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On the optimal tuning of Tuned Mass Dampers in structural systems
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ABSTRACT

This note deals with the optimal tuning of the free parameters of the so-called Tuned Mass Dampers (TMDs), devices which are conceived to be attached to structural systems in order to reduce vibrations induced by external actions. The main point of view and application perspective here is that of civil and seismic engineering, though the basic TMD concept arises in mechanical engineering [1]. A TMD is a device in which an additional auxiliary mass is attached to the primary mass of the structural element by a spring or a spring/dashpot element to counterbalance the vibrations of the primary system. The main tuning parameters of the device are the mass, frequency and damping coefficient ratios between the parameters of the auxiliary mass and those of the primary system. The paper reports the initial results of an on-going research at the University of Bergamo [2].

First, the basic concepts of classical Den Hartog's tuning [1] are briefly reviewed and analysed systematically in the general context of damping of the main system and of the TMD device, both for harmonic excitation at the principal mass and at the support. Analytical results are summarised in table form, whereby the different explicit expressions of the normalised amplitude x_1/x_{st} of the primary mass oscillation are given as a function of the various parameters of the system. Some 2D and 3D maps are also provided to show the role of the tuning parameters.

Optimal tuning is then investigated numerically on a SDOF + TMD system by a minimax procedure. Comparisons to the fundamental tuning by Den Hartog are provided and the effect of structural damping is explored. Results are also compared to classical contributions from the literature [3,4]. Useful abacuses are also compiled in the spirit of [4], for easy access to the best tuning.

The dynamical response of a 10-storey shear-type building with a TMD added on top subjected to El Centro input ground motion is then evaluated numerically. Results display the considerable reduction of the top floor displacement after the insertion of the TMD. Plots are presented together with a resuming table showing the percentage of vibration reduction as a function of the damping involved.

This contribution attempts a preliminary study on the usefulness of TMD devices in the field of seismic engineering. Further results on tuning at seismic input should help in clearing if TMDs could work effectively for seismic isolation. Detailed practical applications are also left for future work.

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ON THE OPTIMAL TUNING OF TUNED MASS DAMPERS IN STRUCTURAL SYSTEMS

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Keywords: Tuned Mass Damper (TMD), optimal tuning, minimax procedure, abacus of optimal parameters, control of structural vibration, seismic response.

Abstract. *This note deals with the optimal tuning of the free parameters of the so-called Tuned Mass Dampers (TMDs), devices which are conceived to be attached to structural systems in order to reduce the vibrations induced by external actions. First, the basic concepts of classical Den Hartog's tuning are briefly reviewed and analysed systematically in the general context of damping of the main system and of the TMD device, both for harmonic excitation at the principal mass or at the support. Analytical results are summarised in table form, whereby the different explicit expressions of the normalised amplitude* x_1/x_{st} *of the primary mass oscillation are given as a function of the various system parameters. Some 2D and 3D maps are also provided to show the role played by the tuning parameters. Optimal tuning is then investigated numerically on a SDOF + TMD system by a minimax procedure. Comparisons to the fundamental tuning by Den Hartog are provided and the effect of structural damping is explored. Useful abacuses are also compiled, for easier access to the best tuning. The dynamical response of a 10-storey shear-type building with a TMD added on top, subjected to El Centro input ground motion is then evaluated numerically. Results show and quantify the reduction of the top-floor displacement after the insertion of the TMD.*

1 INTRODUCTION

This paper considers the task of finding the best tuning of the free parameters of the Tuned Mass Dampers (TMDs). These devices are conceived to be attached to structural elements or systems in view of containing the structural vibrations that may be induced by the presence of dynamical external actions. The main point of view and application perspective here is that of civil and seismic engineering, though the basic TMD concept seems to arise in naval and mechanical engineering [1]. A TMD is a device in which an additional, auxiliary mass is attached to the primary mass of a structural element by a spring or a spring/dashpot element, to counterbalance the vibrations of the primary system. The main tuning parameters of the device are the mass, frequency and damping coefficient ratios between the parameters of the auxiliary mass and those of the primary system. The paper reports the preliminary results of an on-going research investigation at the University of Bergamo [2].

According to Den Hartog [1], the original concept of the TMD absorber was introduced by Frahm in 1909, with the goal of reducing the rolling vibrations in ships. Being conceived without any inherent damping, its effectiveness holds true under resonance conditions, when the frequency of the external action approaches the natural frequency of the system, within the so-called *"operating range"* of the TMD device. On the contrary, the absorber does not work properly when the forcing frequency starts to deviate from the natural frequency of the structure and reaches the resonant frequencies of the system with added TMD, leading theoretically to unbounded oscillations. Due to these limitations, Ormondroyd and Den Hartog in 1928 have shown that, by introducing some damping in the TMD device, the response of the Frahm's absorber improves at variable frequency of the external action. Later, the concept was further analysed and presented systematically and codified in Den Hartog's classical text on Mechanical Vibrations [1]. There, explicit formulas were derived for the optimal tuning in the presence of harmonic excitation on the primary system, which was assumed undamped. Implicitly, it was thought that inherent damping of the system would just provide additional help in the effectiveness of the device. Further studies have then inspected the role of the inherent damping of the structural system, which no longer allows closed-form derivations as in Den Hartog's case and needs either numerical or approximate analytical approaches to locate the best tuning (see e.g. Rana and Soong [3] and Ghosh and Basu [4], respectively). The present paper attempts a contribution along these lines.

A number of useful works in the literature have inquired the use of TMD devices in different structural contexts, including that of civil and seismic engineering, where the dynamical external actions easily go beyond the application of a single harmonic excitation [5-8]. A brief review of previous work in the field is presented e.g. by Sadek et al. [8], which show that different conclusions on the effectiveness of the devices in controlling seismic vibrations have been obtained. Some of them appear rather skeptical. For example, Sladek and Klingner [5], by adopting Den Hartog's tuning concept on a TMD located on top of a 25-storey building subjected to El Centro ground motion, have concluded that the TMD turned-out not that useful in taming the response of the structure. Different studies on the effectiveness of TMD devices in seismic applications have also been conducted by Villaverde and coworkers, e.g. Villaverde [6] and Villaverde and Koyama [7]. They claim that the optimal TMD parameters are obtained when the damping factors of the first two complex modes of vibration of the combined system structure + TMD are nearly equal to the average of the separate damping ratios of the structure and of the TMD. It is observed that the use of a TMD in conditions of resonance with the main structure may introduce additional damping and thus reduce the seismic response of a building. The method proved successful in various 2D and 3D numerical and experimental analyses, under different ground movements. A relevant contribution on the applicability of passive TMD devices in seismic engineering is that of Sadek et al. [8]. These authors present an improvement of the analysis of Villaverde [6] by considering both SDOF and MDOF structural systems and present useful plots of the tuning parameters as a function of mass ratio. They conclude that the best tuning is obtained when the first two modes of the structure with TMD will have equal damping ratios, which are greater of the average of inherent and TMD damping ratios. With this, it is claimed that the amplitude in displacement and acceleration responses may reduce significantly, even up to 50%. Thus, Sadek et al. [8] seem rather in favour of the use of TMD devices for taming the seismic response.

In sum, it appears that the literature reveals quite a controversial point of view on the real effectiveness of passive TMD devices in reducing the seismic response of structural systems, whereas this seems to hold true for harmonic excitations and also for other external actions, like for instance wind loadings. This obviously motivates further investigations on the subject. In this direction, the present paper also attempts a numerical simulation of the seismic response of a prototype 10-storey building, which seems to show the usefulness of the device in the present case. The difficult task is obviously that of getting the most appropriate tuning for the selected structure and for the type of seismic input to be expected. This also calls for the concept of active or semi-active TMDs, which are able to change the tuning parameters depending on these characteristics, a concept that is also much investigated in the recent literature (not in this context).

The present paper is organised as follows. First, in Section 2 the basic concepts of classical Den Hartog's tuning [1] are briefly reviewed and analysed systematically in the more general context of damping of the main system and of the TMD device, both for harmonic excitation at the principal mass or at the support. Analytical results are presented in table form, whereby the different explicit expressions of the normalised amplitude x_1/x_{st} of the primary mass oscillation are given as a function of the system's parameters. Some 2D and 3D maps are also provided to show the role of the tuning parameters. Then, in Section 3 the optimal tuning is investigated numerically on a SDOF + TMD system by the implementation of a *minimax* procedure. Comparisons to the fundamental tuning by Den Hartog are provided and the effect of structural damping is explored. Results are presented in plot form where the optimal parameters are depicted as a function of mass ratio, as provided by classical contributions in the literature [5,8]. Useful abacuses are also compiled in the spirit of [3,4,8], for easy access to the best tuning. Finally, in Section 4 the dynamical response of a 10-storey shear-type building with a TMD added on top, subjected to El Centro input ground motion is evaluated numerically. Results display and quantify the reduction of the top-floor displacement after the insertion of the TMD. Plots are presented together with a resuming table showing the percentage of vibration reduction in terms of top-floor displacement, velocity and acceleration, as a function of structural damping. At end, closing Section 5 outlines some first conclusions on the preliminary results obtained here.

2 DEN HARTOG'S TMD CONCEPT AND RELEVANT TUNING

The TMD concept may be illustrated by an elastic SDOF system with primary stiffness and mass K, M, with an attached TMD of secondary stiffness and mass k, m (see Scheme 1 in following Table 1). The secondary mass is usually interpreted as a fraction of the primary mass, through the mass ratio $\mu = m / M$. In the classical analysis by Den Hartog [1], an harmonic force $P_0 \sin \omega t$ with angular frequency ω is let to act on the primary mass. The input frequency ω is confronted to the natural angular frequency of the primary system $\Omega_n = \sqrt{(K / M)}$. Also, in view of reaching the best tuning, the natural angular frequency of the absorber $\omega_a = \sqrt{(k / m)}$ is confronted as well to the previous two. Thus, the two frequency ratios

$f = \omega_a / \Omega_n$ and $g = \omega / \Omega_n$ are naturally defined to interpret the functioning of the system. Typically, the idea is to damp the vibrations of the primary mass when the system is near the condition of resonance $\omega = \Omega_n$, thus for g near 1. The value $f = 1$ is also often assumed, though not necessarily leading to the best tuning. The insertion of the TMD induces two resonant frequencies in the obtained undamped 2-DOF system, which move apart from the singular original resonant frequency and make a fork of operating range for the frequency ratios g around the primary resonance condition (Fig. 1). Indeed, by writing the equations of motions of the 2-DOF system and looking for the particular harmonic solution, one gets the displacement amplitude x_1 normalised to the static displacement of the primary mass $x_{st} = P_0 / K$ (Table 1), which become unbounded for values of g that set to zero the denominator of x_1 / x_{st}.

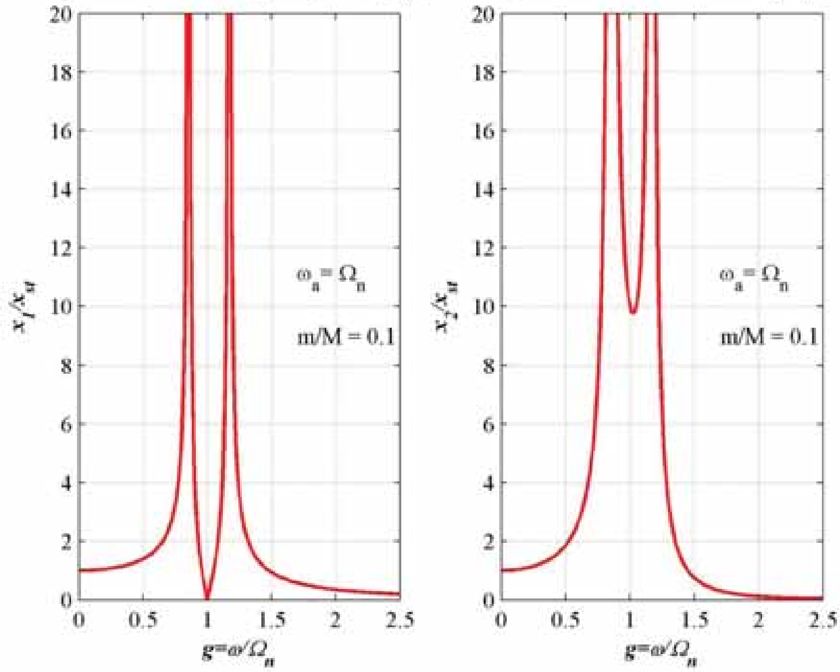


Figure 1: Amplification factor of the undamped absolute displacements x_1, x_2 as a function of g ($f = 1$, $\mu = 0.1$).

