as well to the classical formulation of isotropic damage of the '$(1-D)$' type ('basic' formulation of isotropic damage) and to the 'basic' formulation of anisotropic damage developed in Carol et al. (2001a).

2.1 Secant relations

In the simplest setting of pure elastic degradation (no elastic-plastic coupling), it is assumed that unloading always leads to the origin of the stress/strain curve with a constant secant stiffness/compliance. Fully-reversible reloading also follows back the same linear path until the nonlinear envelope behavior resumes. In other words the material behavior is characterized by a secant linear hyperelastic constitutive law: at any damage state the stress tensor $\boldsymbol{\sigma}$ and (small) strain tensor $\boldsymbol{\epsilon}$ are related by

$$\boldsymbol{\sigma} = \mathbb{E} : \boldsymbol{\epsilon} \; ; \qquad \boldsymbol{\epsilon} = \mathbb{C} : \boldsymbol{\sigma} \; , \tag{1}$$

where $\mathbb{E}$ and $\mathbb{C}$ are the positive-definite fourth-order stiffness and compliance tensors, respectively, which are endowed with both major and minor symmetries and are the inverse of each other, i.e. $\mathbb{E}{:}\mathbb{C}{=}\mathbb{C}{:}\mathbb{E}{=}\mathbb{I}_s$. In the initial (virgin) state the material is characterized by undamaged stiffness and compliance $\mathbb{E}_0$, $\mathbb{C}_0$.

It is also assumed that the compliance tensor $\mathbb{C}$ is a function of a generally-defined damage variable $\boldsymbol{\mathcal{D}}$, which may be scalar, vector- or tensor-valued (in the following of the paper symmetric second-order tensor variables are assumed). Obviously, the damaged compliance also depends explicitly on its initial value $\mathbb{C}_0$. Analogously, in a dual framework, the secant stiffness could be expressed in terms of dual damage variables $\bar{\boldsymbol{\mathcal{D}}}$ and initial value $\mathbb{E}_0$. Variables $\boldsymbol{\mathcal{D}}$ and $\bar{\boldsymbol{\mathcal{D}}}$ may be linked to each other and it is mainly a matter of choice to take one of the two as the underlying damage variable of the model. The distinction between dual variables is not completely pursued in the present summary section: without loss of generality the subsequent equations are mainly confined to the compliance-based formulation in terms of $\boldsymbol{\mathcal{D}}$.

Thereby, the elastic (free) energy of the material per unit reference volume u may be expressed, at any stage of the damage process, by the quadratic forms

$$u = \frac{1}{2}\,\boldsymbol{\epsilon} : \mathbb{E}(\mathbb{E}_0, \bar{\boldsymbol{\mathcal{D}}}) : \boldsymbol{\epsilon} = \frac{1}{2}\,\boldsymbol{\sigma} : \mathbb{C}(\mathbb{C}_0, \boldsymbol{\mathcal{D}}) : \boldsymbol{\sigma} \; . \tag{2}$$

Disregarding effects other than mechanical, the previous relation may be differentiated with respect to time to obtain the incremental energy balance in terms of the (non-negative) dissipation rate $\dot{d}$ and conjugate forces $-\boldsymbol{\mathcal{Y}}$, energetically-associated to the rate of the damage variable $\boldsymbol{\mathcal{D}}$:

$$\dot{u} = \boldsymbol{\sigma}:\dot{\boldsymbol{\epsilon}} - \dot{d}\,; \qquad \dot{d} = (-\boldsymbol{\mathcal{Y}}) : \dot{\boldsymbol{\mathcal{D}}} \geq 0\,, \qquad -\boldsymbol{\mathcal{Y}} = \left.\frac{\partial u}{\partial \boldsymbol{\mathcal{D}}}\right|_{\boldsymbol{\sigma}} = \frac{1}{2}(\boldsymbol{\sigma}\otimes\boldsymbol{\sigma})::\frac{\partial \mathbb{C}}{\partial \boldsymbol{\mathcal{D}}}\,. \tag{3}$$

The conjugate forces $-\boldsymbol{\mathcal{Y}}$ constitute the space in which the hardening/softening loading surface $F(-\boldsymbol{\mathcal{Y}}, \mathbf{p})=0$ is defined, where $\mathbf{p}$ denotes an appropriate set of hardening internal variables, and where the local normal to the damage surface $\boldsymbol{\mathcal{N}}$ and the *'damage rule'* $\boldsymbol{\mathcal{M}}$ entering the damage rate $\dot{\boldsymbol{\mathcal{D}}}$ can be established:

$$\boldsymbol{\mathcal{N}} = \left.\frac{\partial F}{\partial(-\boldsymbol{\mathcal{Y}})}\right|_{\lambda}\;; \qquad \dot{\boldsymbol{\mathcal{D}}} = \dot{\lambda}\;\boldsymbol{\mathcal{M}}\,. \tag{4}$$

Here, similar to plastic flow rules, $\dot{\lambda}$ is the (non-negative) inelastic multiplier defining the magnitude of the damage rate. This brings in a convenient analogy to plasticity formulations.

2.2 Plasticity-like concepts

The above-mentioned thermodynamic framework of elastic damage (formulation in terms of a free energy potential) is indeed perfectly compatible with well-established plasticity-like concepts allowing dual, and, in a sense, more intuitive derivations in stress (or strain) space. A first step is to consider the (fourth-order) space of forces $-\mathbb{Y}$ conjugate to the compliance rate $\dot{\mathbb{C}}$, in which we can rephrase the dissipation rate $\dot{d}$ as

$$\dot{d} = \frac{1}{2}\,\boldsymbol{\sigma} : \dot{\mathbb{C}} : \boldsymbol{\sigma} = (-\mathbb{Y}) :: \dot{\mathbb{C}}\,; \qquad -\mathbb{Y} = \left.\frac{\partial u}{\partial \mathbb{C}}\right|_{\boldsymbol{\sigma}} = \frac{1}{2}\,\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}\,; \qquad -\boldsymbol{\mathcal{Y}} = (-\mathbb{Y}) :: \frac{\partial \mathbb{C}}{\partial \boldsymbol{\mathcal{D}}}\,, \tag{5}$$

and loading function $F(-\mathbb{Y}, \mathbf{p})$, relevant normal $\mathbb{N}$ and fourth-order *'compliance rule'* $\mathbb{M}$ as

$$\mathbb{N} = \left.\frac{\partial F}{\partial(-\mathbb{Y})}\right|_{\lambda} = \frac{\partial \mathbb{C}}{\partial \boldsymbol{\mathcal{D}}} : \boldsymbol{\mathcal{N}}\,; \qquad \dot{\mathbb{C}} = \dot{\lambda}\;\mathbb{M}\,; \qquad \mathbb{M} = \frac{\partial \mathbb{C}}{\partial \boldsymbol{\mathcal{D}}} : \boldsymbol{\mathcal{M}}\,. \tag{6}$$

These quantities may be finally related to the ones more usually adopted in stress space where the loading function is given as $F(\boldsymbol{\sigma}, \mathbf{p})$, with normal to the surface $\mathbf{n}$, degrading strain rate $\dot{\boldsymbol{\epsilon}}_d$ and *'flow rule'* $\mathbf{m}$:

$$\mathbf{n} = \left.\frac{\partial F}{\partial \boldsymbol{\sigma}}\right|_{\lambda} = \mathbb{N} : \boldsymbol{\sigma}\,; \qquad \dot{\boldsymbol{\epsilon}}_d = \dot{\lambda}\;\mathbf{m}\,; \qquad \mathbf{m} = \mathbb{M} : \boldsymbol{\sigma}\,. \tag{7}$$

Using these concepts, the constitutive equations of elastic degradation may be rephrased in terms of rates and become formally identical to the ones classically derived in the context of non-holonomic elastic-plasticity. A basic difference is that the current secant stiffness $\mathbb{E}$ enters the tangent operator $\mathbb{E}_{\tan}$ instead of the initial stiffness $\mathbb{E}_0$. Indeed, for further inelastic loading, the inelastic multiplier $\dot{\lambda}$, incremental stress/strain law and tangent stiffness are obtained from the consistency condition as follows:

$$\dot{\lambda} = \frac{1}{\bar{H}}\, \mathbf{n} : \mathbb{E} : \dot{\boldsymbol{\epsilon}} \; ; \qquad \dot{\boldsymbol{\sigma}} = \mathbb{E}_{\tan} : \dot{\boldsymbol{\epsilon}} \; ; \qquad \mathbb{E}_{\tan} = \mathbb{E} - \frac{1}{\bar{H}}\, \mathbb{E} : \mathbf{m} \otimes \mathbf{n} : \mathbb{E} \; , \tag{8}$$

where the hardening parameters in strain and stress space, $\bar{H}$ and H, respectively, are defined as:

$$\bar{H} = -\left. \frac{\partial F}{\partial \lambda} \right|_{\boldsymbol{\epsilon}} = H + \mathbf{n} : \mathbb{E} : \mathbf{m} \; ; \qquad H = -\left. \frac{\partial F}{\partial \lambda} \right|_{\boldsymbol{\sigma}} \; . \tag{9}$$

Similar to elastic-plasticity, H, $\mathbf{n}$ and $\mathbf{m}$ are restricted in such a way that, to avoid sub-critical softening, the denominator $\bar{H}{=}H{+}\mathbf{n}{:}\mathbb{E}{:}\mathbf{m}$ in eqn (8) remains always positive. The model is called associated *in stress space* (traditional definition) when $\mathbf{m}$ is proportional to $\mathbf{n}$; consequently, the tangent stiffness $\mathbb{E}_{\tan}$ exhibits major symmetry. If $\mathbf{m}$ is derived from a potential Q, associativity may be alternatively stated as $Q{=}F$. Other definitions of associativity may be established *in compliance space*, if $\mathbb{M}$ is parallel to $\mathbb{N}$, which implies the former, or *in damage space*, if $\boldsymbol{\mathcal{M}}$ is parallel to $\boldsymbol{\mathcal{N}}$, which implies all the previous ones. The latter may be termed *full associativity* (Carol et al., 1994).

2.3 CDM concepts

In Continuum Damage Mechanics (CDM) the damage-state relation $\mathbb{C}{=}\mathbb{C}(\mathbb{C}_0, \boldsymbol{\mathcal{D}})$ (or equivalently $\mathbb{E}{=}\mathbb{E}(\mathbb{E}_0, \bar{\boldsymbol{\mathcal{D}}})$) is usually derived indirectly through a series of conceptual steps (see e.g. Carol et al., 2001a; Rizzi and Carol, 2001; and references quoted therein). Although some criticism may be addressed to such a purely phenomenological approach (see for instance Ladevèze, 1983; Rabier, 1989; He and Curnier, 1995), this scheme is adopted here since it provides a modular structure of the constitutive equations, not only useful to derive and interpret the final secant relations obtained, but also eventually convenient to implement more comprehensive generalizations involving e.g. coupling to plasticity, viscoelasticity, or other constitutive models for the undamaged behavior.

A constitutive law for the undamaged material is first introduced, which is expressed in terms of the so-called *'effective'* stress and strain quantities, $\boldsymbol{\sigma}_{\text{eff}}$ and $\boldsymbol{\epsilon}_{\text{eff}}$, namely stress and strain acting at the level of the intact material between microcracks:

$$\boldsymbol{\sigma}_{\text{eff}} = \mathbb{E}_0 : \boldsymbol{\epsilon}_{\text{eff}} \; ; \qquad \boldsymbol{\epsilon}_{\text{eff}} = \mathbb{C}_0 : \boldsymbol{\sigma}_{\text{eff}} \; . \tag{10}$$

Then, one of the relations between nominal and effective (stress or strain) quantities is assumed, often in linear form, by introducing a non-singular fourth-order damage-effect tensor $\mathbb{A}$, as e.g. in $\boldsymbol{\sigma}_{\text{eff}}{=}\mathbb{A}{:}\boldsymbol{\sigma}$ (see e.g. Lam and Zhang, 1995; Zheng and Betten, 1996; Voyiadjis and Park, 1997), together with a second relation expressed through a so-called *'equivalence principle'* ('strain equivalence' if $\boldsymbol{\epsilon}{=}\boldsymbol{\epsilon}_{\text{eff}}$, 'stress equivalence' if $\boldsymbol{\sigma}{=}\boldsymbol{\sigma}_{\text{eff}}$, or 'energy equivalence' if $\boldsymbol{\sigma}{:}\boldsymbol{\epsilon}/2{=}\boldsymbol{\sigma}_{\text{eff}}{:}\boldsymbol{\epsilon}_{\text{eff}}/2$). The *'energy equivalence'* approach (Cordebois and Sidoroff, 1982) is adopted here, since it allows to derive secant stiffness and compliance automatically embedding the major symmetry property. Then, the current elastic energy at any time of the degradation process can be expressed by any of these relations:

$$
\begin{aligned}
u &= \frac{1}{2}\,\boldsymbol{\sigma} \;:\boldsymbol{\epsilon} \;\; = \frac{1}{2}\,\boldsymbol{\epsilon} \;:\boldsymbol{\sigma} \;\; = \frac{1}{2}\,\boldsymbol{\sigma} \;:\mathbb{C}:\boldsymbol{\sigma} \;\; = \frac{1}{2}\,\boldsymbol{\epsilon} \;:\mathbb{E}:\boldsymbol{\epsilon} \\
&= \frac{1}{2}\,\boldsymbol{\sigma}_{\text{eff}}:\boldsymbol{\epsilon}_{\text{eff}} = \frac{1}{2}\,\boldsymbol{\epsilon}_{\text{eff}}:\boldsymbol{\sigma}_{\text{eff}} = \frac{1}{2}\,\boldsymbol{\sigma}_{\text{eff}}:\mathbb{C}_0:\boldsymbol{\sigma}_{\text{eff}} = \frac{1}{2}\,\boldsymbol{\epsilon}_{\text{eff}}:\mathbb{E}_0:\boldsymbol{\epsilon}_{\text{eff}}\;.
\end{aligned}
\tag{11}
$$

The following nominal/effective relations are then consistently assumed/obtained:

$$
\boldsymbol{\epsilon}_{\text{eff}} = \bar{\mathbb{A}}^{\mathrm{T}} : \boldsymbol{\epsilon}\;; \qquad \boldsymbol{\sigma} = \bar{\mathbb{A}} : \boldsymbol{\sigma}_{\text{eff}}\;, \tag{12}
$$

or

$$
\boldsymbol{\sigma}_{\text{eff}} = \mathbb{A} : \boldsymbol{\sigma}\;; \qquad \boldsymbol{\epsilon} = \mathbb{A}^{\mathrm{T}} : \boldsymbol{\epsilon}_{\text{eff}}\;, \tag{13}
$$

and the hyperelastic stiffness and compliance are expressed by:

$$
\mathbb{E} = \bar{\mathbb{A}} : \mathbb{E}_0 : \bar{\mathbb{A}}^{\mathrm{T}}\;; \qquad \mathbb{C} = \mathbb{A}^{\mathrm{T}} : \mathbb{C}_0 : \mathbb{A}\;, \tag{14}
$$

where $\bar{\mathbb{A}}$ and $\mathbb{A}$ are dual fourth-order damage-effect tensors, inverse of each other, that is $\mathbb{A}$:$\bar{\mathbb{A}}$=$\bar{\mathbb{A}}$:$\mathbb{A}$=$\mathbb{I}_s$, and endowed with minor symmetries (but not necessarily major symmetry).

Before arriving at the expressions of the secant stiffness and compliance in eqn (14), the undamaged behavior has to be prescribed. In the present context restriction is made to *initially-isotropic materials*, namely, in terms of undamaged Lamé's constant Λ_0 and shear modulus G_0, or Young's modulus E_0 and Poisson's ratio ν_0:

$$
\mathbb{E}_0 = \Lambda_0\,\mathbf{I}\otimes\mathbf{I} + 2\,G_0\,\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I}\;; \qquad \mathbb{C}_0 = -\frac{\nu_0}{E_0}\,\mathbf{I}\otimes\mathbf{I} + \frac{1+\nu_0}{E_0}\,\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I}\;, \tag{15}
$$

where the two sets of isotropic undamaged parameters are linked by the classical relations

$$
\Lambda_0 = \frac{\nu_0\,E_0}{(1+\nu_0)\,(1-2\nu_0)}\;,\; G_0 = \frac{E_0}{2(1+\nu_0)}\;;\; E_0 = G_0\,\frac{3\Lambda_0 + 2\,G_0}{\Lambda_0 + G_0}\;,\; \nu_0 = \frac{\Lambda_0}{2(\Lambda_0 + G_0)}\;. \tag{16}
$$

Alternatively, by introducing the idempotent fourth-order tensor projection operators $\mathbb{P}_V$=$\mathbf{I}\otimes\mathbf{I}/3$ and $\mathbb{P}_D$=$\mathbb{I}_s$−$\mathbb{P}_V$ (see e.g. Walpole, 1984), the initial isotropic stiffness and compliance tensors can also be conveniently rewritten in spectral form. This highlights directly the decoupling between initial isotropic volumetric and deviatoric responses, which are taken here as reference for the following developments:

$$
\mathbb{E}_0 = 3\,K_0\,\mathbb{P}_V + 2\,G_0\,\mathbb{P}_D\;; \qquad \mathbb{C}_0 = \frac{1}{3\,K_0}\,\mathbb{P}_V + \frac{1}{2\,G_0}\,\mathbb{P}_D\;, \tag{17}
$$

where K_0 is the bulk modulus of the undamaged material, which may be expressed in terms of the previous material constants as:

$$
K_0 = \Lambda_0 + \frac{2}{3}\,G_0 = \frac{E_0}{3\,(1-2\,\nu_0)}\;. \tag{18}
$$

Other representations equivalent to (15) and (17) could also be adopted as a different starting point for the reference isotropic stiffness and compliance. For instance, from eqn (15b) the terms may be regrouped to obtain different binomial expressions of $\mathbb{C}$ in terms of coefficients $1/E_0$ and $-\nu_0/E_0$, or $1/E_0$ and $1/(2G_0)$, or $3/(2E_0)$ and $1/(6K_0)$, etc. (see for example Ladevèze, 1983 and He and Curnier, 1995, which make use of elongation and bulk moduli).

2.4 'Basic' formulation of isotropic and anisotropic damage

The damage-effect tensors $\bar{\mathbb{A}}$, $\mathbb{A}$ should be expressed in terms of the underlying damage variables $\bar{\boldsymbol{\mathcal{D}}}$ or $\boldsymbol{\mathcal{D}}$ adopted in the model. Two notable instances of modelling elastic damage in the present context are the following (Carol et al., 2001a): i) the classical '$(1-D)$' scalar damage model *('basic' isotropic formulation)*, which is founded on a single scalar damage variable $0 \leq D < 1$ varying between 0 (no damage) and 1 (full damage), or on an integrity variable $\bar{\phi} = 1/\phi = \sqrt{1-D}$ with complementary variation between 1 (no damage) and 0 (full damage); ii) the *'basic' anisotropic formulation* of damage established upon Valanis-type positive-definite second-order integrity tensor $\bar{\boldsymbol{\phi}}$ varying between $\mathbf{I}$ (no damage) and $\mathbf{0}$ (full damage) (Valanis, 1990), or upon its inverse $\boldsymbol{\phi} = \bar{\boldsymbol{\phi}}^{-1}$ varying between $\mathbf{I}$ (no damage) and $\boldsymbol{\infty}$ (full damage). In these two cases the damage-effect tensors $\bar{\mathbb{A}}$, $\mathbb{A}$, inverse of each others, are respectively given by the following positive-definite fully-symmetric fourth-order tensors:

$$\bar{\mathbb{A}}_{(1-D)} = \sqrt{1-D} \;\; \mathbb{I}_s = \bar{\phi} \;\; \mathbb{I}_s \,, \qquad \mathbb{A}_{(1-D)} = \frac{1}{\sqrt{1-D}} \;\; \mathbb{I}_s = \phi \;\; \mathbb{I}_s \,; \tag{19}$$

$$\bar{\mathbb{A}}_{\mathrm{bas}} = \sqrt{\bar{\boldsymbol{\phi}}} \;\underline{\overline{\otimes}}\; \sqrt{\bar{\boldsymbol{\phi}}} \,, \qquad \mathbb{A}_{\mathrm{bas}} = \sqrt{\boldsymbol{\phi}} \;\underline{\overline{\otimes}}\; \sqrt{\boldsymbol{\phi}} \,, \tag{20}$$

which, from eqns (12a), (13a) lead respectively to the following nominal to effective relations:

$$\boldsymbol{\epsilon}_{\mathrm{eff},(1-D)} = \sqrt{1-D}\; \boldsymbol{\epsilon} = \bar{\phi}\; \boldsymbol{\epsilon} \,, \qquad \boldsymbol{\sigma}_{\mathrm{eff},(1-D)} = \frac{1}{\sqrt{1-D}}\; \boldsymbol{\sigma} = \phi\; \boldsymbol{\sigma} \,; \tag{21}$$

$$\boldsymbol{\epsilon}_{\mathrm{eff,bas}} = \sqrt{\bar{\boldsymbol{\phi}}} \cdot \boldsymbol{\epsilon} \cdot \sqrt{\bar{\boldsymbol{\phi}}} \,, \qquad \boldsymbol{\sigma}_{\mathrm{eff,bas}} = \sqrt{\boldsymbol{\phi}} \cdot \boldsymbol{\sigma} \cdot \sqrt{\boldsymbol{\phi}} \,. \tag{22}$$

Also, concerning secant relations, the following convenient forms of secant stiffness and compliance are recovered from eqns (14) and (19), and, for initially-isotropic materials, from eqns (14), (15) and (20):

$$\mathbb{E}_{(1-D)} = (1-D)\; \mathbb{E}_0 = \bar{\phi}^2\; \mathbb{E}_0 \,, \qquad \mathbb{C}_{(1-D)} = \frac{1}{1-D}\; \mathbb{C}_0 = \phi^2\; \mathbb{C}_0 \,; \tag{23}$$

$$\mathbb{E}_{\mathrm{bas}} = \Lambda_0\; \bar{\boldsymbol{\phi}} \otimes \bar{\boldsymbol{\phi}} + 2\, G_0\; \bar{\boldsymbol{\phi}} \underline{\overline{\otimes}} \bar{\boldsymbol{\phi}} \,; \qquad \mathbb{C}_{\mathrm{bas}} = -\frac{\nu_0}{E_0}\; \boldsymbol{\phi} \otimes \boldsymbol{\phi} + \frac{1+\nu_0}{E_0}\; \boldsymbol{\phi} \underline{\overline{\otimes}} \boldsymbol{\phi} \,. \tag{24}$$

Notice that eqn (24) presents the Valanis-type structure of orthotropic stiffness and compliance in which $\bar{\boldsymbol{\phi}}$ or $\boldsymbol{\phi}$ simply replace $\mathbf{I}$ in eqn (15) (Valanis, 1990; Zysset and Curnier, 1995). Secant relations (23) and (24) involve two isotropic undamaged elastic constants. Once these two constants are given, one or three degrees of freedom are respectively available to specify the variations of secant stiffness and compliance in the isotropic and anisotropic case. In the anisotropic case this leads to a restricted form of orthotropy based on five parameters.