External damping can then be introduced into the TMD, which, according to [6-8] is really crucial in increasing the overall damping of the system. This corresponds to the insertion of a dash-pot element between the two masses (Scheme 2 in Table 1), with damping coefficient c and damping factor that is either related to critical damping $c_c = 2\, m\, \Omega_n$ or $\tilde{c}_c = 2\, m\, \omega_a$, both often used in the literature (notice that $\tilde{c}_c = f\, c_c$). This leads to the classical fixed-point theory of nodes P and Q (Fig. 2), which are common to all displacement curves regardless of the value of the damping ratio. According to Den Hartog [1], the best tuning of the TMD is reached, in closed-form, for $f_{opt} = 1 / (1+\mu)$, when the nodes P and Q get at the same quote. Correspondingly, the best tuning of the damping coefficient c is that which locates the two maxima in the displacement response curve at almost the same height and around nodes P and Q themselves. Precisely, Den Hartog proposes the average $(c / \tilde{c}_c)_{opt} = \sqrt{(3/8 \cdot \mu / (1+\mu))}$ of the values that lead separately to the stationary condition reached either in P or in Q. The common height of the nodes may also be estimated at best tuning as $(x_1 / x_{st})_{opt} = \sqrt{(1 + 2 / \mu)}$.

As a final systematic development on this line of thought, the damping ratio C/C_c of the primary system, always present in real structures, can be introduced and, following the proposal by Rana and Soong [3], analysed as well for harmonic excitation at the support (according to a flag δ that is equal to 0 for action on the primary mass and to 1 for excitation at the support). This compiles the additional cases reported in Table 1. A map of the amplification factor of the displacement of the primary mass as a function of frequency ratios *f* and *g* is also reported, at no structural damping, in Fig. 3, as a representative example of the system's behaviour. When inherent damping C/C_c is introduced, the closed-form solution for the tuning of *f* is no longer available. Ghosh and Basu [4] have developed an approximate closed-form derivation of f_{opt}, which holds on the hypothesis that points *P* and *Q* are still almost fixed at variable damping. Another useful approach is that of numerical optimization procedures, as put forward in [3], which is further developed and discussed in the next section.

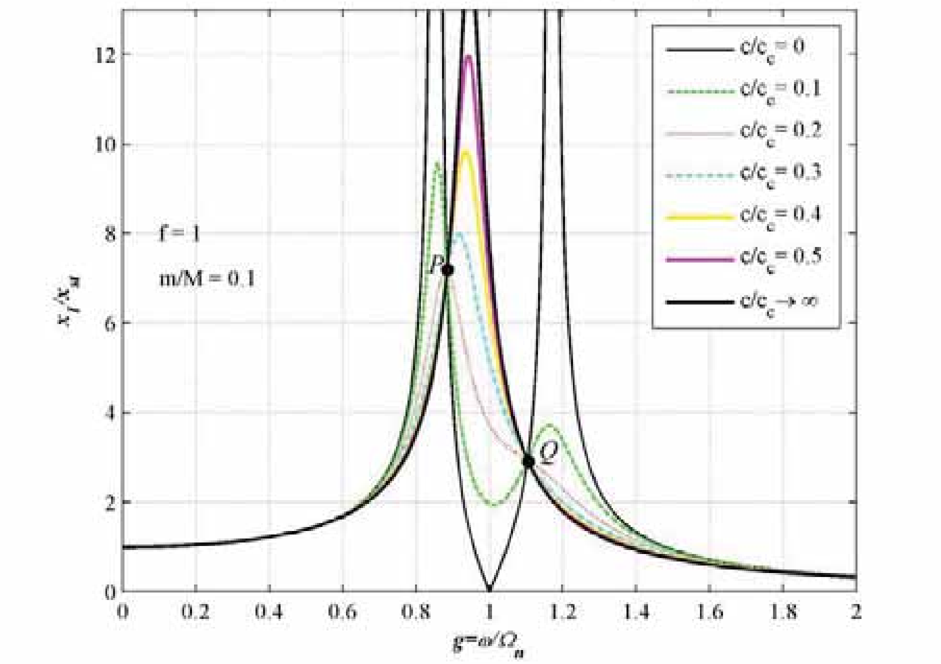


Figure 2: Amplification factor of the absolute displacement of the primary mass x_1 as a function of forcing frequency ratio g, for various damping ratios c/c_c ($f = 1$, $\mu = 0.1$).

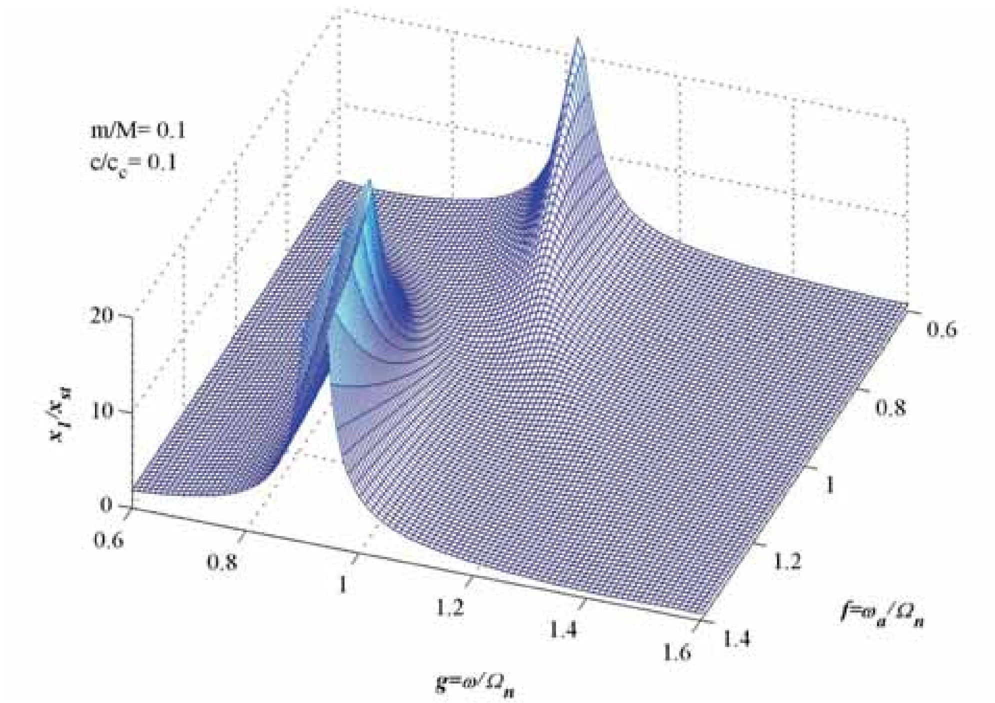
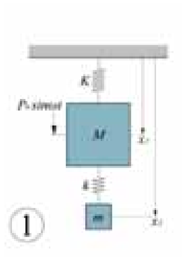
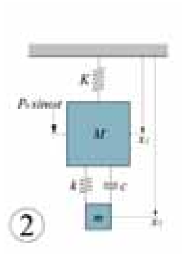
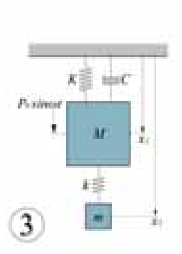
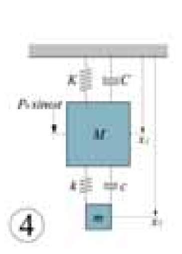
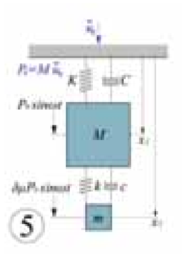


Figure 3: Amplification factor x_1/x_{st} for excitation at the support, $\delta = 1$ ($\mu = 0.1$, $c/c_c = 0.1$, $C/C_c = 0$).

Table 1: Resuming frame with the different cases of SDOF + TMD system (with definition $\tilde{c}_c = 2 \cdot m \cdot \omega_a$).

DISPLACEMENT AMPLITUDE x_1/x_{st} OF A SDOF SYSTEM WITH ADDED TMD

Schemes developed on the basis of Den Hartog's [1] and Rana and Soong's [3] analyses

Scheme	x_1/x_{st}
	$\sqrt{\dfrac{\left(g^2-f^2\right)^2}{\left[\mu \cdot f^2 \cdot g^2-\left(g^2-1\right)\cdot\left(g^2-f^2\right)\right]^2}}$
	$\sqrt{\dfrac{\left(g^2-f^2\right)^2+\left(2\cdot\frac{c}{\tilde{c}_c}\cdot f\cdot g\right)^2}{\left[\mu \cdot f^2 \cdot g^2-\left(g^2-1\right)\cdot\left(g^2-f^2\right)\right]^2+\left(2\cdot\frac{c}{\tilde{c}_c}\cdot f\cdot g\right)^2\cdot\left[\left(1+\mu\right)\cdot g^2-1\right]^2}}$
	$\sqrt{\dfrac{\left(g^2-f^2\right)^2}{\left[\mu \cdot f^2 \cdot g^2-\left(g^2-1\right)\cdot\left(g^2-f^2\right)\right]^2+\left(2\cdot\frac{C}{C_c}\cdot g\right)^2\cdot\left(g^2-f^2\right)^2}}$
	$\sqrt{\dfrac{\left(g^2-f^2\right)^2+\left(2\cdot\frac{c}{\tilde{c}_c}\cdot f\cdot g\right)^2}{\left[\mu \cdot f^2 \cdot g^2-\left(g^2-1\right)\cdot\left(g^2-f^2\right)+4\cdot g^2\cdot f\cdot\frac{C}{C_c}\cdot\frac{c}{\tilde{c}_c}\right]^2+\left(2\cdot\frac{c}{\tilde{c}_c}\cdot f\cdot g\right)^2\cdot\left[\left(1+\mu\right)\cdot g^2-1+\left(\frac{g^2-f^2}{f}\right)\cdot\frac{C}{C_c}\cdot\frac{\tilde{c}_c}{c}\right]^2}}$
	$\sqrt{\dfrac{\left(g^2-f^2-\delta\cdot\mu\cdot f^2\right)^2+\left(2\cdot\frac{c}{\tilde{c}_c}\cdot f\cdot g\right)^2\cdot\left(1+\delta\cdot\mu\right)^2}{\left[\mu \cdot f^2 \cdot g^2-\left(g^2-1\right)\cdot\left(g^2-f^2\right)+4\cdot g^2\cdot f\cdot\frac{C}{C_c}\cdot\frac{c}{\tilde{c}_c}\right]^2+\left(2\cdot\frac{c}{\tilde{c}_c}\cdot f\cdot g\right)^2\cdot\left[\left(1+\mu\right)\cdot g^2-1+\left(\frac{g^2-f^2}{f}\right)\cdot\frac{C}{C_c}\cdot\frac{\tilde{c}_c}{c}\right]^2}}$ $\delta = \begin{cases} 1 & \text{for harmonic excitation at the support} \\ 0 & \text{for harmonic force on the primary mass} \end{cases}$

System Parameters

$$x_{st}=\frac{P_0}{K},\ \omega_a=\sqrt{\frac{k}{m}},\ \Omega_n=\sqrt{\frac{K}{M}},\ \mu=\frac{m}{M},\ \tilde{c}_c=2\cdot m\cdot\omega_a,\ C_c=2\cdot M\cdot\Omega_n,\ f=\frac{\omega_a}{\Omega_n},\ g=\frac{\omega}{\Omega_n}$$

3 OPTIMAL TUNING OF THE TMD PARAMETERS

A tuning procedure is devised within the *MATLAB* environment, by using a *minimax* optimization procedure. The algorithm is based on the existing function "*fminimax*" and solves the task of minimizing the worst case, in terms of maximum values, of a set of multi-valued functions, depending on specific variables (free parameters), possibly subjected to constraint conditions. Within the present concerns, the problem might be basically stated as follows:

$$\min_{\{x\}} \max_{\{F_i\}} \left\{ F_i(\{x\}) \right\}, \quad with \quad \{l_b\} \leq \{x\} \leq \{u_b\} \tag{1}$$

where $\{x\}$ is the vector of the tuning variables, $F_i\{x\}$ are the objective functions and vectors $\{l_b\}$ and $\{u_b\}$ prescribe the lower and upper bounds of the system parameters.