Furthermore, concerning damage rate and conjugate thermodynamic force in the dissipation rate $\dot{d}$, eqn (3b), a logarithmic damage variable $L = -\ln(1-D) = -2\ln\bar{\phi}$ varying between 0 and ∞, with corresponding (holonomic) rate $\dot{L}$, and a (non-holonomic) pseudo-logarithmic rate of damage $\mathring{\mathbf{L}}$ are introduced (in general $\mathring{\mathbf{L}}$ exists only as a rate, while it cannot be expressed as the rate $\dot{\mathbf{L}}$ of a finite quantity $\mathbf{L}$), so that the forces conjugated to such rates are conveniently given by

$$\dot{L} = \frac{\dot{D}}{1-D} = -2\,\frac{\dot{\bar{\phi}}}{\bar{\phi}}\,, \qquad -\mathcal{Y}_{(1-D)} = u\;; \tag{25}$$

$$\overset{\diamond}{\mathbf{L}} = -2\;\mathbb{A}_{\text{bas}} : \dot{\boldsymbol{\phi}} = -2\,\sqrt{\boldsymbol{\phi}}\cdot\dot{\boldsymbol{\phi}}\cdot\sqrt{\boldsymbol{\phi}}\,, \qquad -\boldsymbol{\mathcal{Y}}_{\text{bas}} = \frac{1}{2}\,\boldsymbol{\sigma}_{\text{eff,bas}}\cdot\boldsymbol{\epsilon}_{\text{eff,bas}}\,, \tag{26}$$

where $\boldsymbol{\sigma}_{\text{eff,bas}}$ and $\boldsymbol{\epsilon}_{\text{eff,bas}}$ are the effective quantities of the 'basic' model given in eqn (22).

In the two following sections, 'extended' versions of both the isotropic and anisotropic 'basic' formulations are presented. While retaining the fundamental advantages of their 'basic' counterparts (e.g. the isotropic formulation is a particular version of the anisotropic one, stress-based formulations and their dual strain-based are fully equivalent, introduction of pseudo-logarithmic rate of damage and related advantages, etc., see Carol et al., 2001a,b for the details), the 'extended' formulation developed in the sequel provides a broader representation of the degraded behavior, encompassing for instance pure deviatoric ('von Mises') or mixed volumetric/deviatoric damage, among others.

3 'Extended' volumetric/deviatoric formulation of isotropic damage

In the above-mentioned traditional '$(1-D)$' scalar damage model ('basic' isotropic formulation), a single scalar damage variable D or integrity variable $\bar{\phi}$ introduce a proportional degradation of all components of the secant stiffness: $\mathbb{E}=(1-D)\,\mathbb{E}_0=\bar{\phi}^2\,\mathbb{E}_0$. This is equivalent to reduce Young's modulus $E=(1-D)\,E_0$ (and similarly also bulk and shear moduli K and G, and Lamé's constant Λ), while Poisson's ratio remains constant $\nu\equiv\nu_0$, which implies a restricted form of isotropic degradation (see e.g. Ju, 1990). Other possibilities of damage models with a single scalar damage variable $\lambda(D)$ affecting secant compliance as $\mathbb{C}=\mathbb{C}_0+\lambda(D)\,\mathbb{M}$ with different given constant isotropic directions $\mathbb{M}$, as well as different scalar damage models from the literature framed as particular cases of the general setting of elastic degradation are reviewed in Rizzi (1993). An interesting case is that of purely deviatoric or 'von Mises' damage ($\mathbb{M}=\mathbb{P}_D$), in which only shear modulus G is decreased while bulk modulus K remains unaffected and the associated model is based on a 'von Mises' loading surface (Neilsen and Schreyer, 1992). The proposed extension is based on the idea of allowing independent evolutions of volumetric and deviatoric parts of stiffness and compliance, then including as particular case the above-mentioned 'von Mises' damage formulation.

3.1 Bi-dissipative isotropic model (two damage variables)

Indeed, the most general form of isotropic elastic damage with constant unloading/reloading secant stiffness and compliance would require two independent scalar damage variables acting independently on each of the two elastic material parameters taken as reference. For that purpose, referring here to undamaged elastic bulk and shear moduli K_0 and G_0, it is convenient to adopt the volumetric-deviatoric format of current secant stiffness and compliance similar to eqn (17):

$$\mathbb{E}_{\rm iso} = 3\,K\;\mathbb{P}_V + 2\,G\;\mathbb{P}_D\;; \qquad \mathbb{C}_{\rm iso} = \frac{1}{3\,K}\;\mathbb{P}_V + \frac{1}{2\,G}\;\mathbb{P}_D\;, \tag{27}$$

where, as opposed to eqn (17), K and G are the current (secant) degraded counterparts of K_0 and G_0. Moduli K and G are assumed to decrease independently with two damage variables D_K and D_G evolving separately between 0 and 1 or, alternatively, with integrity variables $\bar{\phi}_K$, $\bar{\phi}_G$, or their inverses ϕ_K, ϕ_G, displaying complementary variations between 1 and 0:

$$\bar{\phi}_K = \frac{1}{\phi_K} = \sqrt{1 - D_K}\;, \qquad \bar{\phi}_G = \frac{1}{\phi_G} = \sqrt{1 - D_G}\;. \tag{28}$$

Then, rephrasing the '$(1-D)$' model, here with two independent variations, the secant moduli entering eqn (27) are given as:

$$K = (1-D_K)\;K_0 = \bar{\phi}_K^2\;K_0\;; \qquad G = (1-D_G)\;G_0 = \bar{\phi}_G^2\;G_0\;. \tag{29}$$

Differentiating the compliance tensor (27b) yields the following compliance rate:

$$\dot{\mathbb{C}} = \frac{\dot{D}_K}{1 - D_K}\;\frac{\mathbb{P}_V}{3\,K} + \frac{\dot{D}_G}{1 - D_G}\;\frac{\mathbb{P}_D}{2\,G} = -2\;\frac{\dot{\bar{\phi}}_K}{\bar{\phi}_K}\;\frac{\mathbb{P}_V}{3\,K} - 2\;\frac{\dot{\bar{\phi}}_G}{\bar{\phi}_G}\;\frac{\mathbb{P}_D}{2\,G}\;, \tag{30}$$

where notice that K and G are the current moduli. Then, as in eqn (25), two independent logarithmic damage rates $\dot{L}_K$ and $\dot{L}_G$ playing directly the role of inelastic multipliers can be alternatively introduced to the rates $\dot{D}_K$, $\dot{D}_G$, or $\dot{\bar{\phi}}_K$, $\dot{\bar{\phi}}_G$, namely:

$$\dot{L}_K = \frac{\dot{D}_K}{1 - D_K} = -2\;\frac{\dot{\bar{\phi}}_K}{\bar{\phi}_K}\;; \qquad \dot{L}_G = \frac{\dot{D}_G}{1 - D_G} = -2\;\frac{\dot{\bar{\phi}}_G}{\bar{\phi}_G}\;. \tag{31}$$

With this change of variables, eqn (30) can just be rewritten as:

$$\dot{\mathbb{C}} = \dot{L}_K\;\frac{\mathbb{P}_V}{3\,K} + \dot{L}_G\;\frac{\mathbb{P}_D}{2\,G}\;. \tag{32}$$

Rates definitions (31) actually correspond to assume the following total relations:

$$L_K = \ln\frac{1}{1 - D_K} = -2\ln\bar{\phi}_K\;; \qquad L_G = \ln\frac{1}{1 - D_G} = -2\ln\bar{\phi}_G\;, \tag{33}$$

where L_K and L_G are logarithmic damage variables varying between 0 and ∞, or, what is equivalent, the inverse relations

$$D_K = 1 - \mathrm{e}^{-L_K}\;, \quad D_G = 1 - \mathrm{e}^{-L_G}\;; \qquad \bar{\phi}_K = \mathrm{e}^{-L_K/2}\;, \quad \bar{\phi}_G = \mathrm{e}^{-L_G/2}\;. \tag{34}$$

This is a convenient departure point for a general bi-dissipative isotropic degradation model based on two damage variables, two inelastic multipliers, two loading surfaces, etc.

3.2 Single-dissipative isotropic model (one damage variable)

Restricting now our attention to single-dissipative models based on a single loading surface and a single logarithmic damage variable L, one may assume the following relation between the two logarithmic damage variables:

$$L_K = \beta_K \, L \; ; \qquad L_G = \beta_G \, L \; , \tag{35}$$

where β_K and β_G are two additional constant parameters. With linear link (35), volumetric and deviatoric degradations are no longer independent, although their values are not necessarily identical as in the 'basic' scalar '(1–D)' model of elastic degradation (β_K=β_G). With hypothesis (35), eqn (32) becomes

$$\dot{\mathbb{C}} = \dot{L} \, \left(\beta_K \, \frac{\mathbb{P}_V}{3K} + \beta_G \, \frac{\mathbb{P}_D}{2G} \right) \; . \tag{36}$$

Taking now $\dot{L}$ as the inelastic multiplier, $\dot{\lambda}$=$\dot{L}$, the following terms of the general theory in eqns (4b), (6d) and (7d) can be identified:

$$\mathcal{M} = 1 \; ; \qquad \mathbb{M} = \frac{\partial \mathbb{C}}{\partial L} \, \mathcal{M} = \beta_K \, \frac{\mathbb{P}_V}{3K} + \beta_G \, \frac{\mathbb{P}_D}{2G} \; ; \qquad \mathbf{m} = \mathbb{M} : \boldsymbol{\sigma} = \beta_K \, \boldsymbol{\epsilon}_V + \beta_G \, \boldsymbol{\epsilon}_D \; , \tag{37}$$

where $\boldsymbol{\epsilon}_V$=$\epsilon_V \, \mathbf{I}$ and ϵ_V=$\mathrm{tr}\,\boldsymbol{\epsilon}/3$ are the volumetric part and volumetric component of the strain tensor $\boldsymbol{\epsilon}$, while $\boldsymbol{\epsilon}_D$ is its deviatoric part.

The dissipation rate equation leads to the force conjugate to the logarithmic damage rate $\dot{L}$:

$$\dot{d} = \frac{1}{2} \, \boldsymbol{\sigma} : \dot{\mathbb{C}} : \boldsymbol{\sigma} = -\mathcal{Y} \, \dot{L} \; ; \qquad -\mathcal{Y} = \beta_K \, u_V + \beta_G \, u_D \; , \tag{38}$$

where u_V and u_D are the current volumetric and deviatoric components of the elastic energy u=u_V+u_D=$\boldsymbol{\sigma}_V$:$\boldsymbol{\epsilon}_V/2$+$\boldsymbol{\sigma}_D$:$\boldsymbol{\epsilon}_D/2$,

$$u_V = \frac{1}{2} \, \frac{\sigma_V^2}{K} = \frac{1}{2} \, 9 \, K \, \epsilon_V^2 \; ; \qquad u_D = \frac{1}{2} \, \frac{\boldsymbol{\sigma}_D : \boldsymbol{\sigma}_D}{2 \, G} = G \;\; \boldsymbol{\epsilon}_D : \boldsymbol{\epsilon}_D \; , \tag{39}$$

and σ_V, $\boldsymbol{\sigma}_V$ and $\boldsymbol{\sigma}_D$ are the volumetric component, volumetric part and deviatoric part of the stress tensor.

In order to obtain an *associated model*, the following loading function is considered,

$$F = \beta_K \, u_V + \beta_G \, u_D - r(L) = -\mathcal{Y} - r(L) \; , \tag{40}$$

whose gradients with respect to $-\mathcal{Y}$, $-\mathbb{Y}$ and $\boldsymbol{\sigma}$ at constant λ are equal to the corresponding rules (37), namely $\mathcal{N}$=$\mathcal{M}$=1, $\mathbb{N}$=$\mathbb{M}$ and $\mathbf{n}$=$\mathbf{m}$. From F, the hardening moduli H and $\bar{H}$=H+$\mathbf{n}$:$\mathbb{E}$:$\mathbf{m}$, eqn (9), can also be derived as follows:

$$H = \frac{\partial r}{\partial L} - \left(\beta_K^2 u_V + \beta_G^2 u_D \right) \; ; \qquad \bar{H} = \frac{\partial r}{\partial L} + \beta_K^2 \, u_V + \beta_G^2 \, u_D \; . \tag{41}$$

The inelastic multiplier $\dot{\lambda}$ and (symmetric) tangent operator $\mathbb{E}_{\mathrm{tan}}$ can be finally obtained from eqn (8) as:

$$\dot{\lambda} = \dot{L} = \frac{1}{\bar{H}} \; (\beta_K \boldsymbol{\sigma}_V + \beta_G \boldsymbol{\sigma}_D) : \dot{\boldsymbol{\epsilon}} \; ; \tag{42}$$

$$\mathbb{E}_{\tan} = \mathbb{E}_{\text{iso}} - \frac{1}{\bar{H}}\ (\beta_K \boldsymbol{\sigma}_V + \beta_G \boldsymbol{\sigma}_D) \otimes (\beta_K \boldsymbol{\sigma}_V + \beta_G \boldsymbol{\sigma}_D)\ , \tag{43}$$

where $\mathbb{E}_{\text{iso}}$ is the current secant stiffness in eqn (27a), with bulk and shear moduli (29).

The model described has only one function to be defined, that is the hardening function $r{=}r(D)$, plus the two constants β_K and β_G. Some particular cases of interest are for $\beta_K{=}\beta_G$, in which the traditional '$(1{-}D)$' scalar damage model is recovered, and for $\beta_K{=}0$, which results in pure deviatoric damage according to a 'von Mises' loading surface, while on the contrary for $\beta_G{=}0$ a pure volumetric damage would be obtained.

3.2.1 The path-parameter η

Actually, since the hypothesis $L_K{=}\beta_K\, L$, $L_G{=}\beta_G\, L$ corresponds to assume constrained straight paths in the plane of the logarithmic damage variables L_K and L_G (Fig. 1), a single path-parameter η with values between -1 and 1 can be introduced to express the constants β_K and β_G, which is sufficient to define uniquely the slope of the linear paths with respect to the bisector at 45°. Taking

$$\beta_K = 1 - \eta\ , \qquad \beta_G = 1 + \eta\ ; \qquad \eta = \frac{\beta_G - \beta_K}{2}\ , \tag{44}$$

we have (Fig. 1):

$$\operatorname{tg}\Theta = \frac{\beta_G}{\beta_K}\ ; \qquad \eta = \operatorname{tg}\theta = \operatorname{tg}\left(\Theta - \frac{\pi}{4}\right) = \frac{\beta_G - \beta_K}{\beta_G + \beta_K}\ . \tag{45}$$

Then, while $-1{\le}\eta{\le}1$, parameters β_K and β_G are linked by the constraint $\beta_K{+}\beta_G{=}2$ and take values between 0 and 2 (Fig. 2). The 'basic' formulation is therefore recovered for $\eta{=}0$ (or $\beta_K{=}\beta_G{=}1$), while pure deviatoric and pure volumetric damage are respectively obtained for $\eta{=}1$ (or $\beta_K{=}0$, $\beta_G{=}2$) and $\eta{=}{-1}$ (or $\beta_K{=}2$, $\beta_G{=}0$).

Figs. 1-2

Consistently with the hypotheses above, the variations of damage variables D_K and D_G and integrity variables $\bar{\phi}_K$ and $\bar{\phi}_G$, eqn (34), and logarithmic damage variables L_K and L_G, eqn (33), are linked to the evolution of single scalar variables D, $\bar{\phi}$ and L by the following power laws:

$$D_K = 1 - (1 - D)^{\beta_K} = 1 - (1 - D)^{1-\eta}\ , \qquad D_G = 1 - (1 - D)^{\beta_G} = 1 - (1 - D)^{1+\eta}\ ; \tag{46}$$

$$\bar{\phi}_K = \bar{\phi}^{\beta_K} = \bar{\phi}^{1-\eta}\ , \qquad \bar{\phi}_G = \bar{\phi}^{\beta_G} = \bar{\phi}^{1+\eta}\ ; \tag{47}$$

$$L_K = \beta_K\ L = (1 - \eta)\ L\ , \qquad L_G = \beta_G\ L = (1 + \eta)\ L\ , \tag{48}$$

where, analogously to eqns (33), (34), variables D, $\bar{\phi}$ and L are related by:

$$D = 1 - \mathrm{e}^{-L}\ , \quad \bar{\phi} = \mathrm{e}^{-L/2}\ ; \qquad L = \ln\frac{1}{1 - D} = -2\ln\bar{\phi}\ . \tag{49}$$

The main relations of the 'extended' isotropic model in terms of β_K and β_G, eqns (27), (37), (38c), (41) and (43), can then be rewritten in terms of the single constant parameter η, which better highlights the difference between contributions from volumetric and deviatoric components of the quantities involved. Also, it may be seen more clearly how, for η=0, the formulation collapses into the 'basic' isotropic model in Carol et al. (2001a):

$$\mathbb{E}_{\rm iso} = \bar{\phi}^{2(1-\eta)}\, 3\, K_0\, \mathbb{P}_V + \bar{\phi}^{2(1+\eta)}\, 2\, G_0\, \mathbb{P}_D \; ; \qquad \mathbb{C}_{\rm iso} = \frac{\phi^{2(1-\eta)}}{3\, K_0}\, \mathbb{P}_V + \frac{\phi^{2(1+\eta)}}{2\, G_0}\, \mathbb{P}_D \; ; \tag{50}$$

$$\mathbb{M} = (1-\eta)\,\frac{\mathbb{P}_V}{3K} + (1+\eta)\,\frac{\mathbb{P}_D}{2G} = \mathbb{C} + \eta\,\left(\frac{\mathbb{P}_D}{2G} - \frac{\mathbb{P}_V}{3K}\right) \; ; \tag{51}$$

$$\mathbf{m} = (1-\eta)\,\boldsymbol{\epsilon}_V + (1+\eta)\,\boldsymbol{\epsilon}_D = \boldsymbol{\epsilon} + \eta\,(\boldsymbol{\epsilon}_D - \boldsymbol{\epsilon}_V) \; ; \tag{52}$$

$$-\mathcal{Y} = (1-\eta)\, u_V + (1+\eta)\, u_D = u + \eta\,(u_D - u_V) \; ; \tag{53}$$

$$\bar{H} = \frac{\partial r}{\partial L} + (1-\eta)^2\, u_V + (1+\eta)^2\, u_D = \frac{\partial r}{\partial L} + (1+\eta^2)\, u + 2\,\eta\,(u_D - u_V) \; ; \tag{54}$$

$$\begin{aligned} \mathbb{E}_{\rm tan} &= \mathbb{E}_{\rm iso} - \frac{1}{\bar{H}}\,\Big((1-\eta)\,\boldsymbol{\sigma}_V + (1+\eta)\,\boldsymbol{\sigma}_D\Big) \otimes \Big((1-\eta)\,\boldsymbol{\sigma}_V + (1+\eta)\,\boldsymbol{\sigma}_D\Big) = \\ &= \mathbb{E}_{\rm iso} - \frac{1}{\bar{H}}\,\Big(\boldsymbol{\sigma} + \eta\,(\boldsymbol{\sigma}_D - \boldsymbol{\sigma}_V)\Big) \otimes \Big(\boldsymbol{\sigma} + \eta\,(\boldsymbol{\sigma}_D - \boldsymbol{\sigma}_V)\Big) \; . \end{aligned} \tag{55}$$

3.3 Secant isotropic parameters

Focusing now on the secant relations obtained from eqns (27)-(29), (46)-(49) and from classical formulas analogous to (16) and (18) with current parameters in place of initial ones, the functional dependencies of secant elastic parameters K, G, Λ, E and ν in the 'extended' isotropic formulation as a function of scalar damage variable D and constant parameter η are given by the following expressions:

$$\begin{aligned} \frac{K}{K_0} &= (1-D)^{1-\eta} \; ; \qquad\qquad \frac{G}{G_0} = (1-D)^{1+\eta} \; ; \\ \frac{\Lambda}{\Lambda_0} &= \frac{1+\nu_0}{3\,\nu_0}\,(1-D)^{1-\eta} - \frac{1-2\nu_0}{3\,\nu_0}\,(1-D)^{1+\eta} \; ; \\ \frac{E}{E_0} &= \frac{3\,(1-D)^2}{2\,(1+\nu_0)\;(1-D)^{1-\eta} + \;(1-2\nu_0)\,(1-D)^{1+\eta}} \; ; \\ \nu &= \frac{(1+\nu_0)\;(1-D)^{1-\eta} - \;(1-2\nu_0)\,(1-D)^{1+\eta}}{2\,(1+\nu_0)\;(1-D)^{1-\eta} + \;(1-2\nu_0)\,(1-D)^{1+\eta}} \; . \end{aligned} \tag{56}$$

To avoid undetermination for ν_0=0 (Λ_0=0), the normalization in eqn (56c) could be alternatively stated as e.g. with respect to E_0. Then eqn (56c) could be replaced by:

$$\frac{\Lambda}{E_0} = \frac{(1-D)^{1-\eta}}{3\,(1-2\nu_0)} - \frac{(1-D)^{1+\eta}}{3\,(1+\nu_0)} \; , \tag{57}$$

which includes as particular case the function $3\,\Lambda/E_0 = (1-D)^{1-\eta} - (1-D)^{1+\eta}$ for $\nu_0 = 0$ ($\Lambda_0 = 0$). The dependence on damage variable $\bar{\phi}$ and logarithmic damage variable L may be read by replacing in eqns (56), (57) the term $(1-D)$ with $\bar{\phi}^2$ and e^{-L}, respectively.

The corresponding variations of K-G, Λ, E and ν are respectively represented in Figs 3, 4, 5 and 6 as a function of both (a) scalar damage variable D and (b) logarithmic damage variable L, for different constant values of $-1 \leq \eta \leq 1$. The asymptotic dependencies on L favor the adoption of the newly-introduced logarithmic damage variable L increasing from 0 to ∞ as damage grows to the limit. Except for Fig. 3 (since not necessary), reporting the evolution of K and G, a reference ('academic') value of Poisson's ratio $\nu_0 = 1/8 = 0.125$ is taken in the drawings, which corresponds to assume $K_0 = G_0 = 3\,\Lambda_0 = 4/9\,E_0$. Similar variations have been also obtained for different 'engineering' values of parameter ν_0 ranging between 0 and 0.49. Taking for concrete the reference value $\nu_0 = 0.18$, secant parameters Λ, E and ν are also displayed in Fig. 7 as a function of L. In the upper plots of Figs 3-6, the case $\eta = 0$ renders the linear variations of K-G, Λ and E with D and the constant value $\nu \equiv \nu_0$ of the 'basic' isotropic formulation. The extreme cases of pure deviatoric ($\eta = 1$) and pure volumetric damage ($\eta = -1$) are represented as well, together with the parameters evolution for some intermediate positive and negative values of η. Variations of K and G in Fig. 3 are represented together since they correspond to each other within a change of sign in η, eqns (56a,b).

Figs. 3-7

Some comments on the functional dependencies of the secant parameters are in order. Variations are all monotonic except for Λ (Fig. 4): extrema may be recorded for both positive and negative values of η. Except for K and G (Fig. 3), the plots in Figs 4-6 also display flex points (change of curvature). The curves of the E evolution also overlap for the negative values of η (Fig. 5). This is the only case with all curves reaching the same (zero) limit value as damage grows to the maximum, for all η values. In contrast, for the other material parameters, a discontinuous behavior is recorded towards full damage: for K and G, Fig. 3, and Λ, Fig. 4, all plots converge to zero, as it is obtained for $\eta = 0$, except for the limit cases with $\eta = \pm 1$. In the latter cases, one of the two reference moduli K and G keeps constant, while the other one decreases linearly with D to zero. Correspondingly, $\Lambda = K - 2/3\,G$ does not approach zero, while reaches $\Lambda_{\mathrm{lim}^+}/\Lambda_0 = 3/(3 - 2\,G_0/K_0)$ (or $3\,\Lambda_{\mathrm{lim}^+} = E_0/(1 - 2\,\nu_0) = 3\,K_0$) for $\eta = 1$, and $\Lambda_{\mathrm{lim}^-}/\Lambda_0 = -2/(3\,K_0/G_0 - 2)$ (or $3\,\Lambda^{\mathrm{lim}} = -E_0/(1+\nu_0) = -2\,G_0$) for $\eta = -1$. For $K_0 = G_0$, $\Lambda_{\mathrm{lim}^+}/\Lambda_0 = 3$ and $\Lambda_{\mathrm{lim}^-}/\Lambda_0 = -2$, see Fig. 4. The same occurs for ν (Fig. 6), where $\nu_{\mathrm{lim}^+} = 0.5$, for $\eta = 1$, and $\nu_{\mathrm{lim}^-} = -1$, for $\eta = -1$, coincide with the limit values of ν inducing loss of positive definiteness of the current secant stiffness. Then, for 'von Mises' damage, an incompressible elastic stiffness is obtained in the limit ($K/G \to \infty$), while for volumetric damage a 'non-deviatoric' secant stiffness is recovered ($G/K \to \infty$). A further discontinuity of the limit value of ν at maximum damage is also recorded for $\eta = 0$ since ν is always constant, $\nu \equiv \nu_0$, while for the positive and negative values of η, ν approaches separately the same definiteness bounds ν_{lim^+} and ν_{lim^-} obtained respectively for $\eta = 1$ and $\eta = -1$.

Finally, notice also the important feature that, for the negative values of η, parameters Λ and ν become negative within certain ranges of the damage variables. For traditional engineering materials the fact that damage may lead to particular microstructural arrangements inducing macroscopically a negative Poisson's ratio seems quite unlikely to occur. In this sense mixed volumetric/deviatoric damage with prevailing volumetric component (negative η) should be disregarded in practical applications under concern here by restricting the actual range of η to the interval [0,1], shifting from the '$(1-D)$' damage model, to mixed volumetric/prevailing-deviatoric damage, to pure deviatoric 'von Mises' damage. Also, the fact that for prevailing deviatoric damage (positive η) ν tends to 0.5 as damage progresses to the limit, seems mainly appropriate for metallic materials in which the deviatoric behavior dominates the inelastic process. For other materials (e.g. concrete, geomaterials, etc.) one would expect that $\nu \rightarrow 0$ at increasing damage. This would be better achieved by alternative 'extended' formulations directly based on E and ν (see end of Section 2.3). This is currently the subject of further research.

A closing comment refers to the relation of this 'extended' isotropic damage formulation with some earlier secant holonomic isotropic models for concrete found in the literature (e.g. Kupfer and Gerstle, 1973; Darwin and Pecknold, 1977). At first sight those models might seem similar to the formulation presented here because of their 'secant' character and of the parameters considered for the evolution of the elastic properties, namely K and G. However, a closer inspection immediately reveals fundamental differences. Those earlier models were generated out of the need of simple explicit (total) isotropic relations that could be inserted in general finite element codes and used in practical applications for 'realistic' predictions at a reasonable computational cost (at that time). Basically, they consisted of the linear isotropic secant relation (27a) or (27b), in which K and G were calculated as explicit functions of the invariants of the prescribed strain. These functions were chosen in order to fit a set of experimental curves, usually monotonic plane stress tests for various fixed ratios of the in-plane principal stresses. However, no theoretical aspects of elastic degradation (e.g. dissipation, irreversibility, etc.) were taken into consideration. As the result, even if the verification examples would look satisfactory, a number of fundamental conditions could be violated, and the response of the model under different loading sequences (especially non-proportional or cyclic) would be unpredictable and, in some cases, would exhibit excessive or negative energy dissipation. In fact, in those models G generally appears as a monotonic descending function of the octhaedral shear strain γ, but in order to capture frictional dilatancy of concrete, K has to be a function of both ϵ_V and γ, exhibiting first an initial decrease but then a recovery up to values higher than K_0. In clear contrast to that, the 'extended' formulation proposed here is theoretically consistent (e.g. moduli K and G only decrease, dissipation is never negative, etc.). At the same time, however, it is recognized that it cannot by itself produce a realistic description of such a complex material behavior as the general multiaxial loading of concrete. Further additions will surely be needed to be incorporated into the model such as anisotropic degradation, irreversible (plastic) deformations and stiffness recovery due to microcrack closure. The first of these additional features is developed in Section 4.

3.4 'Extended' isotropic degradation in the context of CDM

The just described formulation of 'extended' isotropic degradation can also be rephrased in the context of CDM as briefly introduced in Section 2.3. As compared to eqn (19), the damage-effect tensors $\bar{\mathbb{A}}$ and $\mathbb{A}$, here labeled $\bar{\mathbb{A}}_{\text{iso}}$ and $\mathbb{A}_{\text{iso}}$, are generalized as follows:

$$\begin{aligned} \bar{\mathbb{A}}_{\text{iso}} &= \bar{\phi}_K\ \mathbb{P}_V + \bar{\phi}_G\ \mathbb{P}_D = (\bar{\phi}_K - \bar{\phi}_G)\ \mathbb{P}_V + \bar{\phi}_G\ \mathbb{I}_s\ ; \\ \mathbb{A}_{\text{iso}} &= \phi_K\ \mathbb{P}_V + \phi_G\ \mathbb{P}_D = (\phi_K - \phi_G)\ \mathbb{P}_V + \phi_G\ \mathbb{I}_s\ , \end{aligned} \tag{58}$$

where $\bar{\phi}_K$, $\bar{\phi}_G$ and their inverses ϕ_K, ϕ_G are linked to D_K, D_G and L_K, L_G by eqns (28), (33) and (34). The components of the damage-effect tensors may be conveniently expressed in terms of the 6×6 matrix representation of Walpole (1984) (in which the fourth-order symmetric identity $\mathbb{I}_s$ maps to a 6×6 identity matrix):

$$[\bar{\mathbb{A}}_{\text{iso}}] = \begin{bmatrix} \frac{\bar{\phi}_K+2\bar{\phi}_G}{3} & \frac{\bar{\phi}_K-\bar{\phi}_G}{3} & \frac{\bar{\phi}_K-\bar{\phi}_G}{3} & & & \\ \frac{\bar{\phi}_K-\bar{\phi}_G}{3} & \frac{\bar{\phi}_K+2\bar{\phi}_G}{3} & \frac{\bar{\phi}_K-\bar{\phi}_G}{3} & & & \\ \frac{\bar{\phi}_K-\bar{\phi}_G}{3} & \frac{\bar{\phi}_K-\bar{\phi}_G}{3} & \frac{\bar{\phi}_K+2\bar{\phi}_G}{3} & & & \\ & & & \bar{\phi}_G & & \\ & & & & \bar{\phi}_G & \\ & & & & & \bar{\phi}_G \end{bmatrix} ;\ [\mathbb{A}_{\text{iso}}] = \begin{bmatrix} \frac{\phi_K+2\phi_G}{3} & \frac{\phi_K-\phi_G}{3} & \frac{\phi_K-\phi_G}{3} & & & \\ \frac{\phi_K-\phi_G}{3} & \frac{\phi_K+2\phi_G}{3} & \frac{\phi_K-\phi_G}{3} & & & \\ \frac{\phi_K-\phi_G}{3} & \frac{\phi_K-\phi_G}{3} & \frac{\phi_K+2\phi_G}{3} & & & \\ & & & \phi_G & & \\ & & & & \phi_G & \\ & & & & & \phi_G \end{bmatrix} . \tag{59}$$

Instead of being postulated directly, the isotropic stiffness and compliance of the 'extended' isotropic model, eqn (27), may then be derived from eqn (14) by adopting damage-effect tensors (58), that is $\mathbb{E}_{\text{iso}}{=}\bar{\mathbb{A}}_{\text{iso}}{:}\mathbb{E}_0{:}\bar{\mathbb{A}}^{\text{T}}_{\text{iso}}$, $\mathbb{C}_{\text{iso}}{=}\mathbb{A}^{\text{T}}_{\text{iso}}{:}\mathbb{C}_0{:}\mathbb{A}_{\text{iso}}$. Also, from eqns (12), (13) and (58) the corresponding nominal/effective relations for strain and stress, and for their volumetric and deviatoric parts, are obtained as:

$$\begin{aligned} \boldsymbol{\epsilon}_{\text{eff}} &= \bar{\phi}_K\, \boldsymbol{\epsilon}_V + \bar{\phi}_G\, \boldsymbol{\epsilon}_D\ ; & \boldsymbol{\sigma} &= \bar{\phi}_K\, \boldsymbol{\sigma}_{\text{eff},V} + \bar{\phi}_G\, \boldsymbol{\sigma}_{\text{eff},D}\ ; \\ \boldsymbol{\sigma}_{\text{eff}} &= \phi_K\, \boldsymbol{\sigma}_V + \phi_G\, \boldsymbol{\sigma}_D\ ; & \boldsymbol{\epsilon} &= \phi_K\, \boldsymbol{\epsilon}_{\text{eff},V} + \phi_G\, \boldsymbol{\epsilon}_{\text{eff},D}\ ; \end{aligned} \tag{60}$$

$$\boldsymbol{\epsilon}_{\text{eff},V} = \bar{\phi}_K\, \boldsymbol{\epsilon}_V\ , \quad \boldsymbol{\epsilon}_{\text{eff},D} = \bar{\phi}_G\, \boldsymbol{\epsilon}_D\ ; \qquad \boldsymbol{\sigma}_{\text{eff},V} = \phi_K\, \boldsymbol{\sigma}_V\ , \quad \boldsymbol{\sigma}_{\text{eff},D} = \phi_G\, \boldsymbol{\sigma}_D\ . \tag{61}$$

From the latter equation it is apparent that integrity variables $\bar{\phi}_K$, $\bar{\phi}_G$ and their inverses ϕ_K, ϕ_G define separately the linear link between volumetric and deviatoric components of effective and nominal stresses and strains (compare to the 'basic' isotropic case, whereby simply $\boldsymbol{\epsilon}_{\text{eff},(1-D)}{=}\bar{\phi}\ \boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}_{\text{eff},(1-D)}{=}\phi\ \boldsymbol{\sigma}$, eqn (21)). Considering next the single-dissipative model based on constant η, the expressions above could just be rewritten by replacing in eqns (58)-(61) power law relations (47), and similar ones for the inverses $\phi_K{=}\phi^{1-\eta}$ and $\phi_G{=}\phi^{1+\eta}$.

4 'Extended' volumetric/deviatoric formulation of anisotropic damage

Similarly to what has been already commented at the beginning of Section 3 for the 'extended' isotropic formulation, the 'basic' anisotropic damage formulation briefly mentioned in Section 2.4 also displays some limitations since it leads to a restricted form of orthotropic material symmetry. This can be seen in various ways. The simplest is perhaps by considering the particular case in which second-order damage tensors $\bar{\boldsymbol{\phi}}$ and $\boldsymbol{\phi}$ entering the damage-effect tensors (20) and stiffness and compliance (24) take the isotropic forms $\bar{\boldsymbol{\phi}}=\bar{\phi}\,\mathbf{I}$, $\boldsymbol{\phi}=\phi\,\mathbf{I}$. Then, as it appears from eqns (19), (20) and (23), (24), the formulation collapses into the '$(1-D)$' scalar damage model, which is by itself a restricted form of isotropic damage. A second argument may follow from a simple count of the number of material parameters available to characterize anisotropy. Sitting on the principal axes of damage, the 'basic' formulation exhibits five parameters, namely two initial elastic constants plus three principal values of damage, while general orthotropic elasticity would need nine independent evolving material parameters.

Also, notice that a desirable feature of the 'extended' anisotropic formulation based on the volumetric/deviatoric decomposition should be that of encompassing both the 'extended' isotropic degradation model just described in Section 3, as well as the 'basic' anisotropic formulation mentioned in Section 2.4 (Carol et al., 2001a). The 'extended' anisotropic formulation developed next accomplishes both requirements and is based on two alternative product-type decompositions of the damage-effect tensor in isotropic and anisotropic parts. Secant relations are presented first, initially in a double-dissipative context and subsequently in the final single-dissipative setting. Next, the pseudo-logarithmic rate of damage is introduced as an extension of the same rate in the 'basic' model. The conjugate thermodynamic forces are derived and the evolution rules for the pseudo-logarithmic rate are identified so that, finally, all the different terms entering the tangent stiffness are developed.

4.1 Bi-dissipative anisotropic model (two damage variables): secant relations

Given the requirements above, a starting natural choice is to consider two independent second-order integrity tensors, one for the volumetric and the other for the deviatoric components of stiffness and compliance. These new tensors are called here $\bar{\boldsymbol{\phi}}_K$ and $\bar{\boldsymbol{\phi}}_G$, and their respective inverses $\boldsymbol{\phi}_K=\bar{\boldsymbol{\phi}}_K^{-1}$ and $\boldsymbol{\phi}_G=\bar{\boldsymbol{\phi}}_G^{-1}$. A similar approach has been previously put forward by He and Curnier (1995), which, based on micromechanical considerations, introduce two tensor damage variables entering separately the orientation distribution functions of elongation and bulk moduli.

Introducing these new variables and attempting a generalization of both damage-effect tensor $\bar{\mathbb{A}}_{\text{iso}}$, eqn (58a), and $\bar{\mathbb{A}}_{\text{bas}}$, eqn (20a), the following fourth-order damage-effect tensor could be assumed:

$$\bar{\mathbb{A}} = \left(\sqrt{\bar{\boldsymbol{\phi}}_K}\,\overline{\underline{\otimes}}\,\sqrt{\bar{\boldsymbol{\phi}}_K}\right) : \mathbb{P}_V + \left(\sqrt{\bar{\boldsymbol{\phi}}_G}\,\overline{\underline{\otimes}}\,\sqrt{\bar{\boldsymbol{\phi}}_G}\right) : \mathbb{P}_D\,. \tag{62}$$

Indeed, by taking $\bar{\boldsymbol{\phi}}_K=\bar{\phi}_K\,\mathbf{I}$, $\bar{\boldsymbol{\phi}}_G=\bar{\phi}_G\,\mathbf{I}$, this would yield $\bar{\mathbb{A}}_{\text{iso}}$, while $\bar{\boldsymbol{\phi}}_K=\bar{\boldsymbol{\phi}}_G=\bar{\boldsymbol{\phi}}$ would render $\bar{\mathbb{A}}_{\text{bas}}$. Then, the corresponding secant stiffness $\mathbb{E}=\bar{\mathbb{A}}:\mathbb{E}_0:\bar{\mathbb{A}}^{\mathrm{T}}$ would take the form

$$\mathbb{E} = 3K_0\,\left(\frac{1}{3}\,\bar{\boldsymbol{\phi}}_K\otimes\bar{\boldsymbol{\phi}}_K\right) + 2\,G_0\,\left(\bar{\boldsymbol{\phi}}_G\overline{\underline{\otimes}}\,\bar{\boldsymbol{\phi}}_G - \frac{1}{3}\,\bar{\boldsymbol{\phi}}_G\otimes\bar{\boldsymbol{\phi}}_G\right). \tag{63}$$

Here the fourth-order tensor terms within brackets may be interpreted as 'degraded' counterparts of the volumetric and deviatoric projection operators $\mathbb{P}_V$ and $\mathbb{P}_D$. With this general assumption, however, the dual expression of damage-effect tensor $\mathbb{A}$ and compliance $\mathbb{C}$ based on the inverses $\boldsymbol{\phi}_K$ and $\boldsymbol{\phi}_G$ would not correspond to the inverses of $\bar{\mathbb{A}}$ and $\mathbb{E}$ evaluated from eqns (62) and (63). In other words, the arising stiffness and compliance-based formulations would not be equivalent. But equivalence between strain and stress-based versions of the theory was one of the main convenient features of the 'basic' anisotropic formulation that it would be desirable to preserve also in the present context.

As a way to obtain this, it is assumed here that bulk and shear integrity tensors are proportional to each other through the following product-type decomposition assumption:

$$\bar{\boldsymbol{\phi}}_K = \bar{\phi}_K \, \bar{\boldsymbol{\psi}} \; ; \qquad \bar{\boldsymbol{\phi}}_G = \bar{\phi}_G \, \bar{\boldsymbol{\psi}} \; , \tag{64}$$

where, similarly to the isotropic case, $\bar{\phi}_K$ and $\bar{\phi}_G$ are scalar integrity variables decreasing from 1 to 0, and $\bar{\boldsymbol{\psi}}$ is a positive-definite isochoric tensor (namely a tensor with unit determinant: $\det \bar{\boldsymbol{\psi}}{=}1$). Notice that, with product-type decompositions (64), $\bar{\phi}_K$ and $\bar{\phi}_G$ represent respectively the 1/3 powers of the determinant of $\bar{\boldsymbol{\phi}}_K$ and $\bar{\boldsymbol{\phi}}_G$, namely $\bar{\phi}_K{=}(\det \bar{\boldsymbol{\phi}}_K)^{1/3}$ and $\bar{\phi}_G{=}(\det \bar{\boldsymbol{\phi}}_G)^{1/3}$. Initially, $\bar{\boldsymbol{\psi}}{=}\mathbf{I}$, and its evolution must be subject to the necessary constraints, so that, in terms of the principal values of $\bar{\boldsymbol{\psi}}$, the products $\bar{\phi}_K \bar{\psi}_I$ and $\bar{\phi}_G \bar{\psi}_I$ (I=1,2,3) keep decreasing from 1 to zero. In product-type scheme (64), the roles of the damage variables are clearly separated: the scalars $\bar{\phi}_K$ and $\bar{\phi}_G$ are responsible for the isotropic part of the degradation, while $\bar{\boldsymbol{\psi}}$ accounts for the anisotropic part. In the particular case that $\bar{\boldsymbol{\psi}}{=}\mathbf{I}$, the model collapses into the 'extended' isotropic formulation of Section 3.1. If, on the other hand, $\bar{\boldsymbol{\phi}}_K{=}\bar{\boldsymbol{\phi}}_G{=}\bar{\boldsymbol{\phi}}$, the 'basic' anisotropic formulation quoted in Section 2.4 is recovered.

With assumption (64), the damage-effect tensor (62) takes the following new form

$$\bar{\mathbb{A}} = \frac{\bar{\phi}_K - \bar{\phi}_G}{3} \, \bar{\boldsymbol{\psi}} \otimes \mathbf{I} + \bar{\phi}_G \, \sqrt{\bar{\boldsymbol{\psi}}} \, \underline{\overline{\otimes}} \, \sqrt{\bar{\boldsymbol{\psi}}} \; , \tag{65}$$

where the absence of major symmetry should be noted. In the principal axes of damage, this tensor may be represented by the following (non-symmetric) 6×6 matrix:

$$[\bar{\mathbb{A}}] = \begin{bmatrix} \frac{\bar{\phi}_K+2\bar{\phi}_G}{3}\,\bar{\psi}_1 & \frac{\bar{\phi}_K-\bar{\phi}_G}{3}\,\bar{\psi}_1 & \frac{\bar{\phi}_K-\bar{\phi}_G}{3}\,\bar{\psi}_1 & & & \\ \frac{\bar{\phi}_K-\bar{\phi}_G}{3}\,\bar{\psi}_2 & \frac{\bar{\phi}_K+2\bar{\phi}_G}{3}\,\bar{\psi}_2 & \frac{\bar{\phi}_K-\bar{\phi}_G}{3}\,\bar{\psi}_2 & & & \\ \frac{\bar{\phi}_K-\bar{\phi}_G}{3}\,\bar{\psi}_3 & \frac{\bar{\phi}_K-\bar{\phi}_G}{3}\,\bar{\psi}_3 & \frac{\bar{\phi}_K+2\bar{\phi}_G}{3}\,\bar{\psi}_3 & & & \\ & & & \bar{\phi}_G\sqrt{\bar{\psi}_1\bar{\psi}_2} & & \\ & & & & \bar{\phi}_G\sqrt{\bar{\psi}_2\bar{\psi}_3} & \\ & & & & & \bar{\phi}_G\sqrt{\bar{\psi}_3\bar{\psi}_1} \end{bmatrix} . \tag{66}$$

The secant stiffness can either be obtained as $\mathbb{E}{=}\bar{\mathbb{A}}{:}\mathbb{E}_0{:}\bar{\mathbb{A}}^{\mathrm{T}}$, with $\bar{\mathbb{A}}$ from eqn (65), or directly from eqns (63), (64). This leads to

$$\mathbb{E} = 3\,K \, \left(\frac{1}{3}\,\bar{\boldsymbol{\psi}} \otimes \bar{\boldsymbol{\psi}}\right) + 2\,G \, \left(\bar{\boldsymbol{\psi}} \, \underline{\overline{\otimes}} \, \bar{\boldsymbol{\psi}} - \frac{1}{3}\,\bar{\boldsymbol{\psi}} \otimes \bar{\boldsymbol{\psi}}\right) = \Lambda \, \bar{\boldsymbol{\psi}} \otimes \bar{\boldsymbol{\psi}} + 2\,G \, \bar{\boldsymbol{\psi}} \, \underline{\overline{\otimes}} \, \bar{\boldsymbol{\psi}} \; . \tag{67}$$

These expressions are similar in form to both relations of $\mathbb{E}$ in the 'extended' isotropic model, eqn (27a), and in the 'basic' anisotropic damage model, eqn (24a). As in (27a), the secant parameters $K{=}\bar{\phi}_K^2\,K_0$ and $G{=}\bar{\phi}_G^2\,G_0$ are involved in eqn (67a), while the projection operators $\mathbb{P}_V$, $\mathbb{P}_D$ are replaced by 'degraded' counterparts expressed in terms of $\boldsymbol{\psi}$. Compared instead to eqn (24a), eqn (67b) displays integrity tensor $\bar{\boldsymbol{\phi}}{=}\bar{\phi}\,\bar{\boldsymbol{\psi}}$ replaced by its isochoric part $\bar{\boldsymbol{\psi}}$ and initial isotropic elastic moduli Λ_0 and G_0 substituted by their secant counterparts $\Lambda{=}K{-}2/3\,G{=}\bar{\phi}_K^2\,K_0{-}2/3\,\bar{\phi}_G^2\,G_0$ and $G{=}\bar{\phi}_G^2\,G_0$.

Through the first analogy remarked above, the damage-effect tensor (65) can also be expressed in terms of a product-type decomposition in isotropic and anisotropic components $\bar{\mathbb{A}}_{\text{iso}}$ and $\bar{\mathbb{A}}_{\text{ani}}$. In this way, the secant stiffness may be alternatively reexpressed in terms of $\mathbb{E}_{\text{iso}}$ and $\bar{\mathbb{A}}_{\text{ani}}$, instead of $\mathbb{E}_0$ and $\bar{\mathbb{A}}$:

$$\bar{\mathbb{A}} = \bar{\mathbb{A}}_{\text{ani}} : \bar{\mathbb{A}}_{\text{iso}}\ , \quad \bar{\mathbb{A}}_{\text{ani}} = \sqrt{\bar{\boldsymbol{\psi}}}\,\underline{\overline{\otimes}}\,\sqrt{\bar{\boldsymbol{\psi}}}\ ; \qquad \mathbb{E} = \bar{\mathbb{A}} : \mathbb{E}_0 : \bar{\mathbb{A}}^{\text{T}} = \bar{\mathbb{A}}_{\text{ani}} : \mathbb{E}_{\text{iso}} : \bar{\mathbb{A}}_{\text{ani}}^{\text{T}}\ , \tag{68}$$

where $\bar{\mathbb{A}}_{\text{iso}}$ is the damage-effect tensor of the 'extended' isotropic formulation, eqn (58a), and $\mathbb{E}_{\text{iso}}{=}\bar{\mathbb{A}}_{\text{iso}}{:}\mathbb{E}_0{:}\bar{\mathbb{A}}_{\text{iso}}^{\text{T}}$ is the corresponding degraded isotropic stiffness, eqn (27a).