In the present case, the tuning variables of the TMD device are mass ratio $\mu = m / M$, frequency ratio $f = \omega_a / \Omega_n$ and damping ratio $c / \tilde{c}_c$. Within these, the choice was made here to optimize only f and $c / \tilde{c}_c$ at given μ, with $0 \leq \mu \leq 1$, since parameter μ is most often assumed a priori based on the structural characteristics. The objective function is taken as the displacement amplitude of the primary mass.

According to Den Hartog's tuning concept (Section 2), the optimal parameters at no structural damping, $C/C_c = 0$, are found by letting fixed nodes P and Q (Fig. 2) at the same quote, leading to $f_{opt}= 1/(1+\mu)$ and $(x_1 / x_{st})_{opt}= \sqrt{(1 + 2/\mu)}$. Furthermore, $(c / \tilde{c}_c)_{opt}= \sqrt{(3/8 \cdot \mu /(1+\mu))}$ is taken as the average of the damping ratios that lead to local peak in P or in Q. However, it appears that there is no explicit analytical proof of this tuning procedure. Also, it does not apply as-is to real cases where the structural damping C/C_c is not zero. Thus, a comprehensive estimate of the best tuning parameters has been carried-out by the minimax procedure, in order to compare these results to the classical outcomes by Den Hartog's tuning. This is done first for the case $C/C_c = 0$ and further for different damping ratios of the primary system. Also, the effect of flag δ ruling the role of harmonic excitation at the primary system ($\delta = 0$) or at the support ($\delta = 1$) is investigated as well. Abacuses are generated to gather in synoptic form the obtained optimal values f_{opt} and $(c / \tilde{c}_c)_{opt}$, together with the corresponding maximum values of the primary mass displacement $(x_1 / x_{st})_{peak}$, for various values of mass ratios μ and structural damping ratios C/C_c.

3.1 Optimal tuning for harmonic excitation at the primary mass ($\delta = 0$)

Figs. 4–6 represent the achieved results in terms of the optimal parameters f_{opt}, $(c / \tilde{c}_c)_{opt}$ and resulting maximum displacement ratio $(x_1 / x_{st})_{peak}$, at no structural damping ($C/C_c = 0$), as a function of prescribed mass ratio μ. From the exam of Fig. 4 it appears that the minimax estimate of f_{opt} is practically coincident with the outcome of Den Hartog's tuning. One could say that this does provide a sort of numerical proof of the analytical formula conjectured by Den Hartog. As expected, slightly different results are instead obtained for $(c / \tilde{c}_c)_{opt}$. Indeed, Fig. 5 shows that for high values of μ larger than about *25%-30%*, some differences between the two estimates start to be appreciated. On the other hand, such difference seems to have mostly academic, rather than practical implications. Correspondingly, looking at the plots in Fig. 6, the least value of $(x_1 / x_{st})_{peak}$ is obtained by the minimax procedure but differs very slightly from Den Hartog's simple estimate based on the common quote of nodes P and Q.

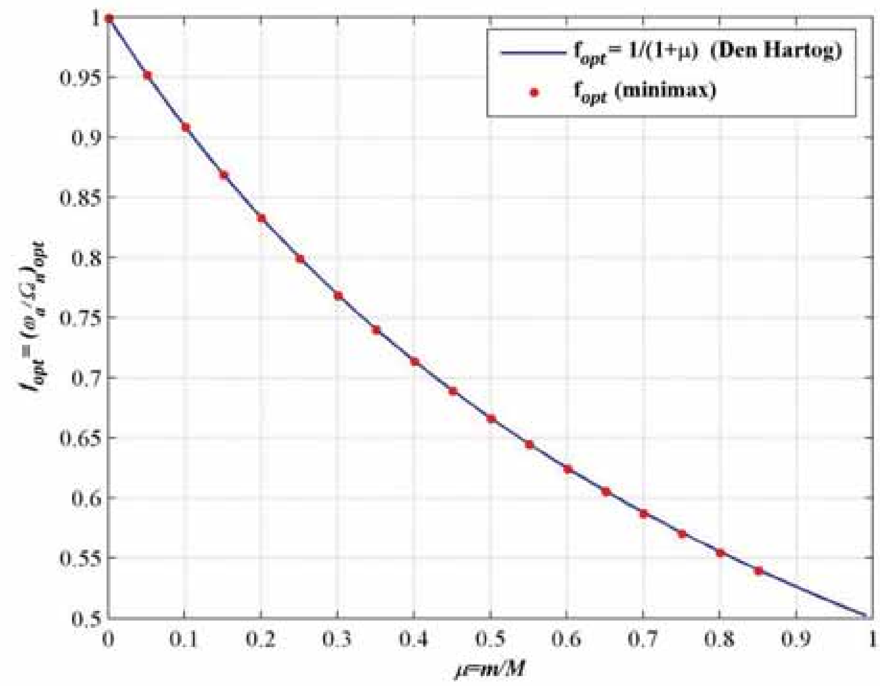


Figure 4: f_{opt} as a function of μ by a minimax procedure and comparison to Den Hartog's tuning ($C/C_c = 0$, $\delta = 0$).

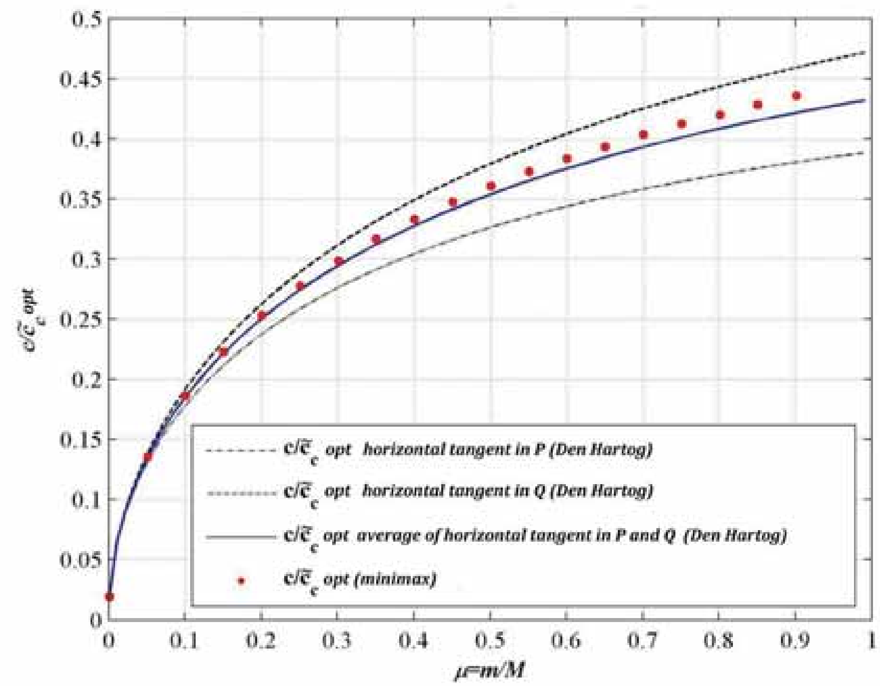


Figure 5: $\left(c/\tilde{c}_c\right)_{opt}$ as a function of μ by a minimax procedure and comparison to Den Hartog's tuning ($C/C_c = 0$, $\delta = 0$).

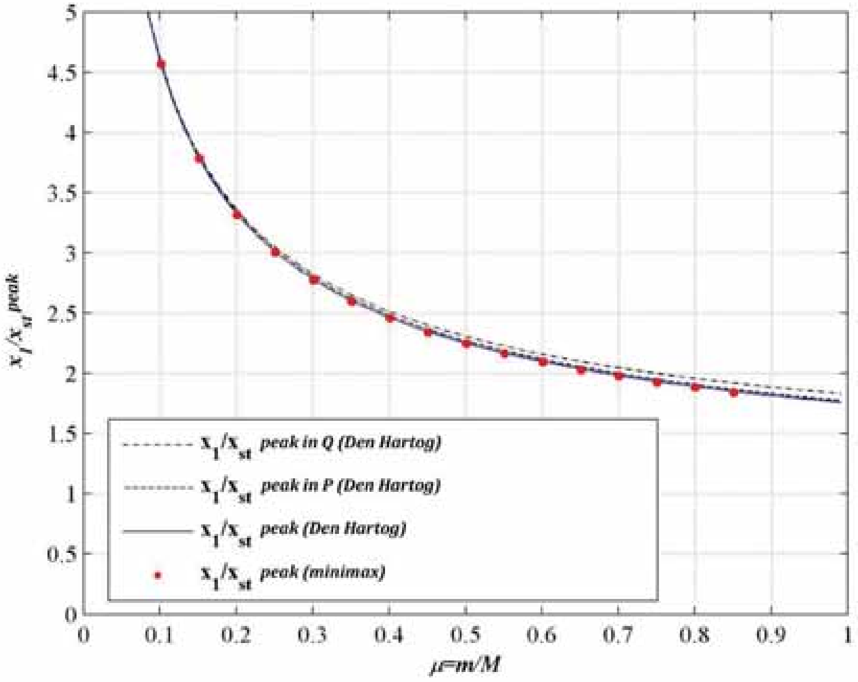


Figure 6: $\left(x_1/x_{st}\right)_{peak}$ as a function of μ by a minimax procedure and comparison to Den Hartog's tuning ($C/C_c = 0$, $\delta = 0$).

Now, further graphs corresponding to the previous ones are obtained as well in the case of structural damping, for a series of inherent damping ratios C/C_c varying between *2%* and *10%*, at step *2%* (Figs. 7–9).

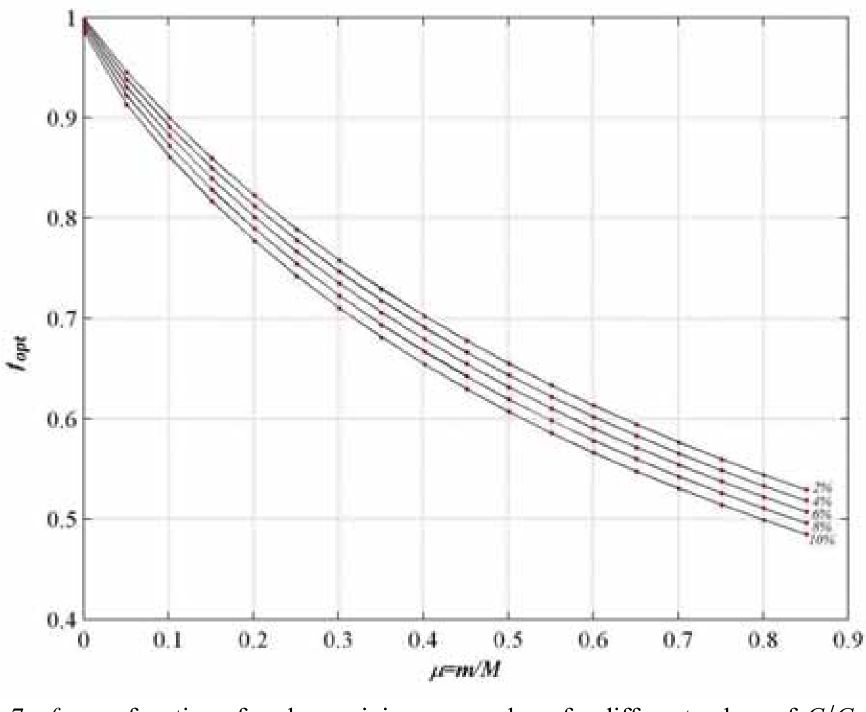


Figure 7: f_{opt} as a function of μ by a minimax procedure, for different values of C/C_c ($\delta = 0$).