Despite being non-symmetric, due to the underlying product-type decomposition in isotropic and anisotropic parts, $\bar{\mathbb{A}}{=}\bar{\mathbb{A}}_{\text{ani}}{:}\bar{\mathbb{A}}_{\text{iso}}$, the damage-effect tensor (65) can be quickly inverted and the inverse $\mathbb{A}{=}\bar{\mathbb{A}}^{-1}$ takes the following dual form in terms of the inverse integrity isochoric tensor $\boldsymbol{\psi}$, inverse of $\bar{\boldsymbol{\psi}}$, and isotropic damage variables ϕ_K and ϕ_G, inverses of $\bar{\phi}_K$ and $\bar{\phi}_G$:

$$\mathbb{A} = \frac{\phi_K - \phi_G}{3}\,\mathbf{I}\otimes\boldsymbol{\psi} + \phi_G\,\sqrt{\boldsymbol{\psi}}\,\underline{\overline{\otimes}}\,\sqrt{\boldsymbol{\psi}}\ , \tag{69}$$

with corresponding (non-symmetric) matrix representation

$$[\mathbb{A}] = \begin{bmatrix} \frac{\phi_K+2\phi_G}{3}\,\psi_1 & \frac{\phi_K-\phi_G}{3}\,\psi_2 & \frac{\phi_K-\phi_G}{3}\,\psi_3 & & & \\ \frac{\phi_K-\phi_G}{3}\,\psi_1 & \frac{\phi_K+2\phi_G}{3}\,\psi_2 & \frac{\phi_K-\phi_G}{3}\,\psi_3 & & & \\ \frac{\phi_K-\phi_G}{3}\,\psi_1 & \frac{\phi_K-\phi_G}{3}\,\psi_2 & \frac{\phi_K+2\phi_G}{3}\,\psi_3 & & & \\ & & & \phi_G\sqrt{\psi_1\psi_2} & & \\ & & & & \phi_G\sqrt{\psi_2\psi_3} & \\ & & & & & \phi_G\sqrt{\psi_3\psi_1} \end{bmatrix}. \tag{70}$$

Since the inverse $\mathbb{A}$ of $\bar{\mathbb{A}}$ takes a dual structure (it is equal to the transpose of the tensor obtained by replacing barred with unbarred quantities), the previous stiffness-based derivation may be equivalently stated in terms of compliance, which involves inverse integrity damage variables decomposed in product form, $\boldsymbol{\phi}_K{=}\phi_K\,\boldsymbol{\psi}$, $\boldsymbol{\phi}_G{=}\phi_G\,\boldsymbol{\psi}$. A dual product-type decomposition of $\mathbb{A}$ in isotropic and anisotropic components $\mathbb{A}_{\text{iso}}$, $\mathbb{A}_{\text{ani}}$ and an alternative representation of the secant compliance in terms of $\mathbb{C}_{\text{iso}}$ and $\mathbb{A}_{\text{ani}}$ instead of $\mathbb{C}_0$ and $\mathbb{A}$ hold as well:

$$\mathbb{A} = \mathbb{A}_{\text{iso}} : \mathbb{A}_{\text{ani}}\ , \quad \mathbb{A}_{\text{ani}} = \sqrt{\boldsymbol{\psi}}\,\underline{\overline{\otimes}}\,\sqrt{\boldsymbol{\psi}}\ ; \qquad \mathbb{C} = \mathbb{A}^{\text{T}} : \mathbb{C}_0 : \mathbb{A} = \mathbb{A}_{\text{ani}}^{\text{T}} : \mathbb{C}_{\text{iso}} : \mathbb{A}_{\text{ani}}\ , \tag{71}$$

where $\mathbb{A}_{\text{iso}}$ is defined in eqn (58b) and $\mathbb{C}_{\text{iso}}$ in eqn (27b). This leads to compliance relations dual to the stiffness expressions (67):

$$\mathbb{C} = \frac{1}{3\,K}\,\left(\frac{1}{3}\boldsymbol{\psi}\otimes\boldsymbol{\psi}\right) + \frac{1}{2\,G}\,\left(\boldsymbol{\psi}\,\underline{\overline{\otimes}}\,\boldsymbol{\psi} - \frac{1}{3}\,\boldsymbol{\psi}\otimes\boldsymbol{\psi}\right) = -\frac{\nu}{E}\,\boldsymbol{\psi}\otimes\boldsymbol{\psi} + \frac{1+\nu}{E}\,\boldsymbol{\psi}\,\underline{\overline{\otimes}}\,\boldsymbol{\psi}\ . \tag{72}$$

Once again these relations bear strict analogy to both expressions (58b) for the 'extended' isotropic model and (24b) for the 'basic' anisotropic model, where K, G and E, ν are secant parameters related by classical elasticity formulas similar to (16c,d) and (18a).

By forming the inner products $\bar{\mathbb{A}}{:}\mathbb{A}{=}\mathbb{A}{:}\bar{\mathbb{A}}{=}\mathbb{I}_s$ and $\mathbb{E}{:}\mathbb{C}{=}\mathbb{C}{:}\mathbb{E}{=}\mathbb{I}_s$ it can be directly checked that the expressions of $\mathbb{A}$ and $\mathbb{C}$, eqns (69), (72), are indeed the inverses of their stiffness-based counterparts, $\bar{\mathbb{A}}$ and $\mathbb{E}$, eqns (65), (67). This confirms that the two derivations are dual versions of the same formulation. It is a convenient property that the final damage-effect tensors $\bar{\mathbb{A}}$ and $\mathbb{A}$ are decomposed in the product of their isotropic and anisotropic parts $\bar{\mathbb{A}}_{\text{iso}}$, $\bar{\mathbb{A}}_{\text{ani}}$ and $\mathbb{A}_{\text{iso}}$, $\mathbb{A}_{\text{ani}}$: in this way, both types of damage effects are clearly separated in the formulation. These convenient properties are due to the postulated product-type decompositions of both second-order tensors $\bar{\boldsymbol{\phi}}_K$, $\bar{\boldsymbol{\phi}}_G$ and $\boldsymbol{\phi}_K$, $\boldsymbol{\phi}_G$, and fourth-order tensors $\bar{\mathbb{A}}$, $\mathbb{A}$. Note also the lack of major symmetry of tensors $\bar{\mathbb{A}}$ and $\mathbb{A}$, and the fact that they nevertheless are inverse of each other and possess dual structures. Tensors $\bar{\mathbb{A}}$ and $\mathbb{A}$ are homogeneous of degree one in $\bar{\boldsymbol{\phi}}_K$, $\bar{\boldsymbol{\phi}}_G$ and $\boldsymbol{\phi}_K$, $\boldsymbol{\phi}_G$. The anisotropic parts $\bar{\mathbb{A}}_{\text{ani}}$ and $\mathbb{A}_{\text{ani}}$ are similar to the damage-effect tensors of the 'basic' formulation, $\bar{\mathbb{A}}_{\text{bas}}$, $\mathbb{A}_{\text{bas}}$, eqn (20), with $\bar{\boldsymbol{\phi}}{=}\bar{\phi}\,\bar{\boldsymbol{\psi}}$ replaced by $\bar{\boldsymbol{\psi}}$, and $\boldsymbol{\phi}{=}\phi\,\boldsymbol{\psi}$ replaced by $\boldsymbol{\psi}$, that is

$$\bar{\mathbb{A}}_{\text{ani}} = (\phi\,\mathbb{I}_s) : \bar{\mathbb{A}}_{\text{bas}} = \phi\,\bar{\mathbb{A}}_{\text{bas}}\ , \qquad \mathbb{A}_{\text{ani}} = (\bar{\phi}\,\mathbb{I}_s) : \mathbb{A}_{\text{bas}} = \bar{\phi}\,\mathbb{A}_{\text{bas}}\ . \tag{73}$$

Substitution of $\bar{\mathbb{A}}$ and $\mathbb{A}$ into eqns (12), (13) leads to the following nominal/effective relations:

$$\begin{aligned}
\boldsymbol{\epsilon}_{\text{eff}} &= \frac{\bar{\phi}_K-\bar{\phi}_G}{3}\,(\bar{\boldsymbol{\psi}}:\boldsymbol{\epsilon})\,\mathbf{I} + \bar{\phi}_G\,\sqrt{\bar{\boldsymbol{\psi}}}\cdot\boldsymbol{\epsilon}\cdot\sqrt{\bar{\boldsymbol{\psi}}}\ ; \quad \boldsymbol{\sigma} = \frac{\bar{\phi}_K-\bar{\phi}_G}{3}\,(\mathrm{tr}\,\boldsymbol{\sigma}_{\text{eff}})\,\bar{\boldsymbol{\psi}} + \bar{\phi}_G\,\sqrt{\bar{\boldsymbol{\psi}}}\cdot\boldsymbol{\sigma}_{\text{eff}}\cdot\sqrt{\bar{\boldsymbol{\psi}}}\ ;\\
\boldsymbol{\sigma}_{\text{eff}} &= \frac{\phi_K-\phi_G}{3}\,(\boldsymbol{\psi}:\boldsymbol{\sigma})\,\mathbf{I} + \phi_G\,\sqrt{\boldsymbol{\psi}}\cdot\boldsymbol{\sigma}\cdot\sqrt{\boldsymbol{\psi}}\ ; \quad \boldsymbol{\epsilon} = \frac{\phi_K-\phi_G}{3}\,(\mathrm{tr}\,\boldsymbol{\epsilon}_{\text{eff}})\,\boldsymbol{\psi} + \phi_G\,\sqrt{\boldsymbol{\psi}}\cdot\boldsymbol{\epsilon}_{\text{eff}}\cdot\sqrt{\boldsymbol{\psi}}\ .
\end{aligned} \tag{74}$$

To rewrite relations (74a,c) in much compact form, let us introduce the following tensor quantities (also needed later in the paper) determined from nominal to effective relations (12a), (13a) through the damage-effect tensors $\bar{\mathbb{A}}_{\text{ani}}$ and $\mathbb{A}_{\text{ani}}$,

$$\boldsymbol{\epsilon}_{\text{ani}} = \bar{\mathbb{A}}^{\mathrm{T}}_{\text{ani}} : \boldsymbol{\epsilon} = \sqrt{\bar{\boldsymbol{\psi}}}\cdot\boldsymbol{\epsilon}\cdot\sqrt{\bar{\boldsymbol{\psi}}}\ ; \qquad \boldsymbol{\sigma}_{\text{ani}} = \mathbb{A}_{\text{ani}} : \boldsymbol{\sigma} = \sqrt{\boldsymbol{\psi}}\cdot\boldsymbol{\sigma}\cdot\sqrt{\boldsymbol{\psi}}\ . \tag{75}$$

They represent stress and strain acting on an intermediate configuration between nominal and effective ones (see Fig. 8 later shown). Notice that energy equivalence holds also in terms of these newly-introduced quantities, namely $\boldsymbol{\sigma}{:}\boldsymbol{\epsilon}/2{=}\boldsymbol{\sigma}_{\text{eff}}{:}\boldsymbol{\epsilon}_{\text{eff}}/2{=}\boldsymbol{\sigma}_{\text{ani}}{:}\boldsymbol{\epsilon}_{\text{ani}}/2$. Also, these stress/strain quantities are related by fictitious direct and inverse isotropic secant laws

$$\boldsymbol{\sigma}_{\text{ani}} = \mathbb{E}_{\text{iso}} : \boldsymbol{\epsilon}_{\text{ani}}\ , \quad \boldsymbol{\epsilon}_{\text{ani}} = \mathbb{C}_{\text{iso}} : \boldsymbol{\sigma}_{\text{ani}}\ ; \qquad \mathbb{E}_{\text{iso}} = \bar{\mathbb{A}}_{\text{iso}} : \mathbb{E}_0 : \bar{\mathbb{A}}^{\mathrm{T}}_{\text{iso}}\ , \quad \mathbb{C}_{\text{iso}} = \mathbb{A}^{\mathrm{T}}_{\text{iso}} : \mathbb{C}_0 : \mathbb{A}_{\text{iso}}\ , \tag{76}$$

where isotropic moduli $\mathbb{E}_{\text{iso}}$, $\mathbb{C}_{\text{iso}}$ are those of eqn (27).

In terms of the intermediate quantities above, the effective strain and stress relations (74a,c) appear to be anisotropic 'extended' versions of both isotropic 'extended' relations (60a,c) and 'basic' relations (22). Indeed, the volumetric and deviatoric components of effective strain and stress become:

$$\boldsymbol{\epsilon}_{\text{eff},V} = \bar{\phi}_K\,\boldsymbol{\epsilon}_{\text{ani},V}\ , \quad \boldsymbol{\epsilon}_{\text{eff},D} = \bar{\phi}_G\,\boldsymbol{\epsilon}_{\text{ani},D}\ ; \qquad \boldsymbol{\sigma}_{\text{eff},V} = \phi_K\,\boldsymbol{\sigma}_{\text{ani},V}\ , \quad \boldsymbol{\sigma}_{\text{eff},D} = \phi_G\,\boldsymbol{\sigma}_{\text{ani},D}\ . \tag{77}$$

Then, the nominal to effective relations for strain and stress can be written in terms of the following volumetric/deviatoric sum-type decompositions of intermediate quantities similar to the effective ones of the 'basic' formulation in eqn (22) (with a change from $\bar{\mathbb{A}}_{\text{bas}}$, $\mathbb{A}_{\text{bas}}$ to $\bar{\mathbb{A}}_{\text{ani}}$, $\mathbb{A}_{\text{ani}}$, eqn (73)):

$$\boldsymbol{\epsilon}_{\text{eff}} = \bar{\phi}_K \, \boldsymbol{\epsilon}_{\text{ani},V} + \bar{\phi}_G \, \boldsymbol{\epsilon}_{\text{ani},D} \; ; \qquad \boldsymbol{\sigma}_{\text{eff}} = \phi_K \, \boldsymbol{\sigma}_{\text{ani},V} + \phi_G \, \boldsymbol{\sigma}_{\text{ani},D} \; . \tag{78}$$

Notice also the close similarity between eqns (77), (78) and the corresponding relations (61), (60a,c) of the 'extended' isotropic model. Due to the non-symmetry of $\bar{\mathbb{A}}$ and $\mathbb{A}$, the inverse effective to nominal relations (74b,d) look instead slightly different and do not embed a similar separation of effects between volumetric and deviatoric components.

4.2 Single-dissipative anisotropic model (one damage variable): secant relations

Similarly to Section 3.1, eqn (33), logarithmic damage variables L_K and L_G are introduced, alternative to $\bar{\phi}_K$ and $\bar{\phi}_G$, namely $L_K{=}-2 \ln \bar{\phi}_K$, $L_G{=}-2 \ln \bar{\phi}_G$. Then, a single-dissipative model can be sequentially formulated when the two logarithmic damage variables are taken proportional to a single logarithmic damage variable L and expressed in terms of constant η, $L_K{=}(1{-}\eta)\, L$, $L_G{=}(1{+}\eta)\, L$, eqn (48). Next, the variations of $\bar{\phi}_K$ and $\bar{\phi}_G$ are given as in eqn (47) by power laws $\bar{\phi}_K{=}\bar{\phi}^{1-\eta}$, $\bar{\phi}_G{=}\bar{\phi}^{1+\eta}$, where $\bar{\phi}$ is related to L by $L{=}-2 \ln \bar{\phi}$, eqn (49d). From eqns (67a), (72a), stiffness and compliance of the anisotropic 'extended' model can then be reexpressed as anisotropic enhanced forms of isotropic relations (50):

$$\begin{aligned} \mathbb{E} &= \bar{\phi}^{2(1-\eta)} \, 3 K_0 \left(\frac{1}{3} \, \bar{\boldsymbol{\psi}} \otimes \bar{\boldsymbol{\psi}} \right) + \bar{\phi}^{2(1+\eta)} \, 2 G_0 \left(\bar{\boldsymbol{\psi}} \,\underline{\overline{\otimes}}\, \bar{\boldsymbol{\psi}} - \frac{1}{3} \, \bar{\boldsymbol{\psi}} \otimes \bar{\boldsymbol{\psi}} \right) \; ; \\ \mathbb{C} &= \frac{\phi^{2(1-\eta)}}{3 K_0} \left(\frac{1}{3} \, \boldsymbol{\psi} \otimes \boldsymbol{\psi} \right) + \frac{\phi^{2(1+\eta)}}{2 G_0} \left(\boldsymbol{\psi} \,\underline{\overline{\otimes}}\, \boldsymbol{\psi} - \frac{1}{3} \, \boldsymbol{\psi} \otimes \boldsymbol{\psi} \right) \; . \end{aligned} \tag{79}$$

This, once more, shows the connection between isotropic and anisotropic versions of the 'extended' model.

Alternatively, the enhancement of the 'extended' model with respect to the 'basic' one may be privileged. Then, by factoring out in (79) the common factors $\bar{\phi}^2$ and ϕ^2 and reintroducing them into the single integrity tensor $\bar{\boldsymbol{\phi}}{=}\bar{\phi}\, \bar{\boldsymbol{\psi}}$ and its inverse $\boldsymbol{\phi}{=}\phi\, \boldsymbol{\psi}$, the following convenient *final forms of stiffness and compliance tensors for the 'extended' anisotropic formulation* can be derived:

$$\begin{aligned} \mathbb{E} &= 3 \hat{K} \left(\frac{1}{3} \, \bar{\boldsymbol{\phi}} \otimes \bar{\boldsymbol{\phi}} \right) + 2 \hat{G} \left(\bar{\boldsymbol{\phi}} \,\underline{\overline{\otimes}}\, \bar{\boldsymbol{\phi}} - \frac{1}{3} \, \bar{\boldsymbol{\phi}} \otimes \bar{\boldsymbol{\phi}} \right) = \hat{\Lambda} \, \bar{\boldsymbol{\phi}} \otimes \bar{\boldsymbol{\phi}} + 2 \hat{G} \, \bar{\boldsymbol{\phi}} \,\underline{\overline{\otimes}}\, \bar{\boldsymbol{\phi}} \; ; \\ \mathbb{C} &= \frac{1}{3 \hat{K}} \left(\frac{1}{3} \, \boldsymbol{\phi} \otimes \boldsymbol{\phi} \right) + \frac{1}{2 \hat{G}} \left(\boldsymbol{\phi} \,\underline{\overline{\otimes}}\, \boldsymbol{\phi} - \frac{1}{3} \, \boldsymbol{\phi} \otimes \boldsymbol{\phi} \right) = -\frac{\hat{\nu}}{\hat{E}} \, \boldsymbol{\phi} \otimes \boldsymbol{\phi} + \frac{1 + \hat{\nu}}{\hat{E}} \, \boldsymbol{\phi} \,\underline{\overline{\otimes}}\, \boldsymbol{\phi} \; , \end{aligned} \tag{80}$$

where, similarly to eqn (24), $\bar{\boldsymbol{\phi}}$ and $\boldsymbol{\phi}$ bring in the anisotropic structure and

$$\hat{K} = \phi^2 \, K = \bar{\phi}^{-2\eta} \, K_0 = \phi^{2\eta} \, K_0 \; , \qquad \hat{G} = \phi^2 \, G = \bar{\phi}^{2\eta} \, G_0 = \phi^{-2\eta} \, G_0 \tag{81}$$

are enhanced secant bulk and shear moduli different than real current isotropic moduli $K=\bar{\phi}^{2(1-\eta)}\,K_0$, $G=\bar{\phi}^{2(1+\eta)}\,G_0$, eqns (29), (47). Secant material parameters

$$\hat{\Lambda} = \hat{K} - \frac{2}{3}\,\hat{G}\,, \qquad \hat{E} = \frac{9\hat{K}\hat{G}}{3\hat{K}+\hat{G}}\,, \qquad \hat{\nu} = \frac{3\hat{K}-2\hat{G}}{2(3\hat{K}+\hat{G})} \tag{82}$$

are also degraded isotropic material parameters corresponding to $\hat{K}$ and $\hat{G}$. Stiffness and compliance are then always expressed in Valanis-type form, but with degraded parameters with hats in place of the undamaged ones (compare eqn (80b,d) to expressions (24) of the 'basic' formulation).

This particular form of orthotropy is characterized by 6 parameters, that is two undamaged isotropic elastic constants, the constant path-parameter η and the three evolving principal values of $\bar{\boldsymbol{\phi}}$ or $\boldsymbol{\phi}$. In the principal axes of damage, the matrix representation of the compliance tensor $\mathbb{C}$ in terms of $\hat{E}$ and $\hat{\nu}$, eqn (80d), may be compared to the 9-parameter matrix representation of an hyperelastic orthotropic compliance $\mathbb{C}_{\rm orth}$. Using the 6×6 matrix representation introduced and discussed by Walpole (1984) in his eqn (45) (in which stress and strain tensors map to six-component vectors embedding the same $\sqrt{2}$ factor in the shear components), the 9 orthotropic engineering coefficients embedded in eqn (80d) may be expressed as follows:

$$[\mathbb{C}_{\rm orth}] = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_2} & -\frac{\nu_{13}}{E_3} & & & \\ -\frac{\nu_{21}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{23}}{E_3} & & & \\ -\frac{\nu_{31}}{E_1} & -\frac{\nu_{32}}{E_2} & \frac{1}{E_3} & & & \\ & & & \frac{1}{2G_{12}} & & \\ & & & & \frac{1}{2G_{23}} & \\ & & & & & \frac{1}{2G_{31}} \end{bmatrix} ; \quad \begin{cases} E_I = \bar{\phi}_I^2\,\hat{E}\,, & I=1,2,3\,; \\ G_{IJ} = \bar{\phi}_I\bar{\phi}_J\,\hat{G}\,, & I,J=1,2\,;\ 2,3\,;\ 3,1\,; \\ \nu_{IJ} = \dfrac{\bar{\phi}_J}{\bar{\phi}_I}\,\hat{\nu}\,, & J\neq I=1,2,3\,. \end{cases} \tag{83}$$

Dependence from η in eqn (83b-d) is hidden in $\hat{E}$, $\hat{G}$, $\hat{\nu}$, according to eqns (81), (82b,c). Note that, by setting η=0 in (81), $\hat{K}=K_0$ and $\hat{G}=G_0$, so that $\hat{E}=E_0$, $\hat{\nu}=\nu_0$ and the material coefficients of the 5-parameter 'basic' formulation are obtained. On the other hand, for $0<\eta<1$, damage progresses faster for the shear modulus than for the bulk modulus, and the opposite occurs for $-1<\eta<0$. In the limit cases η=1 and η=−1, the model only degrades, respectively, the shear and bulk moduli entering stiffness and compliance (79). Also, when η=0 and $\boldsymbol{\phi}$ becomes spherical, $\boldsymbol{\phi}=\phi\,\mathbf{I}$, it is apparent that the 'extended' model collapses into the 'basic' isotropic degradation (namely the classical '(1–D)' scalar damage model).

As a final comment on the secant relations, notice that, although presenting apparently a similar power structure, the present secant relations are different than those of Valanis-type provided in the fabric elasticity model by Zysset and Curnier (1995). In that case an arbitrary power of the fabric tensor is introduced in order to allow a further degree of freedom available e.g. for calibration purposes, while keeping the homogeneity property of the elastic moduli. In our damage formulation this would correspond to a 'basic' formulation in which the integrity tensor, or its inverse, were raised to arbitrary powers. Such option is certainly possible since it is mainly a matter of choice of the basic underlying damage variable. However, this does not bring a change in the structure of the secant stiffness and compliance, which are always of the Valanis-type. So, it can be concluded that Zysset and Curnier (1995) fabric model and our 'basic' secant damage model correspond to each other (in the resulting stiffness and compliance relations) at any given state. On the other hand, the present 'extended' model, as it is apparent from eqn (80), though still resulting in a Valanis-type structure, is different from that of Zysset and Curnier (1995) (and different from the 'basic' one in Carol et al., 2001a), since the additional degree of freedom provided by parameter η allows to assign different weights to isotropic volumetric and deviatoric damage components as discussed above. Also, the two modular product forms of damage and damage-effect tensors reflect themselves in a clear separation of effects between isotropic/anisotropic and 'basic'/'extended' damage components in the final engineering moduli. This latter interpretation of the 'extended' model is further pursued in the next section.

4.3 Alternative decomposition of the damage-effect tensors

Final stiffness and compliance expressions (80) show an underlying second possible decomposition of the damage-effect tensors $\bar{\mathbb{A}}$ and $\mathbb{A}$. While previous decompositions in isotropic and anisotropic parts, eqns (68a), (71a), reflect the extension from 'extended' isotropy to 'extended' anisotropy, this new decomposition represents the enhancement from 'basic' to 'extended' anisotropy and can be formulated only in the single-dissipative context introduced in Section 3.2.

By factoring out factors $\bar{\phi}$ or ϕ from $\bar{\mathbb{A}}_{\text{iso}}$ and $\mathbb{A}_{\text{iso}}$, eqn (58), and sending them to $\bar{\mathbb{A}}_{\text{ani}}$ and $\mathbb{A}_{\text{ani}}$, eqn (73), the damage-effect tensors can be alternatively decomposed in product form as:

$$\bar{\mathbb{A}} = \bar{\mathbb{A}}_{\text{ani}} : \bar{\mathbb{A}}_{\text{iso}} = \bar{\mathbb{A}}_{\text{bas}} : \hat{\bar{\mathbb{A}}}_{\text{iso}} \, ; \qquad \mathbb{A} = \mathbb{A}_{\text{iso}} : \mathbb{A}_{\text{ani}} = \hat{\mathbb{A}}_{\text{iso}} : \mathbb{A}_{\text{bas}} \, , \tag{84}$$

where damage-effect tensors $\bar{\mathbb{A}}_{\text{bas}}$, $\mathbb{A}_{\text{bas}}$ are those entering the 'basic' formulation, eqn (20), and $\hat{\bar{\mathbb{A}}}_{\text{iso}}$, $\hat{\mathbb{A}}_{\text{iso}}$ form, with $\bar{\mathbb{A}}_{\text{iso}}$, $\mathbb{A}_{\text{iso}}$, an enhanced isotropic damage-effect counterpart of eqn (73):

$$\begin{aligned} \hat{\bar{\mathbb{A}}}_{\text{iso}} &= (\phi \, \mathbb{I}_s) : \bar{\mathbb{A}}_{\text{iso}} = \phi \, \bar{\mathbb{A}}_{\text{iso}} = \bar{\phi}^{-\eta} \, \mathbb{P}_V + \bar{\phi}^{\eta} \, \mathbb{P}_D ; \\ \hat{\mathbb{A}}_{\text{iso}} &= (\bar{\phi} \, \mathbb{I}_s) : \mathbb{A}_{\text{iso}} = \bar{\phi} \, \mathbb{A}_{\text{iso}} = \phi^{-\eta} \, \mathbb{P}_V + \phi^{\eta} \, \mathbb{P}_D . \end{aligned} \tag{85}$$

Tensors $\hat{\bar{\mathbb{A}}}_{\text{iso}}$ and $\hat{\mathbb{A}}_{\text{iso}}$ would yield isotropic stiffness $\hat{\mathbb{E}}_{\text{iso}}$ and compliance $\hat{\mathbb{C}}_{\text{iso}}$ embedding parameters with hats, eqns (81), (82):

$$\hat{\mathbb{E}}_{\text{iso}} = \hat{\bar{\mathbb{A}}}_{\text{iso}} : \mathbb{E}_0 : \hat{\bar{\mathbb{A}}}^{\mathrm{T}}_{\text{iso}} = 3\hat{K} \, \mathbb{P}_V + 2\hat{G} \, \mathbb{P}_D ; \quad \hat{\mathbb{C}}_{\text{iso}} = \hat{\mathbb{A}}^{\mathrm{T}}_{\text{iso}} : \mathbb{C}_0 : \hat{\mathbb{A}}_{\text{iso}} = \frac{1}{3\hat{K}} \, \mathbb{P}_V + \frac{1}{2\hat{G}} \, \mathbb{P}_D . \tag{86}$$

The final expressions of damage-effect tensors (65) and (69) interpreted on the light of decompositions (84b,d), (85), (20) become:

$$\bar{\mathbb{A}} = \frac{\bar{\phi}^{-\eta} - \bar{\phi}^{\eta}}{3} \, \bar{\boldsymbol{\phi}} \otimes \mathbf{I} + \bar{\phi}^{\eta} \, \sqrt{\bar{\boldsymbol{\phi}}} \, \overline{\underline{\otimes}} \, \sqrt{\bar{\boldsymbol{\phi}}} \, ; \qquad \mathbb{A} = \frac{\phi^{-\eta} - \phi^{\eta}}{3} \, \mathbf{I} \otimes \boldsymbol{\phi} + \phi^{\eta} \, \sqrt{\boldsymbol{\phi}} \, \overline{\underline{\otimes}} \, \sqrt{\boldsymbol{\phi}} \, . \tag{87}$$

Comparing eqns (80b,d) and (15), the effects of the 'basic' contribution (introducing integrity tensor $\bar{\boldsymbol{\phi}}$ and its inverse $\boldsymbol{\phi}$ in place of $\mathbf{I}$) and isotropic-enhanced contribution (inserting hatted parameters in place of undamaged ones) become apparent. Indeed, through product decompositions (84), secant stiffness and compliance may be alternatively represented as:

$$\begin{aligned} \mathbb{E} &= \bar{\mathbb{A}} : \mathbb{E}_0 : \bar{\mathbb{A}}^{\mathrm{T}} = \bar{\mathbb{A}}_{\mathrm{ani}} : \mathbb{E}_{\mathrm{iso}} : \bar{\mathbb{A}}^{\mathrm{T}}_{\mathrm{ani}} = \bar{\mathbb{A}}_{\mathrm{bas}} : \hat{\mathbb{E}}_{\mathrm{iso}} : \bar{\mathbb{A}}^{\mathrm{T}}_{\mathrm{bas}} ; \\ \mathbb{C} &= \mathbb{A}^{\mathrm{T}} : \mathbb{E}_0 : \mathbb{A} = \mathbb{A}^{\mathrm{T}}_{\mathrm{ani}} : \mathbb{C}_{\mathrm{iso}} : \mathbb{A}_{\mathrm{ani}} = \mathbb{A}^{\mathrm{T}}_{\mathrm{bas}} : \hat{\mathbb{C}}_{\mathrm{iso}} : \mathbb{A}_{\mathrm{bas}} \ , \end{aligned} \tag{88}$$

where in the last decompositions $\bar{\mathbb{A}}_{\mathrm{bas}}$, $\mathbb{A}_{\mathrm{bas}}$ act on isotropic tensors $\hat{\mathbb{E}}_{\mathrm{iso}}$, $\hat{\mathbb{C}}_{\mathrm{iso}}$.

Isotropic stiffness and compliance $\hat{\mathbb{E}}_{\mathrm{iso}}$ and $\hat{\mathbb{C}}_{\mathrm{iso}}$ also enter fictitious isotropic secant relations between additional intermediate stress and strain quantities $\boldsymbol{\sigma}_{\mathrm{bas}}$, $\boldsymbol{\epsilon}_{\mathrm{bas}}$ alternative to $\boldsymbol{\sigma}_{\mathrm{ani}}$, $\boldsymbol{\epsilon}_{\mathrm{ani}}$ and defined with $\bar{\mathbb{A}}_{\mathrm{bas}}$, $\mathbb{A}_{\mathrm{bas}}$ as the effective stress and strain in the 'basic' formulation, eqn (22):

$$\boldsymbol{\sigma}_{\mathrm{bas}} = \hat{\mathbb{E}}_{\mathrm{iso}} : \boldsymbol{\epsilon}_{\mathrm{bas}} \ ; \qquad \boldsymbol{\epsilon}_{\mathrm{bas}} = \hat{\mathbb{C}}_{\mathrm{iso}} : \boldsymbol{\sigma}_{\mathrm{bas}} \ . \tag{89}$$

Compare to constitutive relations between $\boldsymbol{\sigma}_{\mathrm{ani}}$, $\boldsymbol{\epsilon}_{\mathrm{ani}}$, eqn (76a,b). From the alternative decompositions (84), the following relations between effective and intermediate quantities also hold:

$$\boldsymbol{\epsilon}_{\mathrm{eff}} = \bar{\mathbb{A}}^{\mathrm{T}}_{\mathrm{iso}} : \boldsymbol{\epsilon}_{\mathrm{ani}} = \hat{\bar{\mathbb{A}}}^{\mathrm{T}}_{\mathrm{iso}} : \boldsymbol{\epsilon}_{\mathrm{bas}} \ ; \qquad \boldsymbol{\sigma}_{\mathrm{eff}} = \mathbb{A}_{\mathrm{iso}} : \boldsymbol{\sigma}_{\mathrm{ani}} = \hat{\mathbb{A}}_{\mathrm{iso}} : \boldsymbol{\sigma}_{\mathrm{bas}} \ . \tag{90}$$

Furthermore, since $\hat{\bar{\mathbb{A}}}_{\mathrm{iso}}$, $\hat{\mathbb{A}}_{\mathrm{iso}}$ and $\bar{\mathbb{A}}_{\mathrm{iso}}$, $\mathbb{A}_{\mathrm{iso}}$ are just proportional to each other, eqn (85), intermediate stress and strain quantities $\boldsymbol{\sigma}_{\mathrm{bas}}$, $\boldsymbol{\epsilon}_{\mathrm{bas}}$ and $\boldsymbol{\sigma}_{\mathrm{ani}}$, $\boldsymbol{\epsilon}_{\mathrm{ani}}$ are also just related by:

$$\boldsymbol{\epsilon}_{\mathrm{bas}} = (\bar{\phi}\, \mathbb{I}_s) : \boldsymbol{\epsilon}_{\mathrm{ani}} = \bar{\phi}\, \boldsymbol{\epsilon}_{\mathrm{ani}} \ ; \qquad \boldsymbol{\sigma}_{\mathrm{bas}} = (\phi\, \mathbb{I}_s) : \boldsymbol{\sigma}_{\mathrm{ani}} = \phi\, \boldsymbol{\sigma}_{\mathrm{ani}} \ . \tag{91}$$

This is obviously consistent with relations (85). Then, energy equivalence holds in terms of nominal, effective and of both intermediate stress and strain quantities involved in eqns (90), (91): $\boldsymbol{\sigma}{:}\boldsymbol{\epsilon}/2{=}\boldsymbol{\sigma}_{\mathrm{eff}}{:}\boldsymbol{\epsilon}_{\mathrm{eff}}/2{=}\boldsymbol{\sigma}_{\mathrm{ani}}{:}\boldsymbol{\epsilon}_{\mathrm{ani}}/2{=}\boldsymbol{\sigma}_{\mathrm{bas}}{:}\boldsymbol{\epsilon}_{\mathrm{bas}}/2$. The same is true for the volumetric and deviatoric energy components. Equality $\boldsymbol{\sigma}_{\mathrm{ani}}{\cdot}\boldsymbol{\epsilon}_{\mathrm{ani}}/2{=}\boldsymbol{\sigma}_{\mathrm{bas}}{\cdot}\boldsymbol{\epsilon}_{\mathrm{bas}}/2$ also holds, which will be used later in conjunction with the definition of the thermodynamic force $-\boldsymbol{\mathcal{Y}}$ of the 'extended' model. Another interesting feature is that, since $\boldsymbol{\sigma}_{\mathrm{ani}}$, $\boldsymbol{\epsilon}_{\mathrm{ani}}$ and $\boldsymbol{\sigma}_{\mathrm{bas}}$, $\boldsymbol{\epsilon}_{\mathrm{bas}}$ are respectively related by isotropic stiffness and compliance $\mathbb{E}_{\mathrm{iso}}$, $\mathbb{C}_{\mathrm{iso}}$, eqn (76), and $\hat{\mathbb{E}}_{\mathrm{iso}}$, $\hat{\mathbb{C}}_{\mathrm{iso}}$, eqn (89), all properties of effective stress and strain in the 'basic' formulation (such as e.g. coaxiality) are maintained here also for the intermediate quantities $\boldsymbol{\sigma}_{\mathrm{ani}}$, $\boldsymbol{\epsilon}_{\mathrm{ani}}$ and $\boldsymbol{\sigma}_{\mathrm{bas}}$, $\boldsymbol{\epsilon}_{\mathrm{bas}}$. Such properties are essential to many of the characteristics of the pseudo-logarithmic damage rate, which is introduced next.

A sketch of the different transformations between nominal, intermediate and effective stress/strain quantities is depicted in Fig. 8 for the compliance stress to strain relations in terms of the various damage-effect tensors $\mathbb{A}$. A similar reverse scheme also applies to the stiffness strain to stress relations in terms of the various dual damage-effect tensors $\bar{\mathbb{A}}$.

Fig. 8

4.4 Pseudo-logarithmic rate of damage for the 'extended' formulation

In view of completing the 'extended' formulation of anisotropic damage, the pseudo-logarithmic rate introduced in eqn (26a,b) for the 'basic' formulation is now generalized to the 'extended' case. This allows to derive a conjugate thermodynamic force with properties complementary to those brought about by eqn (26c).

Taking the dissipation rate in eqns (3) and (5a), differentiation of the compliance $\mathbb{C}$, eqn (72), yields:

$$\dot{d} = \frac{1}{2}\,\boldsymbol{\sigma} : \dot{\mathbb{C}} : \boldsymbol{\sigma} = \left.\frac{\partial u}{\partial \phi_K}\right|_{\boldsymbol{\sigma}} \dot{\phi}_K + \left.\frac{\partial u}{\partial \phi_G}\right|_{\boldsymbol{\sigma}} \dot{\phi}_G + \left.\frac{\partial u}{\partial \boldsymbol{\psi}}\right|_{\boldsymbol{\sigma}} : \dot{\boldsymbol{\psi}}\,, \tag{92}$$

where

$$\begin{aligned} \left.\frac{\partial u}{\partial \phi_K}\right|_{\boldsymbol{\sigma}} &= \frac{2}{9}\,\frac{\phi_K}{K_0}\,(\boldsymbol{\psi} : \boldsymbol{\sigma})^2\,; \\ \left.\frac{\partial u}{\partial \phi_G}\right|_{\boldsymbol{\sigma}} &= \frac{\phi_G}{G_0}\left(\mathrm{tr}\,(\boldsymbol{\sigma}\cdot\boldsymbol{\psi})^2 - \frac{1}{3}\,(\boldsymbol{\psi} : \boldsymbol{\sigma})^2\right)\,; \\ \left.\frac{\partial u}{\partial \boldsymbol{\psi}}\right|_{\boldsymbol{\sigma}} &= -\,\frac{\nu}{E}\,(\boldsymbol{\psi} : \boldsymbol{\sigma})\,\boldsymbol{\sigma} + \frac{1+\nu}{E}\,\boldsymbol{\sigma}\cdot\boldsymbol{\psi}\cdot\boldsymbol{\sigma}\,. \end{aligned} \tag{93}$$

Logarithmic rates of bulk and shear isotropic scalar damage variables can be evaluated as in eqn (31): $\dot{\phi}_K{=}\phi_K\,\dot{L}_K/2$, $\dot{\phi}_G{=}\phi_G\,\dot{L}_G/2$, where, in the single-dissipative setting, L_K and L_G are taken proportional to the single logarithmic isotropic scalar damage variable L, eqn (48), so that $\dot{L}_K{=}(1{-}\eta)\,\dot{L}$, $\dot{L}_G{=}(1{+}\eta)\,\dot{L}$. In a similar way, the (non-holonomic) pseudo-logarithmic rate $\overset{\diamond}{\boldsymbol{\mu}}$ of the isochoric integrity tensor $\boldsymbol{\psi}$ is introduced analogously to eqn (26a,b):

$$\overset{\diamond}{\boldsymbol{\mu}} = 2\,\bar{\mathbb{A}}_{\mathrm{ani}} : \dot{\boldsymbol{\psi}} = 2\,\sqrt{\bar{\boldsymbol{\psi}}}\cdot\dot{\boldsymbol{\psi}}\cdot\sqrt{\bar{\boldsymbol{\psi}}}\,; \qquad \dot{\boldsymbol{\psi}} = \frac{1}{2}\,\mathbb{A}_{\mathrm{ani}} : \overset{\diamond}{\boldsymbol{\mu}} = \frac{1}{2}\,\sqrt{\boldsymbol{\psi}}\cdot\overset{\diamond}{\boldsymbol{\mu}}\cdot\sqrt{\boldsymbol{\psi}}\,. \tag{94}$$

Replacing rates $\dot{\phi}_K$, $\dot{\phi}_G$ and $\dot{\boldsymbol{\psi}}$, eqn (94d), and making use of eqn (48), the dissipation rate $\dot{d}$, eqns (92), (93), may be written as

$$\dot{d} = \Big(u + \eta\,(u_D - u_V)\Big)\,\dot{L} + \left(\frac{1}{2}\,\boldsymbol{\sigma}_{\mathrm{bas}}\cdot\boldsymbol{\epsilon}_{\mathrm{bas}}\right) : \overset{\diamond}{\boldsymbol{\mu}}\,, \tag{95}$$

where

$$\begin{aligned} \frac{1}{2}\,\boldsymbol{\sigma}_{\mathrm{bas}}\cdot\boldsymbol{\epsilon}_{\mathrm{bas}} &= \frac{1}{2}\left(-\frac{\hat{\nu}}{\hat{E}}\;\mathrm{tr}\,\boldsymbol{\sigma}_{\mathrm{bas}}\;\boldsymbol{\sigma}_{\mathrm{bas}} + \frac{1+\hat{\nu}}{\hat{E}}\;\boldsymbol{\sigma}^2_{\mathrm{bas}}\right) \\ &= \frac{1}{2}\left(\;\hat{\Lambda}\;\;\mathrm{tr}\,\boldsymbol{\epsilon}_{\mathrm{bas}}\;\boldsymbol{\epsilon}_{\mathrm{bas}} + \;2\,\hat{G}\;\;\boldsymbol{\epsilon}^2_{\mathrm{bas}}\right), \end{aligned} \tag{96}$$

embedding parameters with hats, replaces $\boldsymbol{\sigma}_{\mathrm{ani}}{\cdot}\boldsymbol{\epsilon}_{\mathrm{ani}}/2$, since they coincide, and, due to energy equivalence (Section 4.3), volumetric and deviatoric elastic energies may be also defined in terms of volumetric components and deviatoric parts of effective quantities $\boldsymbol{\sigma}_{\mathrm{bas}}$, $\boldsymbol{\epsilon}_{\mathrm{bas}}$ and in terms of elastic moduli with hats $\hat{K}$, $\hat{G}$:

$$u_V = \frac{1}{2}\,\frac{\sigma^2_{\mathrm{bas},V}}{\hat{K}} = \frac{1}{2}\,9\hat{K}\,\epsilon^2_{\mathrm{bas},V}\,; \qquad u_D = \frac{1}{2}\,\frac{\boldsymbol{\sigma}_{\mathrm{bas},D} : \boldsymbol{\sigma}_{\mathrm{bas},D}}{2\,\hat{G}} = \hat{G}\,\boldsymbol{\epsilon}_{\mathrm{bas},D} : \boldsymbol{\epsilon}_{\mathrm{bas},D}\,. \tag{97}$$

Alternatively $u_V{=}u_{V,0}$, $u_D{=}u_{D,0}$ and $u{=}u_V{+}u_D{=}u_0{=}u_{V,0}{+}u_{D,0}$ could be used instead to introduce in the first terms of eqn (95) the real effective stress and strain quantities (75), (77), (78) and the initial elastic moduli K_0, G_0. Expressions (95), (97) privilege to show the enhancement of the 'extended' model with respect to the previous 'basic' one.

Now, the property is crucial that, since $\boldsymbol{\psi}$ remains isochoric, the pseudo-logarithmic rate $\overset{\diamond}{\boldsymbol{\mu}}$ is purely deviatoric, i.e. $\mathrm{tr}\,\overset{\diamond}{\boldsymbol{\mu}}{=}0$ (see Carol et al., 2001a, Appendix A). Indeed, from the derivative of the third invariant of $\boldsymbol{\psi}$, namely $\partial\,(\det\boldsymbol{\psi})/\partial\boldsymbol{\psi}{=}(\det\boldsymbol{\psi})\ \boldsymbol{\psi}^{-\mathrm{T}}$, one obtains $(\det\boldsymbol{\psi})^{\dot{}}{=}(\det\boldsymbol{\psi})\ \mathrm{tr}\,(\sqrt{\bar{\boldsymbol{\psi}}}{\cdot}\dot{\boldsymbol{\psi}}{\cdot}\sqrt{\bar{\boldsymbol{\psi}}})$, so that, since $\det\boldsymbol{\psi}{\equiv}1$, $(\det\boldsymbol{\psi})^{\dot{}}{=}0$ and eqn (94b) yields $\mathrm{tr}\,\overset{\diamond}{\boldsymbol{\mu}}{=}0$. Furthermore, the rate $\overset{\diamond}{\boldsymbol{\mu}}$ turns out to be the deviatoric part of the pseudo-logarithmic rate (26b), which can also be introduced in the present context. Indeed, the product-type decompositions of the integrity tensor $\bar{\boldsymbol{\phi}}{=}\bar{\phi}\,\bar{\boldsymbol{\psi}}$ and of its inverse $\boldsymbol{\phi}{=}\phi\,\boldsymbol{\psi}$ yield $\dot{\boldsymbol{\phi}}{=}\dot{\phi}\,\boldsymbol{\psi}{+}\phi\,\dot{\boldsymbol{\psi}}$. Also, similarly to eqn (31), eqn (49) renders $2\,\dot{\phi}/\phi{=}\dot{L}$ and definition (94b) leads to:

$$\overset{\diamond}{\mathbf{L}} = 2\ \bar{\mathbb{A}}_{\mathrm{bas}} : \dot{\boldsymbol{\phi}} = 2\ \sqrt{\bar{\boldsymbol{\phi}}} \cdot \dot{\boldsymbol{\phi}} \cdot \sqrt{\bar{\boldsymbol{\phi}}} = \dot{L}\ \mathbf{I} + \overset{\diamond}{\boldsymbol{\mu}}\ . \tag{98}$$

Then, the previous rates $\dot{L}$ and $\overset{\diamond}{\boldsymbol{\mu}}$ turn out to be respectively the volumetric component and deviatoric part of $\overset{\diamond}{\mathbf{L}}$, namely:

$$\dot{L}_V = \frac{\mathrm{tr}\,\overset{\diamond}{\mathbf{L}}}{3} = \dot{L}\ ; \qquad \overset{\diamond}{\mathbf{L}}_D = \overset{\diamond}{\mathbf{L}} - \dot{L}_V\ \mathbf{I} = \overset{\diamond}{\boldsymbol{\mu}}\ . \tag{99}$$

Notice that, as in the 'basic' formulation, the volumetric component and deviatoric part of the pseudo-logarithmic rate $\overset{\diamond}{\mathbf{L}}$ are respectively attached to the isotropic and to the anisotropic parts of stiffness degradation (Carol et al., 2001a). Then, product-type decompositions of isotropic and anisotropic effects in damage and damage-effect tensors mirror themselves in the classical (sum-type) decomposition in volumetric and deviatoric parts of the convenient pseudo-logarithmic rate. Notice that, while $\dot{L}$ is an exact rate, $\overset{\diamond}{\boldsymbol{\mu}}$ and $\overset{\diamond}{\mathbf{L}}$ are not. Furthermore, since $\overset{\diamond}{\boldsymbol{\mu}}$ is purely deviatoric, its double contraction with another second-order symmetric tensor, such as the force $\boldsymbol{\sigma}_{\mathrm{bas}}{\cdot}\boldsymbol{\epsilon}_{\mathrm{bas}}/2$ in eqn (95), will only activate the deviatoric part of that tensor, i.e. any volumetric part may be added to that tensor without changing the resulting contraction. Based on this property, the force tensor $\boldsymbol{\sigma}_{\mathrm{bas}}{\cdot}\boldsymbol{\epsilon}_{\mathrm{bas}}/2$ in the second product addend of the dissipation rate (95) can be replaced by the deviatoric part of a single force tensor which possesses as volumetric component the same force term $u{+}\eta\,(u_D{-}u_V)$ already appearing in the first product addend. This leads to the right definition of the force associated to the pseudo-logarithmic rate $\overset{\diamond}{\mathbf{L}}{=}\dot{L}\,\mathbf{I}{+}\overset{\diamond}{\boldsymbol{\mu}}$.