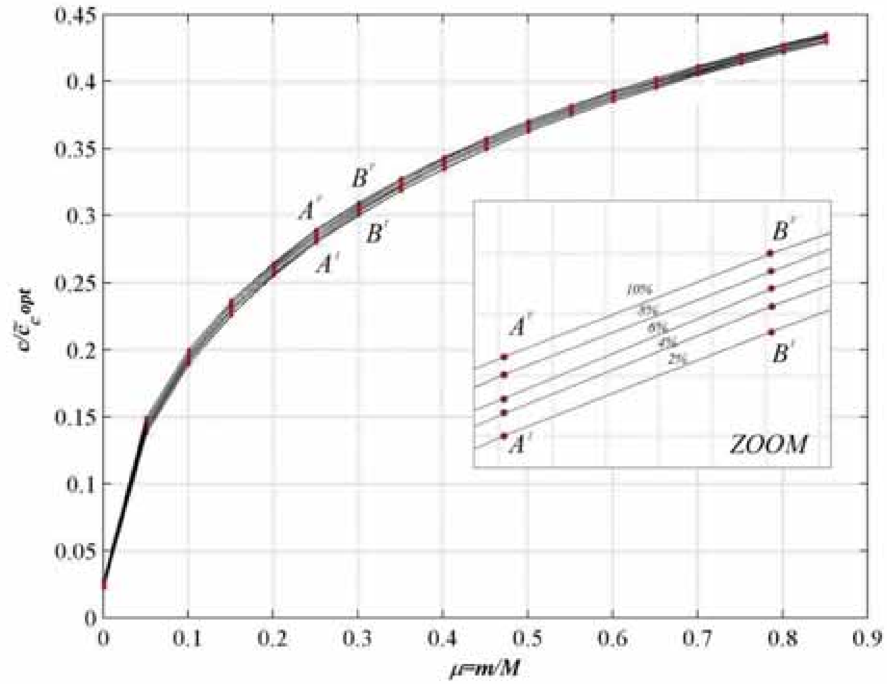


Figure 8: $\left(c/\tilde{c}_c\right)_{opt}$ as a function of μ by a minimax procedure, for different values of C/C_c ($\delta = 0$).

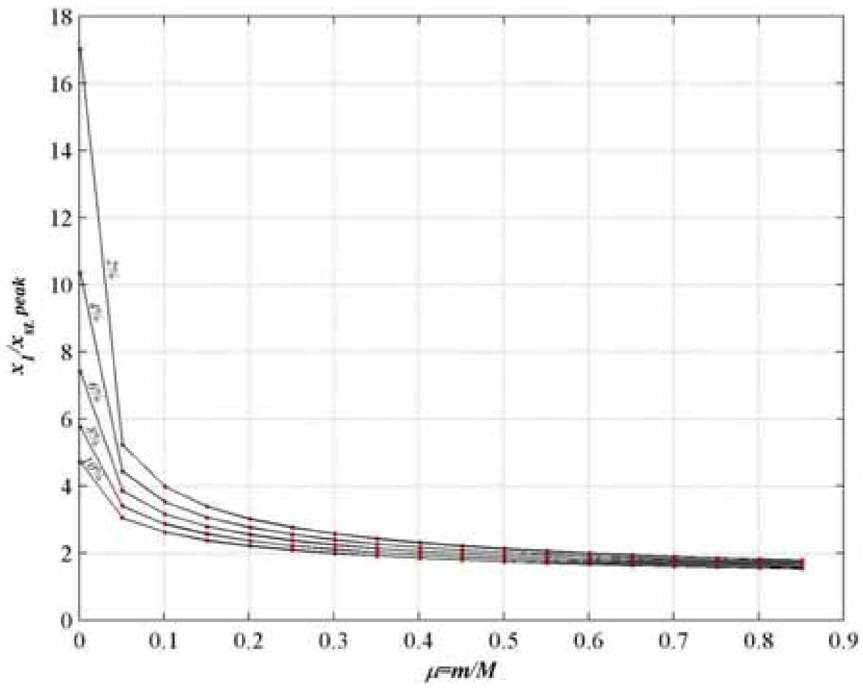


Figure 9: $\left(x_1/x_{st}\right)_{peak}$ as a function of μ by a minimax procedure, for different values of C/C_c ($\delta = 0$).

From the analysis of Figs. 7 and 9 it appears that for increasing C/C_c, at given μ, a decrease of f_{opt} and $(x_1 / x_{st})_{peak}$ is recorded. On the contrary, Fig. 8 shows that, still at given μ, $(c / \tilde{c}_c)_{opt}$ increases very slightly. Though some differences can be appreciated, the various curves appear rather near to each other, especially for the last cited case of Fig. 8, Thus, it may be concluded that the optimal tuning is weakly influenced by inherent damping C/C_c, except for the dependence of $(x_1 / x_{st})_{peak}$ in Fig. 9 at small values of μ, where the role of structural damping is more apparent in reducing the displacement amplitude at increasing C/C_c.

The following Table 2 reports an abacus that allows to conveniently get f_{opt}, $(c/\tilde{c}_c)_{opt}$ and corresponding $(x_1/x_{st})_{peak}$ at given μ and C/C_c. As shown as well by Fig. 9, the maximum displacement ratio seems to decrease asymptotically for high values of μ approaching 1. Notice that the optimization procedure really brings at the same height the two displacement peaks that appear in the plot of x_1/x_{st} as a function of g. This can be appreciated by 2-D and 3-D maps, that represent x_1/x_{st} at the achieved optimal tuning. Two samples of these are reported in Figs. 10–11 and 12–13 for extreme values in the range of the abacus parameters.

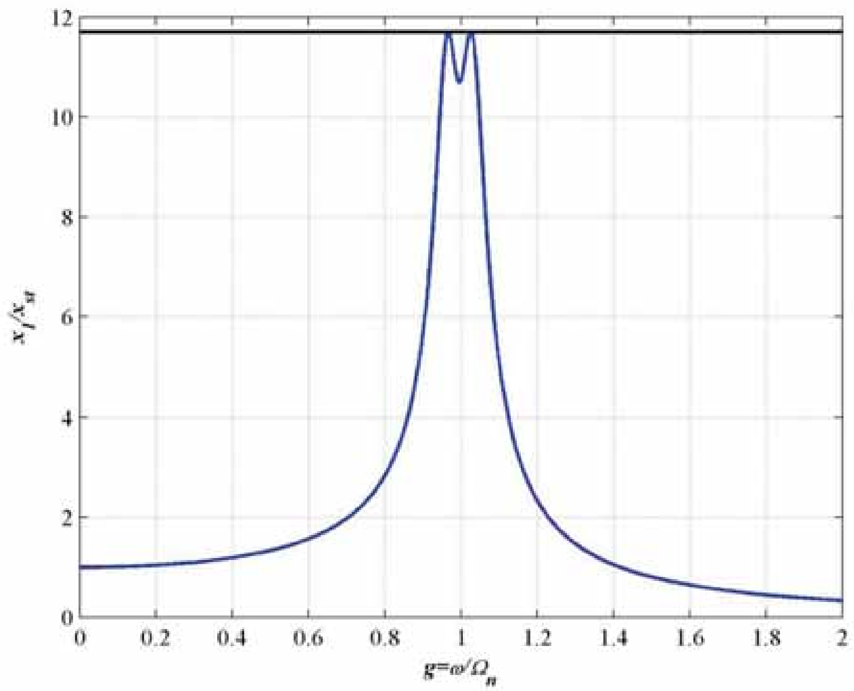


Figure 10: 2-D map of the displacement ratio as a function of g at optimal tuning, for $\mu = 0.5\%$ and $C/C_c = 2\%$, with quote of the two peaks.

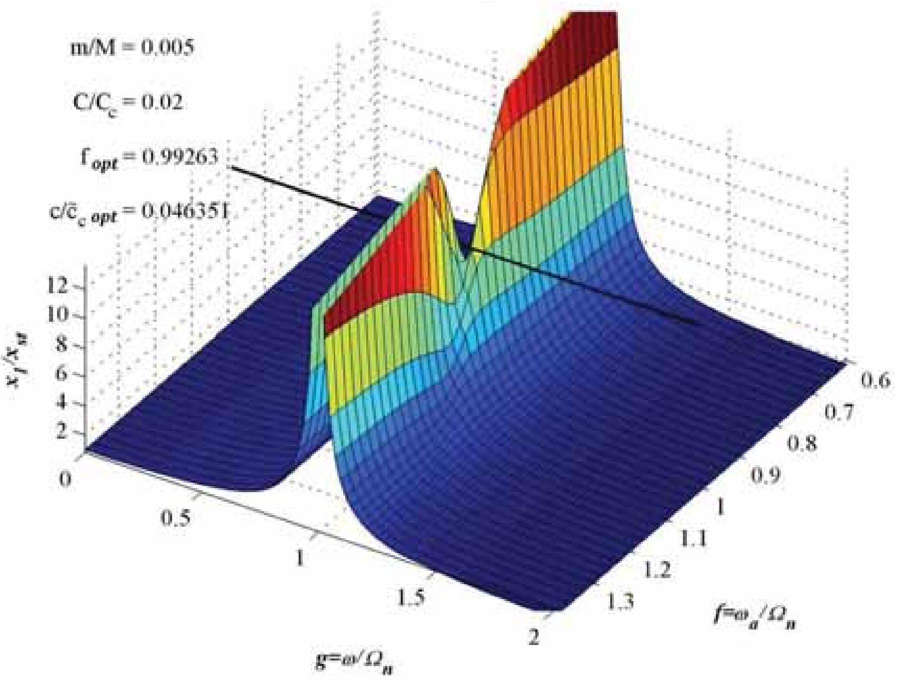


Figure 11: 3-D map of the displacement ratio as a function of g and f at optimal tuning, for $\mu = 0.5\%$ and $C/C_c = 2\%$, with quote of the two peaks.

Table 2: Abacus with estimate of f_{opt}, $\left(c/\tilde{c}_c\right)_{opt}$ and $\left(x_1/x_{st}\right)_{peak}$ for different values of μ and C/C_c (case $\delta = 0$).

	C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$		C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$
	0.0	0.995027	0.043120	20.022772		0.0	0.990109	0.060586	14.179084
	0.5	0.994497	0.043946	17.051707		0.5	0.989370	0.061840	12.629496
	1.0	0.993927	0.044542	14.819509		1.0	0.988591	0.062860	11.372360
	1.5	0.993314	0.045321	13.085709		1.5	0.987777	0.063335	10.334655
$\mu = 0.5\%$	2.0	0.992632	0.046351	11.703664	$\mu = 1.0\%$	2.0	0.986917	0.063745	9.464838
	2.5	0.991924	0.046840	10.577683		2.5	0.985990	0.064633	8.725481
	3.0	0.991149	0.047480	9.644010		3.0	0.985019	0.065718	8.090034
	3.5	0.990337	0.048207	8.857815		3.5	0.984013	0.066136	7.538492
	4.0	0.989469	0.048678	8.187712		4.0	0.982954	0.066437	7.055547
	4.5	0.988554	0.049512	7.609550		4.5	0.981840	0.066943	6.629457
	5.0	0.987586	0.049544	7.106376		5.0	0.980675	0.067940	6.250854

	C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$		C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$
	0.0	0.985221	0.074454	11.592367		0.0	0.980389	0.085829	10.052684
	0.5	0.984352	0.075028	10.539526		0.5	0.979401	0.086306	9.254163
	1.0	0.983434	0.075640	9.655237		1.0	0.978354	0.087059	8.568664
	1.5	0.982444	0.076835	8.902345		1.5	0.977251	0.087937	7.973915
$\mu = 1.5\%$	2.0	0.981418	0.077912	8.254765	$\mu = 2.0\%$	2.0	0.976092	0.089128	7.453842
	2.5	0.980365	0.078082	7.691927		2.5	0.974928	0.089215	6.995306
	3.0	0.979244	0.079104	7.198727		3.0	0.973688	0.089992	6.588249
	3.5	0.978084	0.079827	6.763355		3.5	0.972399	0.090829	6.224735
	4.0	0.976879	0.080350	6.376367		4.0	0.971067	0.091554	5.898265
	4.5	0.975623	0.080870	6.030269		4.5	0.969690	0.092166	5.603571
	5.0	0.974318	0.081393	5.719050		5.0	0.968268	0.092664	5.336338

	C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$		C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$
	0.0	0.975624	0.095457	9.003327		0.0	0.970875	0.104536	8.229602
	0.5	0.974504	0.096389	8.359564		0.5	0.969642	0.105919	7.690412
	1.0	0.973347	0.097231	7.798583		1.0	0.968418	0.106112	7.214920
	1.5	0.972141	0.098077	7.305425		1.5	0.967104	0.107268	6.792845
$\mu = 2.5\%$	2.0	0.970889	0.098904	6.868911	$\mu = 3.0\%$	2.0	0.965794	0.107398	6.415782
	2.5	0.969593	0.099719	6.480098		2.5	0.964399	0.108302	6.077126
	3.0	0.968250	0.100526	6.131685		3.0	0.962959	0.109205	5.771476
	3.5	0.966860	0.101333	5.817849		3.5	0.961476	0.110007	5.494345
	4.0	0.965430	0.101871	5.533805		4.0	0.959960	0.110375	5.241990
	4.5	0.963951	0.102490	5.275598		4.5	0.958392	0.110952	5.011309
	5.0	0.962419	0.103244	5.039934		5.0	0.956773	0.111674	4.799688