Indeed, the thermodynamic force conjugate to the rate $\overset{\diamond}{\mathbf{L}}$ can be promptly evaluated from the derivative of the elastic energy at constant stress, eqn (3c). In this respect, due to the structure of the final secant stiffness and compliance (80) embedding parameters with hats and tensors $\bar{\boldsymbol{\phi}}$, $\boldsymbol{\phi}$, one has first terms as in the 'basic' formulation by keeping constant the parameters with hats (which replace the undamaged material parameters) and second new additional terms accounting for the variation of the parameters above. In that respect, notice that these latter derivatives may be worked-out through the chain rule from the definitions of $\bar{\phi}{=}(\det\bar{\boldsymbol{\phi}})^{1/3}$ and $\phi{=}(\det\boldsymbol{\phi})^{1/3}$:

$$\frac{\partial\bar{\phi}}{\partial\bar{\boldsymbol{\phi}}} = \frac{1}{3}\ \bar{\phi}\ \boldsymbol{\phi}\ ; \qquad \frac{\partial\phi}{\partial\boldsymbol{\phi}} = \frac{1}{3}\ \phi\ \bar{\boldsymbol{\phi}}\ . \tag{100}$$

So, for $\hat{K}{=}\bar{\phi}^{-2\eta}\,K_0$ and $1/\hat{K}{=}\phi^{-2\eta}/K_0$:

$$\frac{\partial \hat{K}}{\partial \bar{\boldsymbol{\phi}}} = \frac{\partial \hat{K}}{\partial \bar{\phi}} : \frac{\partial \bar{\phi}}{\partial \bar{\boldsymbol{\phi}}} = -\frac{2}{3}\,\eta\,\hat{K}\,\boldsymbol{\phi}\,; \qquad \frac{\partial (1/\hat{K})}{\partial \boldsymbol{\phi}} = \frac{\partial (1/\hat{K})}{\partial \phi} : \frac{\partial \phi}{\partial \boldsymbol{\phi}} = -\frac{2}{3}\,\eta\,\frac{1}{\hat{K}}\,\bar{\boldsymbol{\phi}}\,. \tag{101}$$

Similarly, for $\hat{G}{=}\bar{\phi}^{2\eta}\,G_0$ and $1/\hat{G}{=}\phi^{2\eta}/G_0$:

$$\frac{\partial \hat{G}}{\partial \bar{\boldsymbol{\phi}}} = \frac{\partial \hat{G}}{\partial \bar{\phi}} : \frac{\partial \bar{\phi}}{\partial \bar{\boldsymbol{\phi}}} = \frac{2}{3}\,\eta\,\hat{G}\,\boldsymbol{\phi}\,; \qquad \frac{\partial (1/\hat{G})}{\partial \boldsymbol{\phi}} = \frac{\partial (1/\hat{G})}{\partial \phi} : \frac{\partial \phi}{\partial \boldsymbol{\phi}} = \frac{2}{3}\,\eta\,\frac{1}{\hat{G}}\,\bar{\boldsymbol{\phi}}\,, \tag{102}$$

and accordingly for the derivatives of $\hat{\Lambda}$ and $\hat{\nu}/\hat{E}$, eqn (82):

$$\frac{\partial \hat{\Lambda}}{\partial \bar{\boldsymbol{\phi}}} = -\frac{2}{3}\,\eta\,\frac{3\,\hat{\Lambda} + 4\,\hat{G}}{3}\,\boldsymbol{\phi}\,; \qquad \frac{\partial (\hat{\nu}/\hat{E})}{\partial \boldsymbol{\phi}} = \frac{2}{3}\,\eta\,\frac{2-\hat{\nu}}{3\,\hat{E}}\,\bar{\boldsymbol{\phi}}\,. \tag{103}$$

Note that, to get the partial derivatives above with respect to $\boldsymbol{\phi}$ instead of $\bar{\boldsymbol{\phi}}$ (or vice versa $\bar{\boldsymbol{\phi}}$ instead of $\boldsymbol{\phi}$) it is enough to change sign in the resulting expressions and substitute there $\boldsymbol{\phi}$ with $\bar{\boldsymbol{\phi}}$ (or vice versa $\bar{\boldsymbol{\phi}}$ with $\boldsymbol{\phi}$), e.g. $\partial\hat{K}/\partial\boldsymbol{\phi}{=}2/3\,\eta\hat{K}\bar{\boldsymbol{\phi}}$.

Then, the derivative of $u{=}\boldsymbol{\sigma}{:}\mathbb{C}{:}\boldsymbol{\sigma}/2$ with respect to $\boldsymbol{\phi}$ at constant stress can be evaluated by taking for $\mathbb{C}$ compliance expression (80d):

$$\begin{aligned} \left.\frac{\partial u}{\partial \boldsymbol{\phi}}\right|_{\boldsymbol{\sigma}} = &-\frac{\hat{\nu}}{\hat{E}}\,(\boldsymbol{\phi}:\boldsymbol{\sigma})\,\boldsymbol{\sigma} + \frac{1+\hat{\nu}}{\hat{E}}\,\boldsymbol{\sigma}\cdot\boldsymbol{\phi}\cdot\boldsymbol{\sigma} \\ &+ \frac{\eta}{3}\left(-\frac{2-\hat{\nu}}{3\,\hat{E}}\,(\boldsymbol{\phi}:\boldsymbol{\sigma})^2 + \frac{1+\hat{\nu}}{\hat{E}}\,\mathrm{tr}\,(\boldsymbol{\sigma}\cdot\boldsymbol{\phi})^2\right)\bar{\boldsymbol{\phi}}\,. \end{aligned} \tag{104}$$

Let us also record for later use the analogous derivative of the volumetric energy component:

$$\left.\frac{\partial u_V}{\partial \boldsymbol{\phi}}\right|_{\boldsymbol{\sigma}} = \frac{1}{9\,\hat{K}}\,\boldsymbol{\phi}:\boldsymbol{\sigma}\,\left(\boldsymbol{\sigma} - \frac{\eta}{3}\,(\boldsymbol{\phi}:\boldsymbol{\sigma})\,\bar{\boldsymbol{\phi}}\right)\,. \tag{105}$$

Now, although $\mathbf{L}$ does not exist in general as a finite quantity, derivatives (104), (105) may be further developed with respect to $\mathbf{L}$ through a formal use of the chain rule and by taking into account that, since $\dot{\boldsymbol{\phi}}{=}\mathbb{A}_{\mathrm{bas}}{:}\overset{\circ}{\mathbf{L}}/2$, $\partial\boldsymbol{\phi}/\partial\mathbf{L}$ may be evaluated as $\mathbb{A}_{\mathrm{bas}}/2$. Then, from eqn (104), the following conjugate force associated to $\overset{\circ}{\mathbf{L}}$ in the dissipation rate $\dot{d}{=}(-\boldsymbol{\mathcal{Y}}){:}\,\overset{\circ}{\mathbf{L}}$ can be derived:

$$-\boldsymbol{\mathcal{Y}} = \left.\frac{\partial u}{\partial \mathbf{L}}\right|_{\boldsymbol{\sigma}} = \frac{1}{2}\,\boldsymbol{\sigma}_{\mathrm{bas}}\cdot\boldsymbol{\epsilon}_{\,\mathrm{bas}} + \frac{\eta}{3}\,(u_D - u_V)\,\mathbf{I}\,, \tag{106}$$

where $\boldsymbol{\sigma}_{\mathrm{bas}}{\cdot}\boldsymbol{\epsilon}_{\,\mathrm{bas}}/2$ is given in eqn (96). Also, for later use, let us record that eqn (105) translates to:

$$\left.\frac{\partial u_V}{\partial \mathbf{L}}\right|_{\boldsymbol{\sigma}} = \frac{1}{2}\,\frac{1}{9\,\hat{K}}\,\mathrm{tr}\,\boldsymbol{\sigma}_{\mathrm{bas}}\,\left(\boldsymbol{\sigma}_{\mathrm{bas}} - \eta\,\boldsymbol{\sigma}_{\mathrm{bas},V}\right)\,. \tag{107}$$

If compared to its counterpart $-\boldsymbol{\mathcal{Y}}_{\mathrm{bas}}$ in the 'basic' formulation, eqn (26c), the conjugate force $-\boldsymbol{\mathcal{Y}}$, eqn (106), exhibits formally only an additional spherical term which reduces its volumetric part depending on the coefficient η. This additional spherical term can be alternatively written solely in terms of u and u_V just by noticing that $u_D - u_V = u - 2\,u_V$. However, notice also that, since $\boldsymbol{\sigma}_{\text{bas}} \cdot \boldsymbol{\epsilon}_{\text{bas}}/2$ is expressed by eqn (96) in terms of parameters with hats, eqns (81), (82), as soon as $\eta \neq 0$, $\boldsymbol{\sigma}_{\text{bas}} \cdot \boldsymbol{\epsilon}_{\text{bas}}/2$ also differs from $-\boldsymbol{\mathcal{Y}}_{\text{bas}} = \boldsymbol{\sigma}_{\text{eff,bas}} \cdot \boldsymbol{\epsilon}_{\text{eff,bas}}/2$, eqn (26a), since in $-\boldsymbol{\mathcal{Y}}_{\text{bas}}$ the undamaged parameters are involved. In the special case that $\eta = 0$, this difference disappears since hatted parameters collapse into undamaged ones. Also, the additional spherical term vanishes and the model collapses into the 'basic' one. Furthermore, by looking at the volumetric and deviatoric components of the conjugate force (106), we have:

$$-\mathcal{Y}_V = \frac{\operatorname{tr}(-\boldsymbol{\mathcal{Y}})}{3} = \frac{u}{3} + \frac{\eta}{3}\,(u_D - u_V)\,; \qquad -\boldsymbol{\mathcal{Y}}_D = \frac{1}{2}\,\boldsymbol{\sigma}_{\text{bas}} \cdot \boldsymbol{\epsilon}_{\text{bas}} - \frac{u}{3}\,\mathbf{I}\,. \tag{108}$$

Then, the deviatoric part of the conjugate force is 'formally' the same as that of the 'basic' formulation, while the volumetric component is modified when $\eta \neq 0$, compare eqn (108) to the volumetric and deviatoric components of $-\boldsymbol{\mathcal{Y}}_{\text{bas}}$ from eqn (26c):

$$-\mathcal{Y}_{\text{bas},V} = \frac{\operatorname{tr}(-\boldsymbol{\mathcal{Y}}_{\text{bas}})}{3} = \frac{u}{3}\,; \qquad -\boldsymbol{\mathcal{Y}}_{\text{bas},D} = \frac{1}{2}\,\boldsymbol{\sigma}_{\text{eff,bas}} \cdot \boldsymbol{\epsilon}_{\text{eff,bas}} - \frac{u}{3}\,\mathbf{I}\,. \tag{109}$$

With respect to this formal analogy, the principal directions of the conjugate force are still those of the 'basic' intermediate stress or strain quantities, while its principal values are modified by the difference in volumetric terms.

Finally, the dissipation rate (95) may also be rewritten as the sum of volumetric and deviatoric contributions, of which only the first one formally changes with respect to the 'basic' formulation:

$$\dot{d} = \dot{d}_V + \dot{d}_D = \Big(u + \eta\,(u_D - u_V)\Big)\,\dot{L} + \left(\frac{1}{2}\,\boldsymbol{\sigma}_{\text{bas}} \cdot \boldsymbol{\epsilon}_{\text{bas}} - \frac{u}{3}\,\mathbf{I}\right) : \mathring{\mathbf{L}}_D\,. \tag{110}$$

Notice that, as commented above, the additional spherical term in the force premultiplying $\mathring{\mathbf{L}}_D$ in eqn (110), as compared to eqn (95), does not change the final value of $\dot{d}$. However, the definition (106) and resulting dissipation rate (110) are the correct expressions for $-\boldsymbol{\mathcal{Y}}$ and $\dot{d}$ in the present 'extended' model.

4.5 Tangent stiffness of the 'extended' anisotropic formulation

To complete the 'extended' formulation, the various ingredients entering the tangent stiffness $\mathbb{E}_{\text{tan}}$ in eqn (8b,c) can now be derived. Notice that what has been already developed about loading function and pseudo-logarithmic damage rule for the 'basic' formulation (Carol et al., 2001a,b) is also valid for the 'extended' formulation. The fact that the conjugate forces slightly differ and have a reduced volumetric component does not change the geometric interpretation of the loading function in the principal force space $(-\mathcal{Y}_1, -\mathcal{Y}_2, -\mathcal{Y}_3)$, the identification of the volumetric and deviatoric parts of the pseudo-logarithmic damage rule $\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{M}}_V + \boldsymbol{\mathcal{M}}_D$ for the isotropic and anisotropic damage growths, and the restrictions on the pseudo-logarithmic damage rule orientation to assure positive dissipation. Consideration of the positiveness condition for the projection of the compliance rate on a fixed orientation, and of the inverse of the Young's orthotropic moduli in the principal axes of damage, leads to the sufficient condition that $\eta \geq 0$. This condition coincides with that already stated in Section 3.3, based on different arguments, and therefore it seems reasonable to restrict the value of η to the interval [0,1].

To develop the tangent stiffness, let us first postulate that the current elastic domain is defined in the space of the thermodynamic forces $-\boldsymbol{\mathcal{Y}}$ by assuming the following rather general isotropic hardening/softening condition generalizing eqn (40b):

$$F(-\boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{D}}) = f(-\boldsymbol{\mathcal{Y}}) - r(\boldsymbol{\mathcal{D}}) \leq 0 \, . \tag{111}$$

Here the function $f(-\boldsymbol{\mathcal{Y}})$ specifies the shape of the damage domain, which is left general at this stage and includes possibly both associative and non-associative flow rules; $r(\boldsymbol{\mathcal{D}})$ is the hardening/softening function, normally expressed in terms of the underlying damage variables $\bar{\boldsymbol{\phi}}$ or $\boldsymbol{\phi}$. The rates of the latter variables depend eventually on $\overset{\circ}{\mathbf{L}} = \dot{\lambda}\, \boldsymbol{\mathcal{M}}$ and thus on the inelastic multiplier $\dot{\lambda}$. So, finally function r bears an implicit dependence on the cumulative value of λ. The model would then be characterized by specific choices of the functions above, according to the material behavior to be described. Concerning $f(-\boldsymbol{\mathcal{Y}})$, as a first approach, such function may be expressed in terms of the invariants of $-\boldsymbol{\mathcal{Y}}$ so that, as remarked above, the surface $F(-\boldsymbol{\mathcal{Y}})=0$ may be represented in the space of the principal thermodynamic forces. Consequently, the gradient $\boldsymbol{\mathcal{N}}$ of the loading function in the conjugate force space is coaxial with $-\boldsymbol{\mathcal{Y}}$ (and so also e.g. with $\boldsymbol{\sigma}_{\text{bas}}$, $\boldsymbol{\epsilon}_{\text{bas}}$).

Note also that, as $-\boldsymbol{\mathcal{Y}}$, eqn (106), F may be expressed in terms of the different intermediate stress or strain quantities. For instance, for the ease of representation and implementation as an update of the 'basic' model, the 'basic' effective quantities $\boldsymbol{\sigma}_{\text{bas}}$, $\boldsymbol{\epsilon}_{\text{bas}}$ may be conveniently adopted. The same 'updating' concept also applies to the various ingredients of the theory: the characteristic terms are obtained by evaluating the new additional addends completing the expressions already available from the 'basic' formulation. As remarked before in conjunction with the development of the thermodynamic forces (104)-(107), this is due to the structure of the final secant stiffness and compliance (80) which embed parameters with hats and tensors $\bar{\boldsymbol{\phi}}$, $\boldsymbol{\phi}$. The expressions of the 'basic' formulation were given in Carol et al. (2001a,b) and Rizzi and Carol (2001) and are recalled here for the sake of completeness (labeled by script $_{\text{bas}}$). The additional terms completing the expressions of the 'extended' model are derived here and marked by superscript *.

The basic ingredients of the tangent stiffness (8c) to be evaluated are the gradients $\mathbf{n}$, $\mathbf{m}$ and the hardening parameters H, $\bar{H}$. The derivation makes use of derivatives (100)-(103) and also of the following 'enhanced' addend derivatives:

$$\left.\frac{\partial(-\boldsymbol{\mathcal{Y}})}{\partial\, \boldsymbol{\sigma}_{\text{bas}}}\right|_{\lambda}^{\star} = \frac{\eta}{3}\, \mathbf{I} \otimes \left(\boldsymbol{\epsilon}_{\text{bas}} - 2\, \boldsymbol{\epsilon}_{\text{bas},V}\right) = \frac{\eta}{3}\, \mathbf{I} \otimes \left(-\frac{2-\hat{\nu}}{3\,\hat{E}} \text{tr}\, \boldsymbol{\sigma}_{\text{bas}}\, \mathbf{I} + \frac{1+\hat{\nu}}{\hat{E}}\, \boldsymbol{\sigma}_{\text{bas}}\right); \tag{112}$$

$$\left.\frac{\partial \mathbb{C}}{\partial \mathbf{L}}\right|^{\star} = \left.\frac{\partial \mathbb{C}}{\partial \boldsymbol{\phi}}\right|^{\star} : \frac{1}{2}\, \mathbb{A}_{\text{bas}} = \frac{\eta}{3} \left(-\frac{2-\hat{\nu}}{3\,\hat{E}}\, \boldsymbol{\phi} \otimes \boldsymbol{\phi} + \frac{1+\hat{\nu}}{\hat{E}}\, \boldsymbol{\phi} \overline{\underline{\otimes}} \boldsymbol{\phi}\right) \otimes \mathbf{I}\, ; \tag{113}$$

$$
\begin{aligned}
\left.\frac{\partial(-\boldsymbol{\mathcal{Y}})}{\partial\,\mathbf{L}}\right|_{\boldsymbol{\sigma}}^{\star} &= \left.\frac{\partial(-\boldsymbol{\mathcal{Y}})}{\partial\,\boldsymbol{\phi}}\right|_{\boldsymbol{\sigma}}^{\star} : \frac{1}{2}\,\mathbb{A}_{\mathrm{bas}} \\
&= \frac{\eta}{6}\,\left(-\frac{2-\hat{\nu}}{3\,\hat{E}}\,\mathrm{tr}\,\boldsymbol{\sigma}_{\mathrm{bas}}\,\boldsymbol{\sigma}_{\mathrm{bas}} + \frac{1+\hat{\nu}}{\hat{E}}\,\boldsymbol{\sigma}_{\mathrm{bas}}^{2}\right)\otimes\mathbf{I} \\
&+ \frac{\eta}{6}\,\mathbf{I}\otimes\left(-\frac{2-\hat{\nu}}{3\,\hat{E}}\,\mathrm{tr}\,\boldsymbol{\sigma}_{\mathrm{bas}}\,\boldsymbol{\sigma}_{\mathrm{bas}} + \frac{1+\hat{\nu}}{\hat{E}}\,\boldsymbol{\sigma}_{\mathrm{bas}}^{2}\right) + \frac{\eta^2}{9}\,u\,\mathbf{I}\otimes\mathbf{I}\,,
\end{aligned} \tag{114}
$$

which are used respectively to develop the expressions of $\mathbf{n}$, $\mathbf{m}$ and H. Notice that the last derivative has been developed from eqn (106) by taking into account that $u_D - u_V = u - 2\,u_V$ and then by using derivatives (106), (107).

Let us consider first the gradient $\mathbf{n}$ of the loading surface in stress space, as obtained from eqn (7a):

$$
\mathbf{n} = \left.\frac{\partial F}{\partial\boldsymbol{\sigma}}\right|_{\lambda} = \left.\frac{\partial F}{\partial(-\boldsymbol{\mathcal{Y}})}\right|_{\lambda} : \left.\frac{\partial(-\boldsymbol{\mathcal{Y}})}{\partial\,\boldsymbol{\sigma}_{\mathrm{bas}}}\right|_{\lambda} : \left.\frac{\partial\boldsymbol{\sigma}_{\mathrm{bas}}}{\partial\,\boldsymbol{\sigma}}\right|_{\lambda} = \boldsymbol{\mathcal{N}} : \left.\frac{\partial(-\boldsymbol{\mathcal{Y}})}{\partial\,\boldsymbol{\sigma}_{\mathrm{bas}}}\right|_{\lambda} : \mathbb{A}_{\mathrm{bas}} = \mathbf{n}_{\mathrm{bas}} + \mathbf{n}^{\star}\,; \tag{115}
$$

$$
\begin{aligned}
\mathbf{n}_{\mathrm{bas}} = &-\frac{\hat{\nu}}{2\,E_0}\,\left((\boldsymbol{\mathcal{N}}:\boldsymbol{\sigma}_{\mathrm{bas}})\,\boldsymbol{\phi} + \mathrm{tr}\,\boldsymbol{\sigma}_{\mathrm{bas}}\,\sqrt{\boldsymbol{\phi}}\cdot\boldsymbol{\mathcal{N}}\cdot\sqrt{\boldsymbol{\phi}}\right) \\
&+ \frac{1+\hat{\nu}}{2\,\hat{E}}\,\left(\sqrt{\boldsymbol{\phi}}\cdot\boldsymbol{\mathcal{N}}\cdot\boldsymbol{\sigma}_{\mathrm{bas}}\cdot\sqrt{\boldsymbol{\phi}} + \sqrt{\boldsymbol{\phi}}\cdot\boldsymbol{\sigma}_{\mathrm{bas}}\cdot\boldsymbol{\mathcal{N}}\cdot\sqrt{\boldsymbol{\phi}}\right);
\end{aligned} \tag{116}
$$

$$
\mathbf{n}^{\star} = \frac{\eta}{3}\,\mathrm{tr}\boldsymbol{\mathcal{N}}\,\left(-\frac{2-\hat{\nu}}{3\,\hat{E}}\,\mathrm{tr}\,\boldsymbol{\sigma}_{\mathrm{bas}}\,\boldsymbol{\phi} + \frac{1+\hat{\nu}}{\hat{E}}\,\sqrt{\boldsymbol{\phi}}\cdot\boldsymbol{\sigma}_{\mathrm{bas}}\cdot\sqrt{\boldsymbol{\phi}}\right)\,. \tag{117}
$$

Next, the flow rule $\mathbf{m}$ of the degrading strain rate can be determined from eqns (6d) and (7d):

$$
\mathbf{m} = \left(\frac{\partial\mathbb{C}}{\partial\mathbf{L}} : \boldsymbol{\mathcal{M}}\right) : \boldsymbol{\sigma} = \mathbf{m}_{\mathrm{bas}} + \mathbf{m}^{\star}\,; \tag{118}
$$

$$
\begin{aligned}
\mathbf{m}_{\mathrm{bas}} = &-\frac{\hat{\nu}}{2\,\hat{E}}\,\left((\boldsymbol{\mathcal{M}}:\boldsymbol{\sigma}_{\mathrm{bas}})\,\boldsymbol{\phi} + \mathrm{tr}\,\boldsymbol{\sigma}_{\mathrm{bas}}\,\sqrt{\boldsymbol{\phi}}\cdot\boldsymbol{\mathcal{M}}\cdot\sqrt{\boldsymbol{\phi}}\right) \\
&+ \frac{1+\hat{\nu}}{2\,\hat{E}}\,\left(\sqrt{\boldsymbol{\phi}}\cdot\boldsymbol{\mathcal{M}}\cdot\boldsymbol{\sigma}_{\mathrm{bas}}\cdot\sqrt{\boldsymbol{\phi}} + \sqrt{\boldsymbol{\phi}}\cdot\boldsymbol{\sigma}_{\mathrm{bas}}\cdot\boldsymbol{\mathcal{M}}\cdot\sqrt{\boldsymbol{\phi}}\right);
\end{aligned} \tag{119}
$$

$$
\mathbf{m}^{\star} = \frac{\eta}{3}\,\mathrm{tr}\boldsymbol{\mathcal{M}}\,\left(-\frac{2-\hat{\nu}}{3\,\hat{E}}\,\mathrm{tr}\,\boldsymbol{\sigma}_{\mathrm{bas}}\,\boldsymbol{\phi} + \frac{1+\hat{\nu}}{\hat{E}}\,\sqrt{\boldsymbol{\phi}}\cdot\boldsymbol{\sigma}_{\mathrm{bas}}\cdot\sqrt{\boldsymbol{\phi}}\right)\,. \tag{120}
$$

Notice that, according to eqns (6b), (7b) and (6d), (7d), it is confirmed that $\mathbf{n}$ and $\mathbf{m}$, eqns (115)-(117) and (118)-(120), are expressed by the same formulas within a shift between rules $\boldsymbol{\mathcal{N}}$ and $\boldsymbol{\mathcal{M}}$.

Finally, the hardening parameter H of a stress-based formulation can be determined from eqn (9c) through the use of the chain rule:

$$
H = -\left.\frac{\partial F}{\partial\lambda}\right|_{\boldsymbol{\sigma}} = \frac{\partial r}{\partial\lambda} + H_f\,; \quad H_f = -\left.\frac{\partial f}{\partial\lambda}\right|_{\boldsymbol{\sigma}} = -\boldsymbol{\mathcal{N}} : \left.\frac{\partial(-\boldsymbol{\mathcal{Y}})}{\partial\mathbf{L}}\right|_{\boldsymbol{\sigma}} : \boldsymbol{\mathcal{M}} = H_f^{\mathrm{bas}} + H_f^{\star}\,; \tag{121}
$$

$$
\begin{aligned}
H_f^{\mathrm{bas}} = &\ \frac{\hat{\nu}}{4\,\hat{E}}\,\left((\boldsymbol{\mathcal{N}}:\boldsymbol{\sigma}_{\mathrm{bas}})\,(\boldsymbol{\mathcal{M}}:\boldsymbol{\sigma}_{\mathrm{bas}}) + \mathrm{tr}\,\boldsymbol{\sigma}_{\mathrm{bas}}\;\mathrm{tr}\,(\boldsymbol{\mathcal{N}}\cdot\boldsymbol{\sigma}_{\mathrm{bas}}\cdot\boldsymbol{\mathcal{M}})\right) \\
&- \frac{1+\hat{\nu}}{2\,\hat{E}}\;tr(\boldsymbol{\mathcal{N}}\cdot\boldsymbol{\sigma}_{\mathrm{bas}}^{2}\cdot\boldsymbol{\mathcal{M}})\,;
\end{aligned} \tag{122}
$$

$$
\begin{aligned}
H^\star_f = &- \frac{\eta}{6}\,\mathrm{tr}\,\boldsymbol{\mathcal{M}}\left(-\frac{2-\hat\nu}{3\,\hat E}\,\mathrm{tr}\,\boldsymbol{\sigma}_{\mathrm{bas}}\ \boldsymbol{\mathcal{N}}:\boldsymbol{\sigma}_{\mathrm{bas}} + \frac{1+\hat\nu}{\hat E}\ \boldsymbol{\mathcal{N}}:\boldsymbol{\sigma}^2_{\mathrm{bas}}\right)\\
&- \frac{\eta}{6}\,\mathrm{tr}\,\boldsymbol{\mathcal{N}}\left(-\frac{2-\hat\nu}{3\,\hat E}\,\mathrm{tr}\,\boldsymbol{\sigma}_{\mathrm{bas}}\,\boldsymbol{\mathcal{M}}:\boldsymbol{\sigma}_{\mathrm{bas}} + \frac{1+\hat\nu}{\hat E}\,\boldsymbol{\mathcal{M}}:\boldsymbol{\sigma}^2_{\mathrm{bas}}\right)\\
&- \frac{\eta^2}{9}\,u\ \ \mathrm{tr}\,\boldsymbol{\mathcal{M}}\ \ \mathrm{tr}\,\boldsymbol{\mathcal{N}}\,.
\end{aligned}
\tag{123}
$$

Similarly, the strain-based hardening parameter $\bar H$ may be obtained either from eqn (9b), as $\bar H{=}H{+}\mathbf{n}{:}\ \mathbb{E}{:}\mathbf{m}$, or from eqn (9a), through a dual strain-based derivation (Carol et al., 1994):

$$
\bar H = -\left.\frac{\partial F}{\partial\lambda}\right|_{\boldsymbol\epsilon} = \frac{\partial r}{\partial\lambda} + \bar H_f\,;\quad \bar H_f = -\left.\frac{\partial f}{\partial\lambda}\right|_{\boldsymbol\epsilon} = -\boldsymbol{\mathcal{N}}:\left.\frac{\partial(-\boldsymbol{\mathcal{Y}})}{\partial\mathbf{L}}\right|_{\boldsymbol\epsilon}:\boldsymbol{\mathcal{M}} = \bar H^{\mathrm{bas}}_f + \bar H^\star_f\,;
\tag{124}
$$

$$
\begin{aligned}
\bar H^{\mathrm{bas}}_f = &\ \frac{\hat\Lambda}{4}\ \Big((\boldsymbol{\mathcal{N}}:\boldsymbol{\epsilon}_{\mathrm{bas}})\ (\boldsymbol{\mathcal{M}}:\boldsymbol{\epsilon}_{\mathrm{bas}}) + \mathrm{tr}\,\boldsymbol{\epsilon}_{\mathrm{bas}}\ \ \mathrm{tr}\,(\boldsymbol{\mathcal{N}}\cdot\boldsymbol{\epsilon}_{\mathrm{bas}}\cdot\boldsymbol{\mathcal{M}})\Big)\\
&+\ \hat G\ \ \mathrm{tr}\,(\boldsymbol{\mathcal{N}}\cdot\boldsymbol{\epsilon}^2_{\mathrm{bas}}\cdot\boldsymbol{\mathcal{M}})\,;
\end{aligned}
\tag{125}
$$

$$
\begin{aligned}
\bar H^\star_f = &\ \frac{\eta}{6}\,\mathrm{tr}\,\boldsymbol{\mathcal{M}}\left(-\frac{3\,\hat\Lambda+4\,\hat G}{3}\,\mathrm{tr}\,\boldsymbol{\epsilon}_{\mathrm{bas}}\ \boldsymbol{\mathcal{N}}:\boldsymbol{\epsilon}_{\mathrm{bas}} + 2\,\hat G\ \boldsymbol{\mathcal{N}}:\boldsymbol{\epsilon}^2_{\mathrm{bas}}\right)\\
&+ \frac{\eta}{6}\,\mathrm{tr}\,\boldsymbol{\mathcal{N}}\left(-\frac{3\,\hat\Lambda+4\,\hat G}{3}\,\mathrm{tr}\,\boldsymbol{\epsilon}_{\mathrm{bas}}\,\boldsymbol{\mathcal{M}}:\boldsymbol{\epsilon}_{\mathrm{bas}} + 2\,\hat G\,\boldsymbol{\mathcal{M}}:\boldsymbol{\epsilon}^2_{\mathrm{bas}}\right)\\
&+ \frac{\eta^2}{9}\,u\ \ \mathrm{tr}\,\boldsymbol{\mathcal{M}}\ \ \mathrm{tr}\,\boldsymbol{\mathcal{N}}\,.
\end{aligned}
\tag{126}
$$

The interchange of roles between the hatted elastic parameters and between the intermediate 'basic' stress and strain in the relations of H and $\bar H$ may be read directly through the correspondences in the dual expressions of $\boldsymbol{\sigma}_{\mathrm{bas}}{\cdot}\boldsymbol{\epsilon}_{\mathrm{bas}}/2$ in eqn (96).

In sum, the tangent stiffness $\mathbb{E}_{\mathrm{tan}}$ of the 'extended' model is obtained by substituting in eqn (8c) expressions (115)-(117), (118)-(120) and secant stiffness $\mathbb{E}$, eqn (80b), in the numerator and hardening parameter (124)-(126) in the denominator.

5 Concluding remarks

An 'extended' formulation of anisotropic elastic damage in initially-isotropic materials has been presented. It is based on symmetric second-order damage tensors and on the underlying volumetric/deviatoric decomposition of the isotropic reference stiffness and compliance. This new formulation includes as particular cases the 'extended' isotropic version of the model and the previously-developed 'basic' model.

A major new feature of the model is the introduction of a path-parameter η allowing to assign different weights to bulk and shear damage components. Additionally, two isotropic elastic constants and three evolving eigenvalues of the second-order damage tensor form a total of six parameters available to characterize the nine engineering material parameters entering the secant stiffness and compliance relations. The latter correspond to a restricted form of orthotropic material symmetry. Complementary to that, a loading function and hardening/softening function must also be prescribed to complete an associative formulation, otherwise a damage evolution rule must also be provided separately.

A second important characteristic of the model are the underlying product-type decompositions of damage and damage-effect tensors in isotropic and anisotropic parts, which reflect themselves in a similar decomposition of effects of the consequent secant moduli. The product decompositions also allow easy inversion of the tensors involved, which preserve a dual structure between stiffness- and compliance-based versions of the formulation. A pseudo-logarithmic rate of damage is also introduced as in the 'basic' formulation and allows to derive a convenient thermodynamic force useful to base loading functions and damage evolution rules. This allows to interpret the present 'extended' formulation as an update of the 'basic' one.

In this respect, although the present framework has not been finalized to the level of detailing a real constitutive model and accomplishing its implementation in a constitutive driver, the new features presented here are crucial to direct the subsequent development of the constitutive model on many important aspects as e.g. that of selecting the most appropriate isotropic reference moduli to work with, that of seeking loading and hardening/softening functions describing reasonable responses of the model for simple loading cases (e.g. uniaxial tension, pure shear, etc.), that of possibly expressing the model response in analytic form in terms of parameters with clear physical meaning (e.g. peak strength and specific fracture-energy), that of considering the different behaviors in tension/compression, and so on, namely all the aspects that have been previously developed for the 'basic' formulation. These issues are at present the concern of further investigation.

There are close analogies of the present formulation of elastic damage to the affine mappings in finite deformation analysis. Indeed, the relationship between damage and geometrical concepts is not surprising, if the interpretation of damage as reduction of stress-carrying area is considered (e.g. Murakami, 1988; Park and Voyiadjis, 1998; Steinmann and Carol, 1998). According to the product-type decompositions, the various intermediate stress/strain quantities introduced here define stress and strain acting on different intermediate configurations that involve isotropic material behavior (Fig. 8). Taking the effective as the reference configuration, they might be interpreted formally as the result of a 'pull-back' operation from the nominal configuration or of a 'push-forward' operation from the effective configuration. Furthermore, the pseudo-logarithmic rate of damage has close analogy to the rate of deformation tensor in finite deformations, and so does the thermodynamic force conjugate to such damage rate to the corresponding stress measure. The formal and practical analogies between these two settings have not been the focus of the present paper but certainly could be the subject of future elaborations.

Also, more general forms of orthotropic degradation based on two coaxial damage tensors for the bulk and shear moduli or on alternative propositions of the damage-effect tensors could be explored, still with the requirement to achieve dual derivations for the stress- and strain-based formulations.

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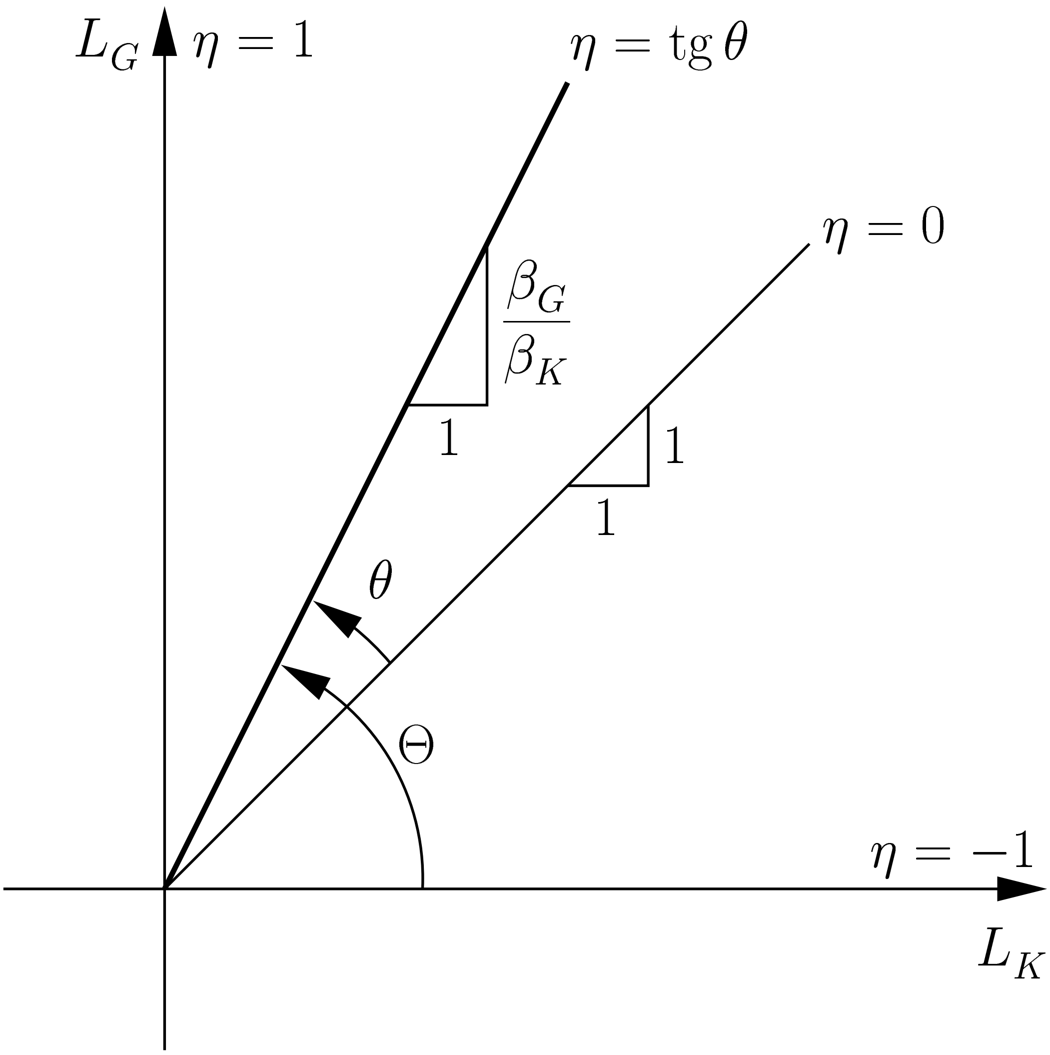

Figure 1: Single-dissipative straight paths in the plane of logarithmic damage variables L_K and L_G. The slope of the linear paths is defined by constant parameter $-1 \leq \eta \leq 1$, where $\eta = (\beta_G - \beta_K)/2$. The 'basic' formulation corresponds to $\eta = 0$ ($\beta_K = \beta_G = 1$), while pure volumetric and pure deviatoric damage are obtained respectively for $\eta = -1$ ($\beta_K = 2$, $\beta_G = 0$) and $\eta = 1$ ($\beta_K = 0$, $\beta_G = 2$).

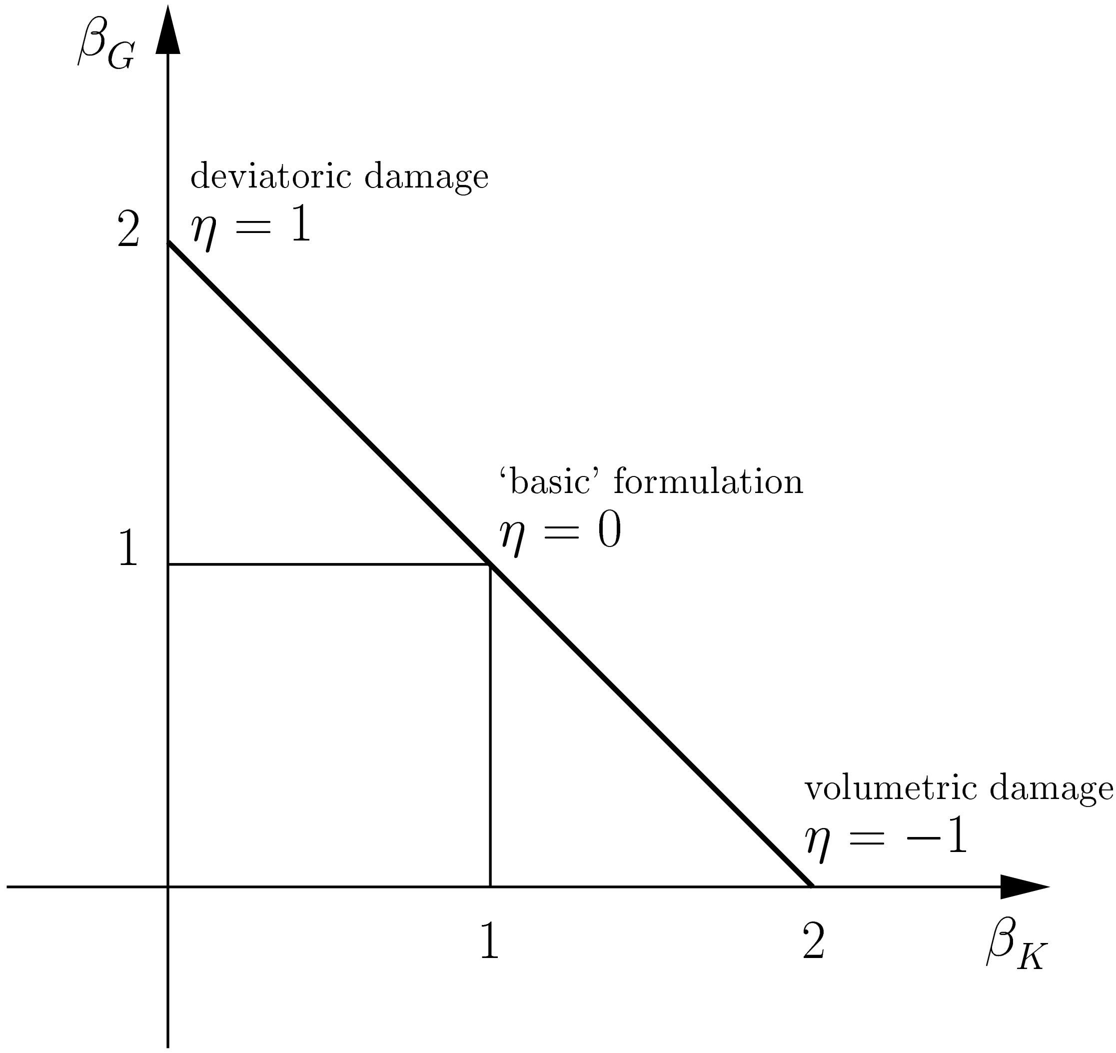

Figure 2: Relation between constants β_K=1−η, β_G=1+η and parameter −1≤η≤1. Parameters β_K and β_G satisfy the constraint $\beta_K+\beta_G=2$. The 'basic' formulation corresponds to η=0 ($\beta_K=\beta_G=1$), while pure volumetric and pure deviatoric damage are obtained respectively for $\eta=-1$ (β_K=2, β_G=0) and η=1 (β_K=0, β_G=2).