	C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$		C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$
	0.0	0.966172	0.112908	7.629036		0.0	0.961533	0.120307	7.145616
	0.5	0.964885	0.113651	7.165082		0.5	0.960162	0.121182	6.738251
	1.0	0.963552	0.114345	6.752126		1.0	0.958763	0.121654	6.373250
	1.5	0.962190	0.114806	6.382671		1.5	0.957294	0.122623	6.044397
$\mu = 3.5\%$	2.0	0.960753	0.115796	6.050235	$\mu = 4.0\%$	2.0	0.955784	0.123541	5.746814
	2.5	0.959271	0.116782	5.749751		2.5	0.954250	0.124090	5.476303
	3.0	0.957762	0.117307	5.476926		3.0	0.952646	0.125075	5.229484
	3.5	0.956218	0.117606	5.228148		3.5	0.951013	0.125690	5.003454
	4.0	0.954609	0.118367	5.000479		4.0	0.949341	0.126204	4.795707
	4.5	0.952944	0.119380	4.791423		4.5	0.947613	0.126988	4.604215
	5.0	0.951239	0.120151	4.598852		5.0	0.945823	0.128088	4.427200

	C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$		C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$
	0.0	0.956922	0.127502	6.745735		0.0	0.952374	0.133895	6.407835
	0.5	0.955492	0.128076	6.382691		0.5	0.950848	0.134996	6.080437
	1.0	0.953992	0.129137	6.055415		1.0	0.949338	0.135174	5.783720
	1.5	0.952487	0.129561	5.758949		1.5	0.947721	0.136344	5.513729
$\mu = 4.5\%$	2.0	0.950896	0.130679	5.489349	$\mu = 5.0\%$	2.0	0.946109	0.136640	5.267146
	2.5	0.949311	0.130795	5.243196		2.5	0.944432	0.137343	5.041039
	3.0	0.947651	0.131545	5.017567		3.0	0.942686	0.138501	4.833119
	3.5	0.945926	0.132680	4.810141		3.5	0.940899	0.139567	4.641326
	4.0	0.944172	0.133431	4.618863		4.0	0.939102	0.139971	4.463862
	4.5	0.942393	0.133799	4.441928		4.5	0.937246	0.140719	4.299253
	5.0	0.940544	0.134754	4.277862		5.0	0.935351	0.141233	4.146210

	C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$
$\mu = 5.5\%$	**0.0**	0.947862	0.140119	6.117509
	0.5	0.946264	0.141342	5.819347
	1.0	0.944711	0.141246	5.547965
	1.5	0.943032	0.142438	5.299979
	2.0	0.941330	0.143259	5.072668
	2.5	0.939620	0.143504	4.863510
	3.0	0.937809	0.144742	4.670547
	3.5	0.935973	0.145606	4.492025
	4.0	0.934121	0.145981	4.326375
	4.5	0.932186	0.147130	4.172368
	5.0	0.930238	0.147571	4.028789

	C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$
$\mu = 6.0\%$	**0.0**	0.943373	0.146225	5.864651
	0.5	0.941769	0.146764	5.590894
	1.0	0.940068	0.148023	5.340796
	1.5	0.938416	0.147923	5.111469
	2.0	0.936631	0.149190	4.900491
	2.5	0.934817	0.150219	4.705863
	3.0	0.933017	0.150355	4.525738
	3.5	0.931128	0.151160	4.358657
	4.0	0.929223	0.151547	4.203267
	4.5	0.927223	0.152834	4.058416
	5.0	0.925240	0.152980	3.923130

	C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$
$\mu = 6.5\%$	**0.0**	0.938956	0.151700	5.641751
	0.5	0.937264	0.152646	5.388806
	1.0	0.935580	0.152944	5.156867
	1.5	0.933781	0.154231	4.943480
	2.0	0.931982	0.154885	4.746668
	2.5	0.930146	0.155490	4.564517
	3.0	0.928274	0.155915	4.395611
	3.5	0.926337	0.156754	4.238499
	4.0	0.924347	0.157670	4.092081
	4.5	0.922348	0.158067	3.955306
	5.0	0.920268	0.159085	3.827306

	C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$
$\mu = 7.0\%$	**0.0**	0.934541	0.157431	5.443543
	0.5	0.932849	0.157703	5.208365
	1.0	0.931060	0.158750	4.992086
	1.5	0.929270	0.159183	4.792609
	2.0	0.927409	0.160057	4.608048
	2.5	0.925476	0.161360	4.436918
	3.0	0.923586	0.161381	4.277758
	3.5	0.921601	0.162131	4.129464
	4.0	0.919587	0.162748	3.990955
	4.5	0.917509	0.163630	3.861346
	5.0	0.915412	0.164180	3.739815

	C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$
$\mu = 7.5\%$	**0.0**	0.930189	0.162609	5.265663
	0.5	0.928463	0.162760	5.045990
	1.0	0.926600	0.164127	4.843387
	1.5	0.924775	0.164427	4.656055
	2.0	0.922859	0.165405	4.482294
	2.5	0.920935	0.165879	4.320819
	3.0	0.918934	0.166907	4.170359
	3.5	0.916908	0.167630	4.029879
	4.0	0.914843	0.168357	3.898427
	4.5	0.912758	0.168649	3.775209
	5.0	0.910589	0.169640	3.659487

	C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$
$\mu = 8.0\%$	**0.0**	0.925919	0.167077	5.104961
	0.5	0.924098	0.167859	4.898828
	1.0	0.922240	0.168564	4.708305
	1.5	0.920351	0.169156	4.531654
	2.0	0.918360	0.170555	4.367521
	2.5	0.916390	0.171072	4.214640
	3.0	0.914385	0.171497	4.071917
	3.5	0.912331	0.172056	3.938414
	4.0	0.910204	0.172998	3.813298
	4.5	0.908062	0.173604	3.695817
	5.0	0.905875	0.174170	3.585334

	C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$
$\mu = 8.5\%$	**0.0**	0.921619	0.172263	4.958791
	0.5	0.919756	0.173013	4.764719
	1.0	0.917863	0.173666	4.584824
	1.5	0.915932	0.174232	4.417755
	2.0	0.913930	0.175238	4.262176
	2.5	0.911955	0.175283	4.116997
	3.0	0.909855	0.176455	3.981218
	3.5	0.907773	0.176830	3.854012
	4.0	0.905600	0.177880	3.734612
	4.5	0.903421	0.178445	3.622331
	5.0	0.901198	0.178979	3.516590

	C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$
$\mu = 9.0\%$	**0.0**	0.917419	0.176518	4.825197
	0.5	0.915463	0.177912	4.641792
	1.0	0.913573	0.178040	4.471451
	1.5	0.911566	0.179054	4.312925
	2.0	0.909561	0.179582	4.165018
	2.5	0.907531	0.179846	4.026802
	3.0	0.905383	0.181163	3.897287
	3.5	0.903276	0.181348	3.775794
	4.0	0.901057	0.182513	3.661573
	4.5	0.898851	0.182927	3.554026
	5.0	0.896586	0.183541	3.452610

	C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$
$\mu = 9.5\%$	**0.0**	0.913230	0.180928	4.702447
	0.5	0.911270	0.181936	4.528578
	1.0	0.909333	0.182144	4.366852
	1.5	0.907262	0.183517	4.216003
	2.0	0.905216	0.184049	4.075078
	2.5	0.903090	0.185134	3.943127
	3.0	0.900997	0.185239	3.819338
	3.5	0.898825	0.185793	3.703029
	4.0	0.896574	0.186919	3.593517
	4.5	0.894350	0.187075	3.490311
	5.0	0.892035	0.187925	3.392851

	C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$
$\mu = 10.0\%$	**0.0**	0.909056	0.185508	4.589121
	0.5	0.907097	0.186038	4.423919
	1.0	0.905053	0.187122	4.269917
	1.5	0.903050	0.187231	4.126089
	2.0	0.900917	0.188393	3.991458
	2.5	0.898802	0.188888	3.865222
	3.0	0.896601	0.189910	3.746653
	3.5	0.894390	0.190607	3.635072
	4.0	0.892147	0.191116	3.529916
	4.5	0.889830	0.192132	3.430666
	5.0	0.887513	0.192591	3.336839

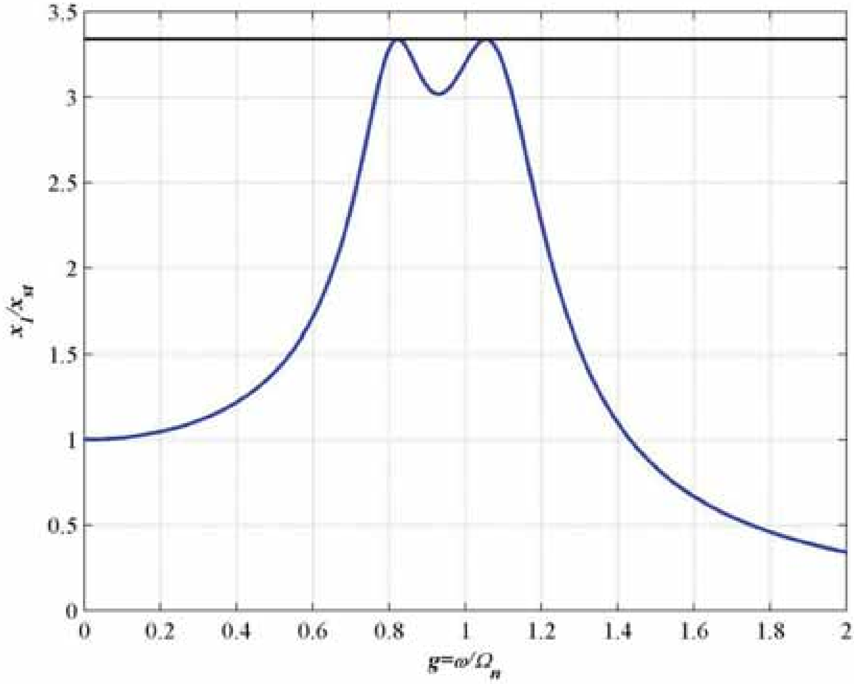


Figure 12: 2-D map of the displacement ratio as a function of g at optimal tuning, for $\mu = 10\%$ and $C/C_c = 5\%$, with quote of the two peaks.

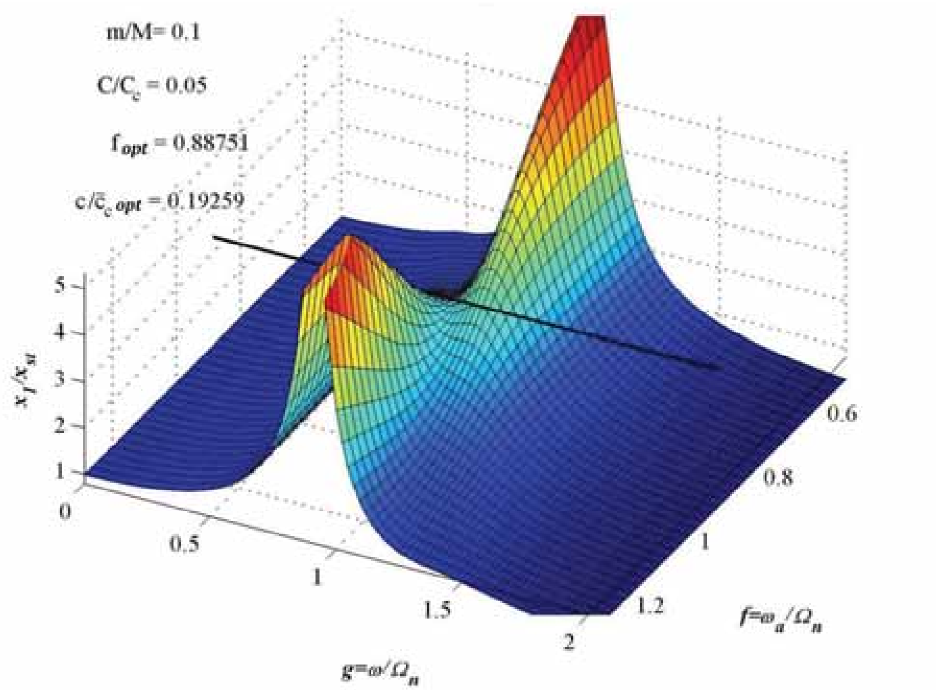


Figure 13: 3-D map of the displacement ratio as a function of g and f at optimal tuning, for $\mu = 10\%$ and $C/C_c = 5\%$, with quote of the two peaks.

3.2 Optimal tuning for harmonic excitation at the support ($\delta = 1$)

The previous study is repeated for the case of harmonic excitation at the support. Corresponding plots are presented in what follows. Fig. 14 shows that, even in this case, the minimax tuning of f_{opt} is practically coincident with Den Hartog's-type estimate that can be obtained from a fixed-points analysis of Scheme 5 in Table 1, as reported by Rana and Soong [3]:

$f_{opt} = 1/(1+\mu)\cdot\sqrt{((2-\mu)/2)}$. The same basically applies for the evaluation of $(x_1 / x_{st})_{peak}$, which should compare to $(x_1 / x_{st})_{peak} = (1+\mu)\cdot\sqrt{(2 / \mu)}$, as reported in Fig. 16. Again, the estimate of $(c / \tilde{c}_c)_{opt}$, comparing to $(c / \tilde{c}_c)_{opt} = \sqrt{(3/8 \cdot \mu /(1+\mu))}\cdot\sqrt{(2/(2-\mu))}$ slightly diverges for high μ values larger than about $25\% - 30\%$ (Fig. 15).

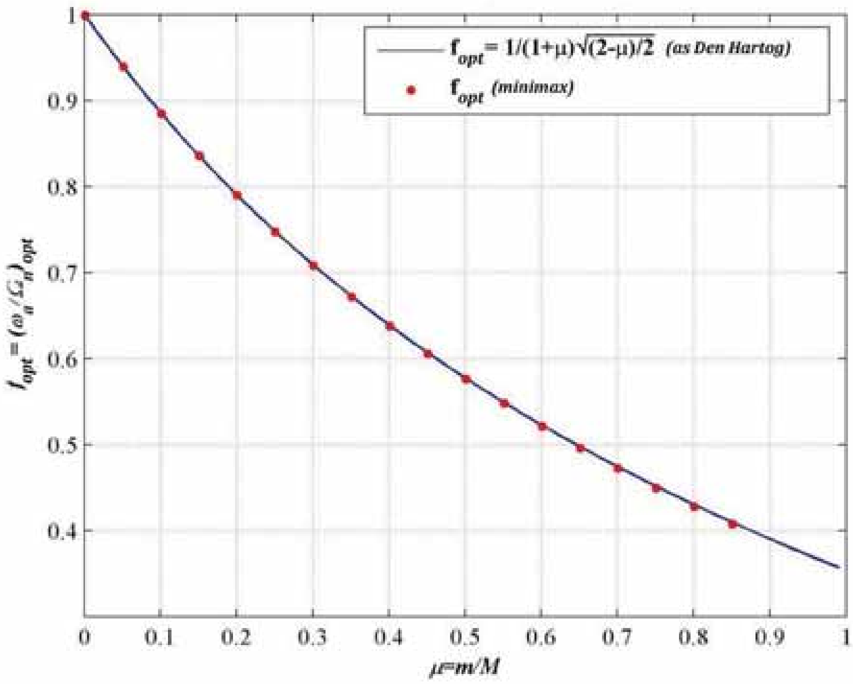


Figure 14: f_{opt} as a function of μ by a minimax procedure and comparison to Den Hartog's-type tuning ($C/C_c = 0$, $\delta = 1$).

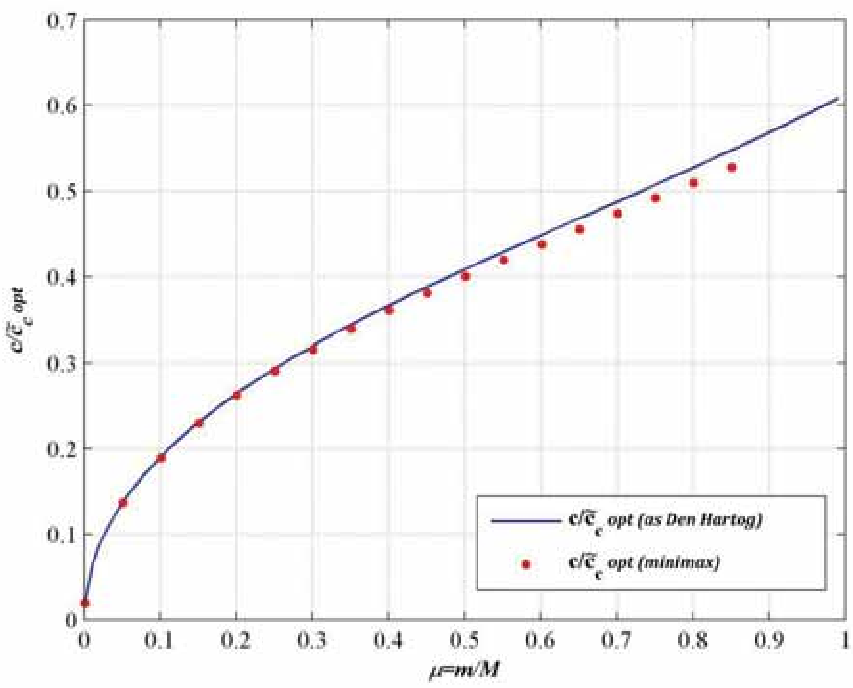


Figure 15: $(c/\tilde{c}_c)_{opt}$ as a function of μ by a minimax procedure and comparison to Den Hartog's-type tuning ($C/C_c = 0$, $\delta = 1$).

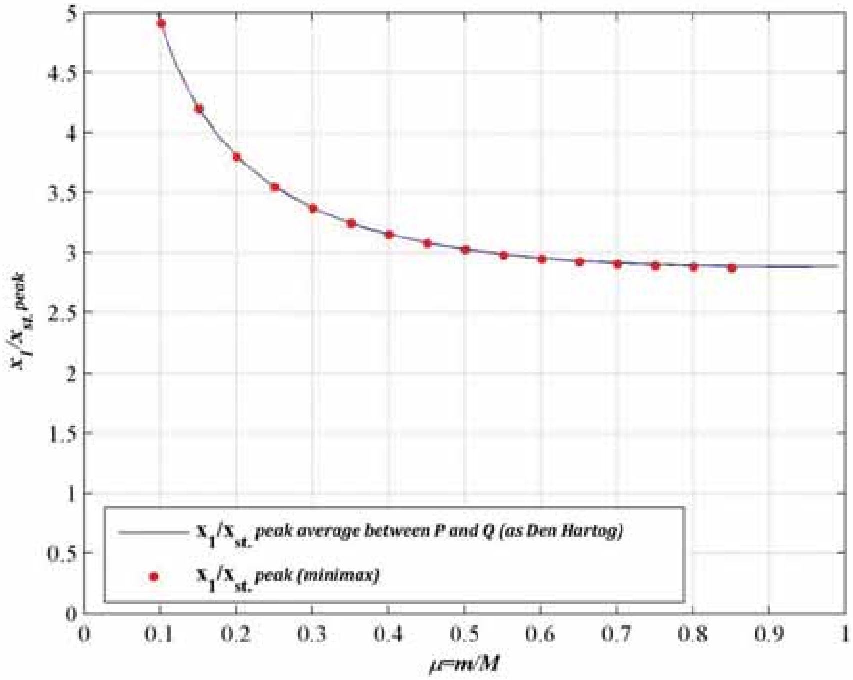


Figure 16: $(x_1/x_{st})_{peak}$ as a function of μ by a minimax procedure and comparison to Den Hartog's-type tuning ($C/C_c = 0$, $\delta = 1$).

The influence of structural damping ratio C/C_c is studied as well as before. The corresponding plots are reported in following Figs. 17–19.

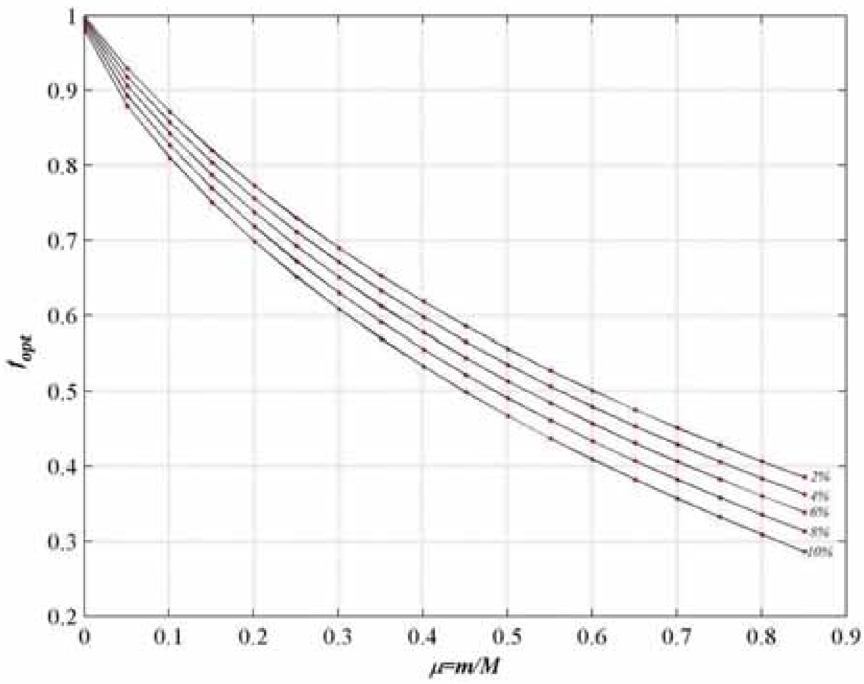


Figure 17: f_{opt} as a function of μ by a minimax procedure, for different values of C/C_c ($\delta = 1$).

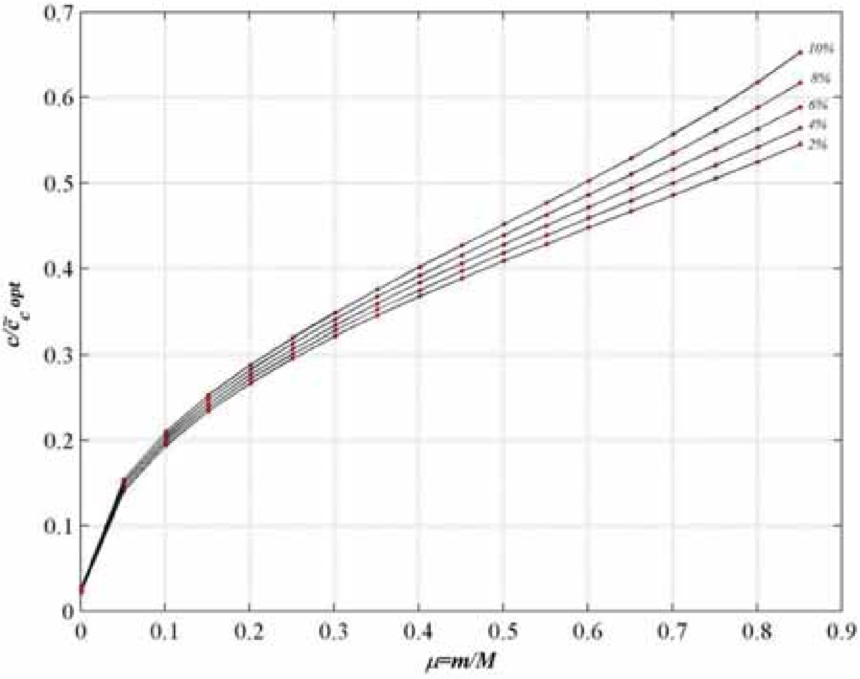


Figure 18: $\left(c/\tilde{c}_c\right)_{opt}$ as a function of μ by a minimax procedure, for different values of C/C_c ($\delta = 1$).