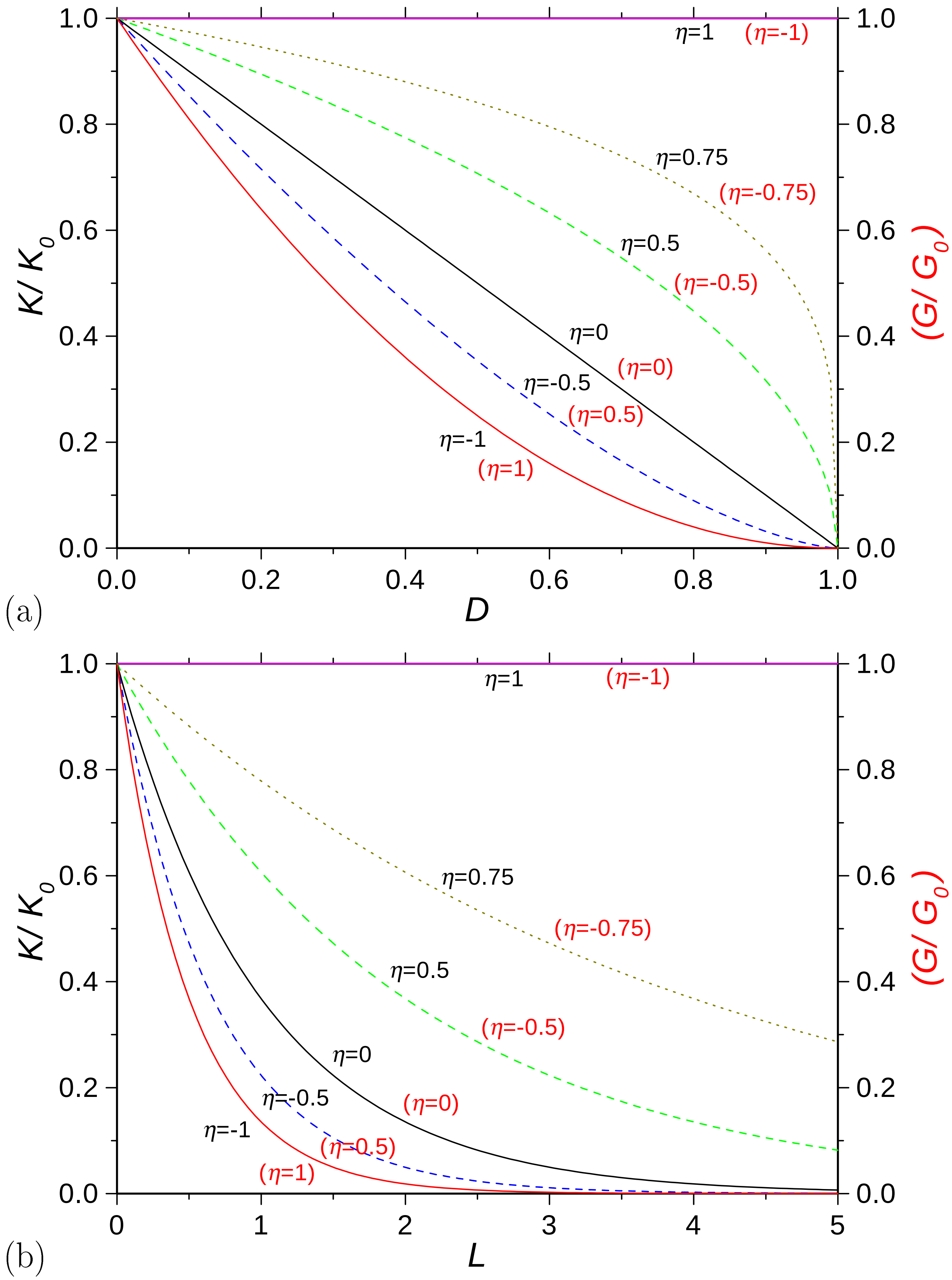

Figure 3: Variation of bulk modulus $K/K_0{=}(1{-}D)^{1-\eta}{=}\mathrm{e}^{-(1-\eta)L}$ (left axis) and shear modulus $G/G_0{=}(1{-}D)^{1+\eta}{=}\mathrm{e}^{-(1+\eta)L}$ (right axis) with respect to (a) scalar damage variable $D{=}1{-}\mathrm{e}^{-L}$ and (b) logarithmic damage variable $L{=}{-}\ln(1{-}D)$, for different values of parameter $-1{\le}\eta{\le}1$ ($\eta{=}1$: pure deviatoric damage; $\eta{=}0$: 'basic' or '$(1-D)$' damage; $\eta{=}{-}1$: pure volumetric damage).

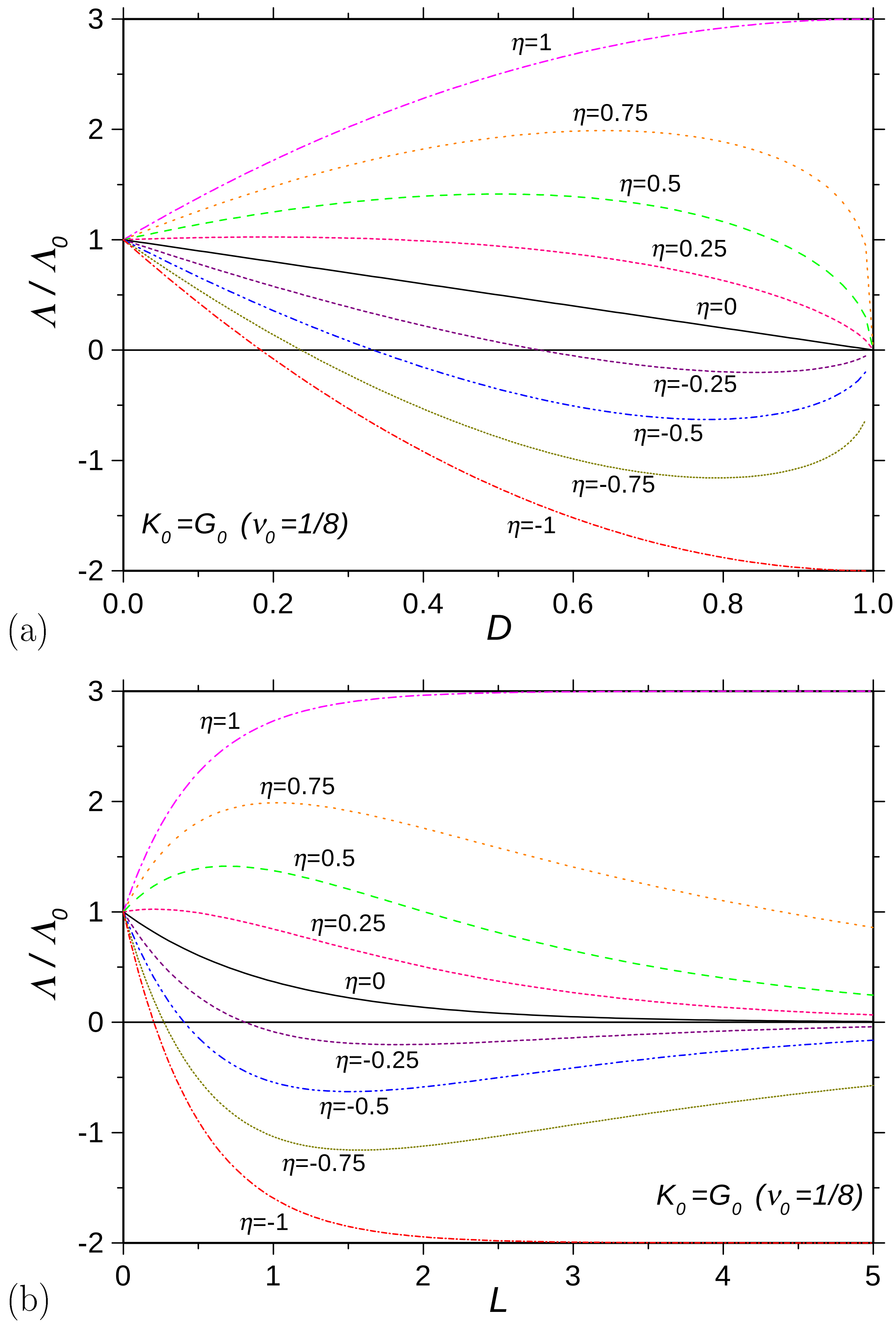

Figure 4: Variation of Lamé's constant Λ/Λ_0 with respect to (a) scalar damage variable $D{=}1{-}\mathrm{e}^{-L}$, eqn (56c), and (b) logarithmic damage variable $L{=}{-}\ln(1{-}D)$, for different values of parameter $-1{\leq}\eta{\leq}1$ ($\eta{=}1$: pure deviatoric damage; $\eta{=}0$: 'basic' or '$(1-D)$' damage; $\eta{=}{-}1$: pure volumetric damage); $K_0{=}G_0{=}3\,\Lambda_0{=}4/9\,E_0$, $\nu_0{=}0.125$.

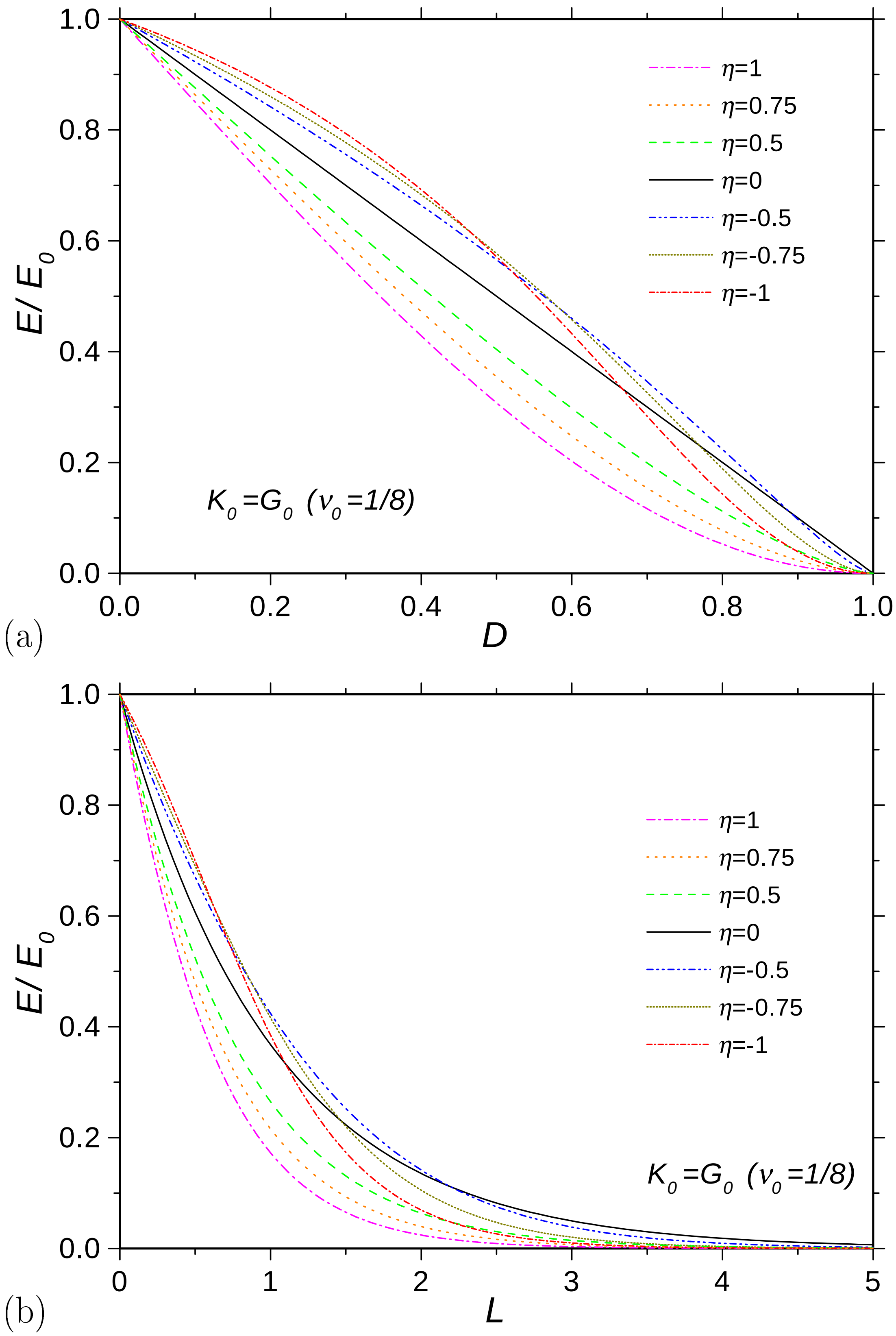

Figure 5: Variation of Young's modulus E/E_0 with respect to (a) scalar damage variable $D{=}1{-}\mathrm{e}^{-L}$, eqn (56d), and (b) logarithmic damage variable $L{=}{-}\ln(1{-}D)$, for different values of parameter $-1{\leq}\eta{\leq}1$ ($\eta{=}1$: pure deviatoric damage; $\eta{=}0$: 'basic' or '$(1-D)$' damage; $\eta{=}{-}1$: pure volumetric damage); $K_0{=}G_0{=}3\,\Lambda_0{=}4/9\,E_0$, $\nu_0{=}0.125$.

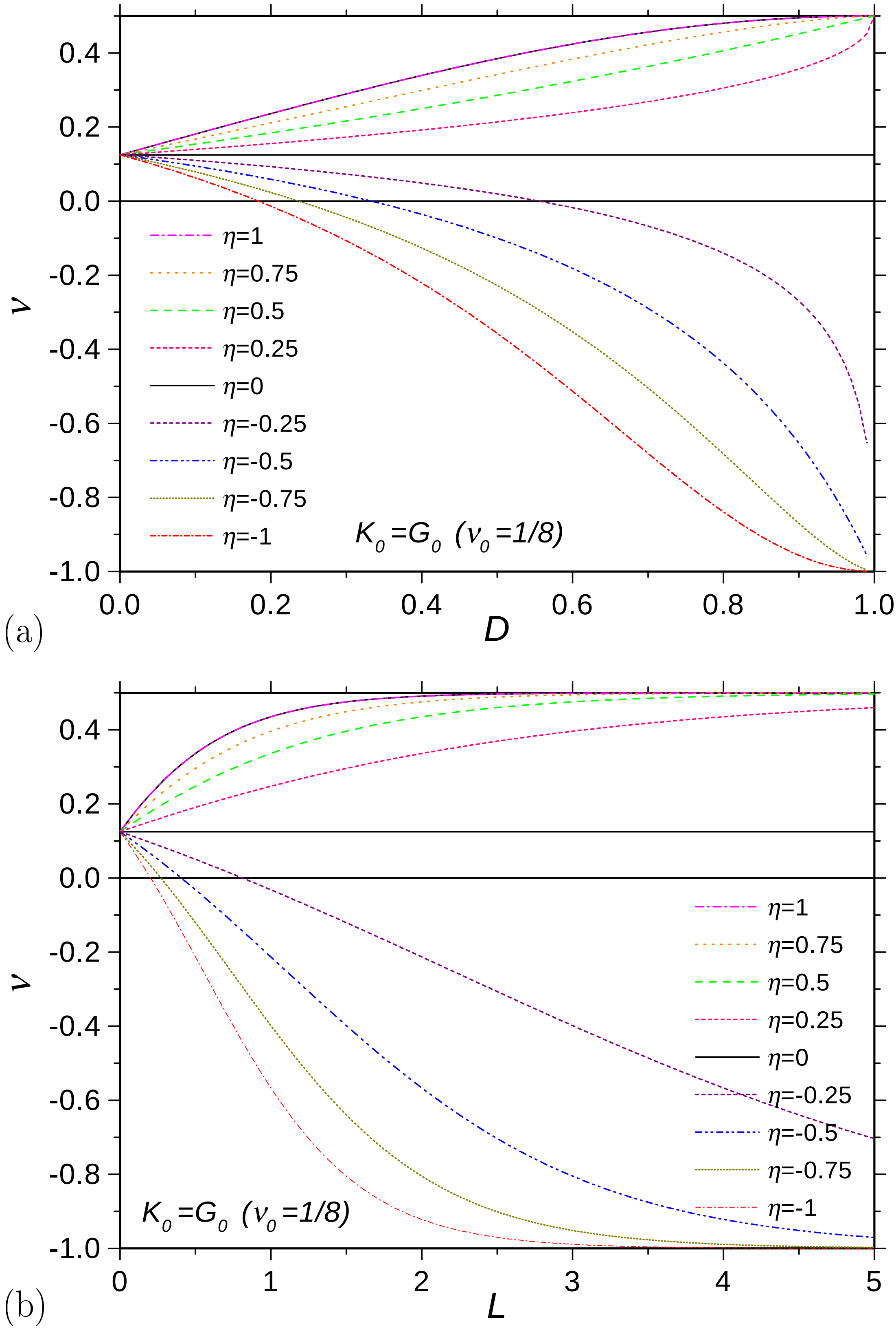

Figure 6: Variation of Poisson's ratio ν with respect to (a) scalar damage variable $D{=}1{-}\mathrm{e}^{-L}$, eqn (56e), and (b) logarithmic damage variable $L{=}{-}\ln(1{-}D)$, for different values of parameter $-1{\leq}\eta{\leq}1$ ($\eta{=}1$: pure deviatoric damage; $\eta{=}0$: 'basic' or '$(1-D)$' damage; $\eta{=}{-}1$: pure volumetric damage); $K_0{=}G_0{=}3\,\Lambda_0{=}4/9\,E_0$, $\nu_0{=}0.125$.

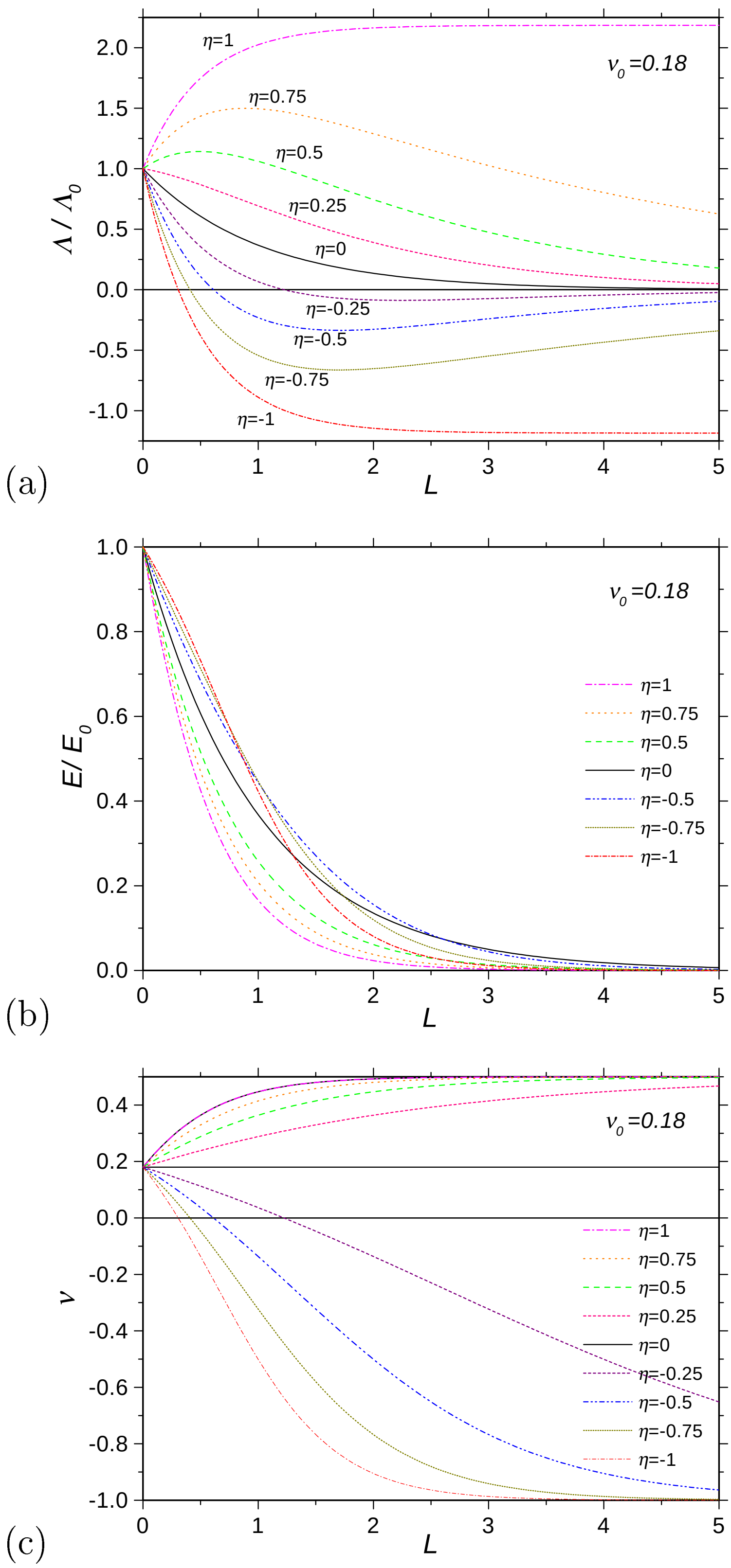

Figure 7: Variation of (a) Lamé's constant Λ/Λ_0, (b) Young's modulus E/E_0 and Poisson's ratio ν with respect to logarithmic damage variable $L{=}{-}\ln(1{-}D)$, for different values of parameter $-1{\leq}\eta{\leq}1$ ($\eta{=}1$: pure deviatoric damage; $\eta{=}0$: 'basic' or '$(1-D)$' damage; $\eta{=}{-}1$: pure volumetric damage); $\nu_0{=}0.18$.

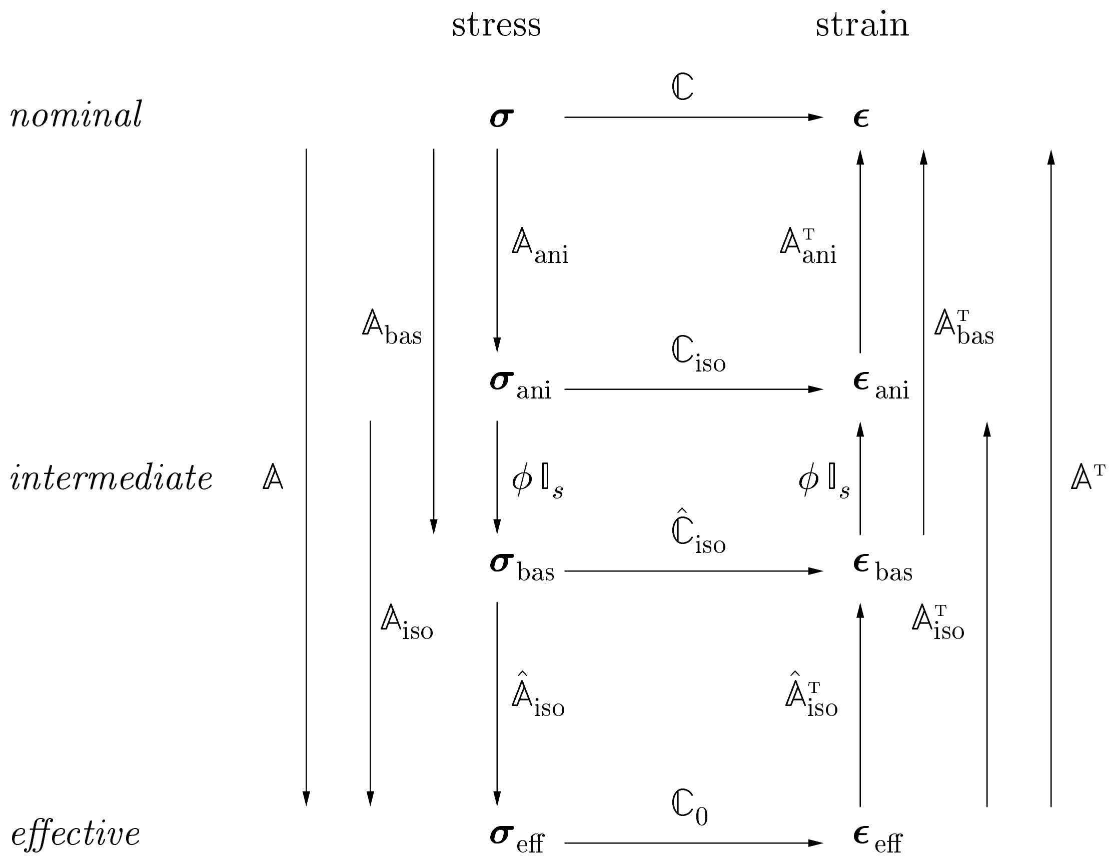

Figure 8: Compliance scheme of the stress to strain relations involving nominal, intermediate and effective stress and strain quantities. A dual stiffness scheme of the opposite strain to stress relations would apply with reverse arrows and the various damage-effect tensors $\mathbb{A}$ replaced by $\bar{\mathbb{A}}=\mathbb{A}^{-1}$.