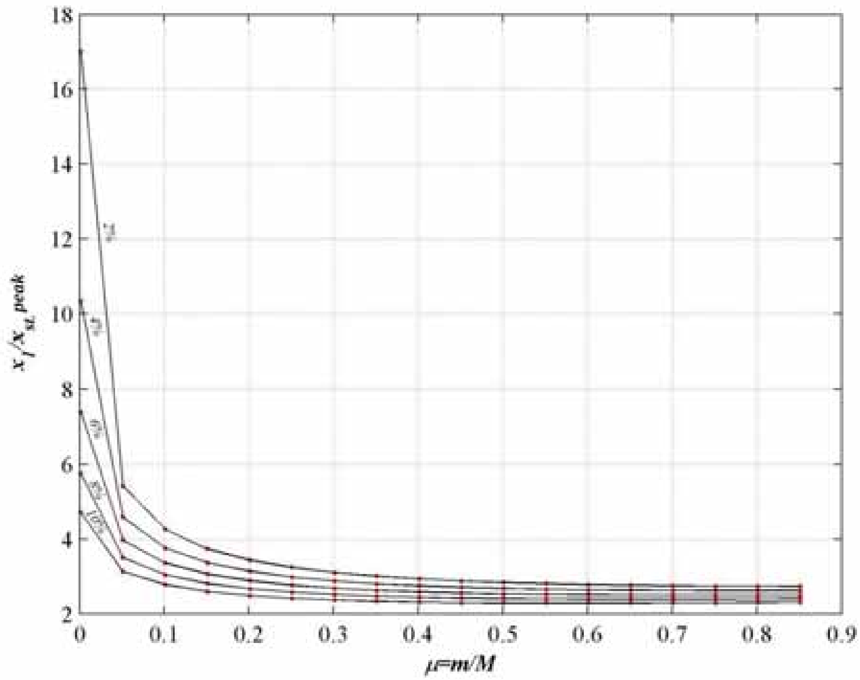


Figure 19: $\left(x_1/x_{st}\right)_{peak}$ as a function of μ by a minimax procedure, for different values of C/C_c ($\delta = 1$).

It can be noticed that, with respect to the previous results ($\delta = 0$), the scatter of parameters f_{opt}, $(c\,/\tilde{c}_c)_{opt}$ and resulting $(x_1\,/\,x_{st})_{peak}$ at given μ and changing C/C_c is more apparent. On the other hand, the global behaviour shares similar features: at given μ, f_{opt} and $(x_1\,/\,x_{st})_{peak}$ decrease at increasing C/C_c, while $(c\,/\tilde{c}_c)_{opt}$ increases. The trends depicted in Figs. 17−19 appear in line with the results obtained by other authors, e.g. [5,8]. Our results are also summarised quantitatively by the abacus reported in following Table 3, which compares as well to similar output from the literature, e.g. [3,4,8].

Table 3: Abacus with estimate of f_{opt}, $(c/\tilde{c}_c)_{opt}$ and $(x_1/x_{st})_{peak}$ for different values of μ and C/C_c (case $\delta = 1$).

	C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$
$\mu = 0.5\%$	0.0	0.993785	0.043613	20.100778
	0.5	0.992962	0.044314	17.113400
	1.0	0.992060	0.044759	14.870498
	1.5	0.991117	0.045645	13.129361
	2.0	0.990082	0.046237	11.740895
	2.5	0.989013	0.047259	10.610556
	3.0	0.987868	0.047945	9.672749
	3.5	0.986648	0.048460	8.883095
	4.0	0.985346	0.048801	8.209837
	4.5	0.983979	0.049142	7.629475
	5.0	0.982617	0.050221	7.124186

	C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$
$\mu = 1.0\%$	0.0	0.987621	0.060966	14.285077
	0.5	0.986480	0.061901	12.720794
	1.0	0.985274	0.062783	11.452965
	1.5	0.984001	0.063626	10.406006
	2.0	0.982665	0.064443	9.528327
	2.5	0.981233	0.064870	8.782425
	3.0	0.979760	0.065618	8.141345
	3.5	0.978226	0.066361	7.584975
	4.0	0.976618	0.066966	7.097853
	4.5	0.974933	0.067429	6.668049
	5.0	0.973200	0.068019	6.286204

	C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$
$\mu = 1.5\%$	0.0	0.981514	0.074467	11.722659
	0.5	0.980142	0.075597	10.655167
	1.0	0.978705	0.076696	9.758574
	1.5	0.977143	0.076705	8.995786
	2.0	0.975577	0.077776	8.339178
	2.5	0.973941	0.078695	7.768848
	3.0	0.972233	0.079516	7.269264
	3.5	0.970451	0.080187	6.828158
	4.0	0.968595	0.080726	6.435999
	4.5	0.966735	0.081846	6.085444
	5.0	0.964774	0.082607	5.770192

	C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$
$\mu = 2.0\%$	0.0	0.975463	0.085855	10.203001
	0.5	0.973885	0.086723	9.389907
	1.0	0.972261	0.088002	8.691742
	1.5	0.970517	0.088416	8.086333
	2.0	0.968751	0.089395	7.556712
	2.5	0.966909	0.090245	7.090034
	3.0	0.964996	0.090968	6.675765
	3.5	0.963011	0.091606	6.305837
	4.0	0.960940	0.092034	5.973619
	4.5	0.958910	0.093423	5.673736
	5.0	0.956718	0.093942	5.401837

	C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$
$\mu = 2.5\%$	0.0	0.969504	0.096396	9.171291
	0.5	0.967727	0.096835	8.512927
	1.0	0.965926	0.097981	7.939024
	1.5	0.964054	0.099003	7.434879
	2.0	0.962101	0.099713	6.988459
	2.5	0.960063	0.100244	6.590936
	3.0	0.958010	0.101336	6.234633
	3.5	0.955843	0.101867	5.913799
	4.0	0.953595	0.102270	5.623515
	4.5	0.951380	0.103510	5.359580
	5.0	0.949018	0.104050	5.118755

	C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$
$\mu = 3.0\%$	0.0	0.963538	0.104774	8.413815
	0.5	0.961653	0.106001	7.859577
	1.0	0.959690	0.107009	7.371219
	1.5	0.957662	0.107926	6.937555
	2.0	0.955561	0.108767	6.550388
	2.5	0.953385	0.109467	6.202613
	3.0	0.951161	0.110299	5.888734
	3.5	0.948827	0.110709	5.604178
	4.0	0.946537	0.112133	5.345096
	4.5	0.944057	0.112403	5.108285
	5.0	0.941606	0.113422	4.891068

	C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$
$\mu = 3.5\%$	0.0	0.957695	0.113574	7.827996
	0.5	0.955627	0.114041	7.349066
	1.0	0.953507	0.114801	6.923062
	1.5	0.951378	0.116216	6.541805
	2.0	0.949111	0.116708	6.198868
	2.5	0.946815	0.117627	5.888951
	3.0	0.944479	0.118761	5.607471
	3.5	0.942026	0.119336	5.350932
	4.0	0.939539	0.120177	5.116187
	4.5	0.936974	0.120915	4.900637
	5.0	0.934327	0.121475	4.702156

	C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$
$\mu = 4.0\%$	0.0	0.951851	0.120872	7.358422
	0.5	0.949690	0.121886	6.936085
	1.0	0.947462	0.122820	6.557658
	1.5	0.945154	0.123575	6.216922
	2.0	0.942774	0.124248	5.908600
	2.5	0.940428	0.125918	5.628374
	3.0	0.937914	0.126492	5.372633
	3.5	0.935361	0.127290	5.138509
	4.0	0.932737	0.128026	4.923358
	4.5	0.930054	0.128801	4.725063
	5.0	0.927314	0.129619	4.541756

	C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$
$\mu = 4.5\%$	0.0	0.946094	0.128198	6.971467
	0.5	0.943813	0.129142	6.593445
	1.0	0.941461	0.129976	6.252702
	1.5	0.939059	0.130952	5.944143
	2.0	0.936570	0.131680	5.663553
	2.5	0.934019	0.132434	5.407385
	3.0	0.931428	0.133427	5.172630
	3.5	0.928737	0.134044	4.956877
	4.0	0.926038	0.135159	4.757875
	4.5	0.923226	0.135811	4.573888
	5.0	0.920388	0.136763	4.403256

	C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$
$\mu = 5.0\%$	0.0	0.940375	0.134857	6.645894
	0.5	0.938013	0.136136	6.303451
	1.0	0.935551	0.136922	5.993203
	1.5	0.933026	0.137730	5.710946
	2.0	0.930439	0.138565	5.453145
	2.5	0.927792	0.139432	5.216839
	3.0	0.925089	0.140334	4.999583
	3.5	0.922315	0.141179	4.799143
	4.0	0.919512	0.142260	4.613773
	4.5	0.916587	0.142884	4.441850
	5.0	0.913644	0.143836	4.281978

	C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$		C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$
	0.0	0.934714	0.141225	6.367274		**0.0**	0.929129	0.147678	6.125520
	0.5	0.932217	0.142076	6.054078		**0.5**	0.926532	0.148439	5.836765
	1.0	0.929693	0.143396	5.769061		**1.0**	0.923913	0.149656	5.572980
	1.5	0.927070	0.144232	5.508655		**1.5**	0.921212	0.150654	5.331115
$\mu = 5.5\%$	**2.0**	0.924358	0.144787	5.269955	$\mu = 6.0\%$	**2.0**	0.918410	0.151277	5.108657
	2.5	0.921622	0.145756	5.050444		**2.5**	0.915516	0.151627	4.903463
	3.0	0.918829	0.146765	4.847897		**3.0**	0.912681	0.153095	4.713589
	3.5	0.915982	0.147816	4.660551		**3.5**	0.909696	0.153754	4.537491
	4.0	0.913072	0.148834	4.486788		**4.0**	0.906709	0.154907	4.373747
	4.5	0.910043	0.149431	4.325185		**4.5**	0.903622	0.155759	4.221142
	5.0	0.907027	0.150552	4.174597		**5.0**	0.900460	0.156499	4.078625

	C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$		C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$
	0.0	0.923594	0.153785	5.913482		**0.0**	0.918039	0.158778	5.725573
	0.5	0.920921	0.154688	5.645415		**0.5**	0.915349	0.160489	5.475347
	1.0	0.918158	0.155300	5.399693		**1.0**	0.912505	0.161127	5.245224
	1.5	0.915321	0.155805	5.173713		**1.5**	0.909581	0.161590	5.032976
$\mu = 6.5\%$	**2.0**	0.912470	0.156819	4.965263	$\mu = 7.0\%$	**2.0**	0.906664	0.162754	4.836715
	2.5	0.909574	0.157981	4.772407		**2.5**	0.903626	0.163401	4.654713
	3.0	0.906618	0.159124	4.593552		**3.0**	0.900616	0.164779	4.485513
	3.5	0.903508	0.159508	4.427298		**3.5**	0.897452	0.165375	4.327856
	4.0	0.900410	0.160489	4.272302		**4.0**	0.894231	0.166064	4.180659
	4.5	0.897280	0.161665	4.127588		**4.5**	0.891013	0.167200	4.042909
	5.0	0.894081	0.162761	3.992179		**5.0**	0.887726	0.168256	3.913788

	C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$		C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$
	0.0	0.912598	0.164362	5.557676		**0.0**	0.907198	0.169643	5.406639
	0.5	0.909732	0.164967	5.322940		**0.5**	0.904299	0.170697	5.185446
	1.0	0.906901	0.166620	5.106459		**1.0**	0.901343	0.171814	4.980967
	1.5	0.903892	0.167041	4.906261		**1.5**	0.898252	0.172193	4.791441
$\mu = 7.5\%$	**2.0**	0.900905	0.168278	4.720706	$\mu = 8.0\%$	**2.0**	0.895191	0.173427	4.615363
	2.5	0.897770	0.168767	4.548254		**2.5**	0.891971	0.173835	4.451411
	3.0	0.894677	0.170104	4.387585		**3.0**	0.888797	0.175133	4.298336
	3.5	0.891493	0.171163	4.237601		**3.5**	0.885558	0.176355	4.155216
	4.0	0.888130	0.171360	4.097305		**4.0**	0.882162	0.176850	4.021108
	4.5	0.884894	0.172908	3.965796		**4.5**	0.878867	0.178487	3.895205
	5.0	0.881445	0.173422	3.842318		**5.0**	0.875330	0.178880	3.776806

	C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$		C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$
	0.0	0.901837	0.174652	5.269916		**0.0**	0.896519	0.179434	5.145406
	0.5	0.898827	0.175287	5.060683		**0.5**	0.893510	0.180818	4.946895
	1.0	0.895822	0.176645	4.866833		**1.0**	0.890363	0.181415	4.762540
	1.5	0.892662	0.177074	4.686826		**1.5**	0.887204	0.182470	4.590997
$\mu = 8.5\%$	**2.0**	0.889547	0.178488	4.519199	$\mu = 9.0\%$	**2.0**	0.883931	0.183085	4.431001
	2.5	0.886360	0.179733	4.362829		**2.5**	0.880671	0.184300	4.281451
	3.0	0.882990	0.180004	4.216615		**3.0**	0.877365	0.185618	4.141443
	3.5	0.879657	0.181077	4.079659		**3.5**	0.873924	0.186331	4.010053
	4.0	0.876274	0.182186	3.951113		**4.0**	0.870379	0.186779	3.886578
	4.5	0.872821	0.183220	3.830280		**4.5**	0.866894	0.188062	3.770351
	5.0	0.869299	0.184178	3.716478		**5.0**	0.863325	0.189150	3.660743

	C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$		C/C_c [%]	f_{opt}	$(c/\tilde{c}_c)_{opt}$	$(x_1/x_{st})_{peak}$
	0.0	0.891235	0.183948	5.031565		**0.0**	0.885996	0.188297	4.926931
	0.5	0.888111	0.184821	4.842580		**0.5**	0.882867	0.189800	4.746533
	1.0	0.884939	0.185869	4.666774		**1.0**	0.879561	0.190117	4.578492
	1.5	0.881752	0.187285	4.502881		**1.5**	0.876333	0.191733	4.421505
$\mu = 9.5\%$	**2.0**	0.878453	0.188197	4.349757	$\mu = 10.0\%$	**2.0**	0.872986	0.192803	4.274640
	2.5	0.875042	0.188723	4.206415		**2.5**	0.869581	0.193848	4.136941
	3.0	0.871654	0.189918	4.071984		**3.0**	0.866042	0.194340	4.007625
	3.5	0.868201	0.191040	3.945683		**3.5**	0.862472	0.195098	3.885976
	4.0	0.864679	0.192089	3.826829		**4.0**	0.858982	0.196787	3.771334
	4.5	0.861088	0.193065	3.714785		**4.5**	0.855280	0.197448	3.663150
	5.0	0.857409	0.193866	3.609027		**5.0**	0.851523	0.198166	3.560922

4 SEISMIC RESPONSE OF A TEN-STOREY SHEAR-TYPE BUILDING WITH TMD ADDED ON TOP

A numerical test on the performance of a TMD device placed on top of a prototype ten-storey shear-type building is developed. The base of the building is subjected to seismic ground motion, with input from El Centro Earthquake, Imperial Valley, 1940. Mass and stiffness distributions of the building without TMD are reported below (Fig. 20). No structural damping is assumed at first. The numerical response of the system is evaluated by the average acceleration method of the Newmark family ($\alpha = 1/4,\ \beta = 1/2$). The time step is taken as $\Delta t = 0.0115\ s$, on a run-time window of $45\ s$.

$$\left[M\right] = 10^3\, diag\left[430,406,382,358,334,310,286,262,238,215\right]\ kg \tag{2}$$

$$\left[K\right] = 10^6 \begin{bmatrix} 661 & -321 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -321 & 623 & -302 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -302 & 585 & -283 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -283 & 548 & -264 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -264 & 510 & -246 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -246 & 472 & -227 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -227 & 434 & -208 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -208 & 397 & -189 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -189 & 359 & -170 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -170 & 170 \end{bmatrix} \frac{N}{m} \tag{3}$$

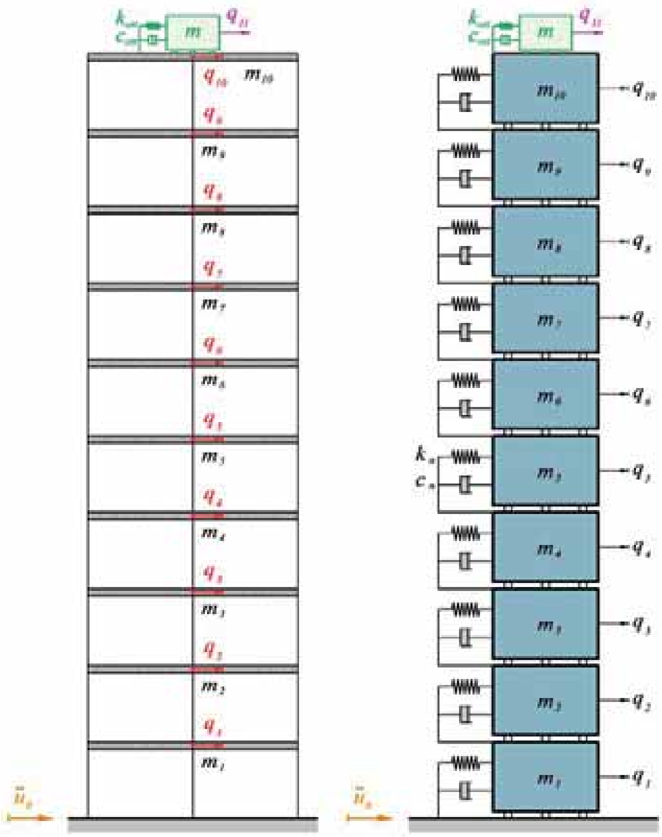


Figure 20: Scheme of a prototype ten-storey shear-type building with TMD added on top (chosen mass ratio $\mu = 3\%$, resulting first mode mass ratio $\mu_1 = 5.26\%$).

For the tuning of the TMD parameters, namely TMD spring stiffness k_{opt} and damping coefficient c_{opt}, the procedure proposed by Rana and Soong [3] is adopted. By assuming a mass ratio $\mu = 3\%$ and knowing the total mass of the structure as $M_{TOT} = 3221000\ kg$, the TMD mass is determined as $m = \mu\ M_{TOT} = 96630\ kg$. The assumed mass ratio, around the upper bound of typical values of μ, is higher than that assumed by Sladek and Klingner [5] ($\mu = 0.65\ \%$), which concluded that such light TMD had practically no effect on the seismic performance. The first modal mass is then obtained as $M_1 = \{\Phi_1^T\}\cdot[M]\cdot\{\Phi_1\}$, where $\{\Phi_1\}$ is the eigenvector corresponding to the fundamental mode of vibration normalised such that to have unitary component at the last floor where the TMD is going to be inserted. From eigenvalue analysis one has $M_1 = 1837400\ kg$, from which the mass ratio associated to the first mode is obtained as $\mu_1 = m/M_1 = 0.0526$. The natural angular frequency of the fundamental mode is also found as $\Omega_1 = 4.7383\ rad/s$. By considering that the structure is assumed at first with no structural damping, one has the following estimates for the best tuning (with input excitation at the support, coherent with case $\delta = 1$ previously treated):

$$f_{opt} = \frac{1}{1+\mu_1}\cdot\sqrt{\frac{2-\mu_1}{2}} = 0.9370\ ; \qquad \left(\frac{c}{\tilde{c}_c}\right)_{opt} = \sqrt{\frac{3\cdot\mu_1}{8\cdot(1+\mu_1)}}\cdot\sqrt{\frac{2}{2-\mu_1}} = 0.1392 \tag{4}$$

and the TMD parameters are finally obtained as:

$$k_{opt} = f_{opt}^2\cdot\Omega_1^2\cdot m \cong 1904742\ N/m\ ; \qquad c_{opt} = \left(\frac{c}{\tilde{c}_c}\right)_{opt}\cdot 2\cdot f_{opt}\cdot\Omega_1\cdot m \cong 119438\ N\cdot s/m\ . \tag{5}$$

The numerical seismic responses of the building with and without TMD are then evaluated and compared. Fig. 21 reports the displacement of the top floor, with and without TMD, for the case of zero structural damping. The plot clearly shows the role of the TMD in reducing the structural response at zero inherent damping.

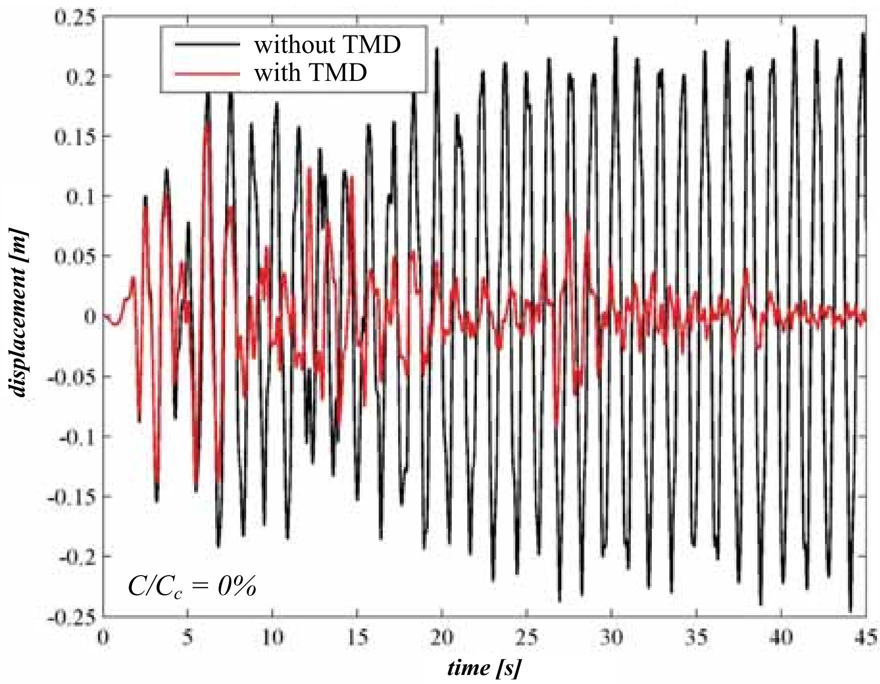


Figure 21: Top-floor displacement of the building with or without TMD. Case with no structural damping.

To enquire the role of damping in the structural response, further analyses are developed for different values of inherent damping. By taking $C/C_c = 2\%$ for the first two modes, with natural angular frequencies $\Omega_1 = 4.7383\ rad/s$ and $\Omega_2 = 12.5784\ rad/s$ and assuming classical Rayleigh damping calibrated on these first two modes, the optimal tuning parameters of the TMD need to be re-estimated. Let us leave unchanged the mass ratio associated to the first mode, $\mu_1 = 0.0526$. By the minimax procedure as reported in the preceding section, the following tuning parameters can be estimated: $f_{opt} = 0.927258$ and $(c/\tilde{c}_c)_{opt} = 0.141758$. Then, as done previously, spring stiffness and damping coefficient of the TMD are evaluated as $k_{opt} \cong 1865510\ N/m$ and $c_{opt} \cong 120410\ N{\cdot}s/m$. The numerical responses in terms of displacement of the top floor with and without TMD are reported in Fig. 22.

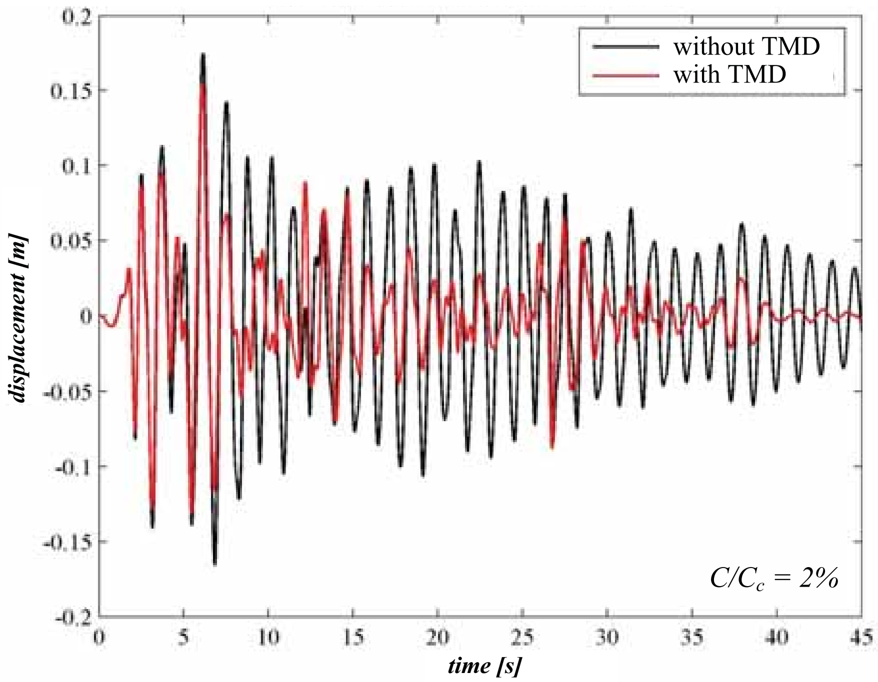


Figure 22: Top-floor displacement of the building with or without TMD. Case with 2% structural damping for the first two modes and classical Rayleigh damping.

Last, the same procedure is re-applied for the case $C/C_c = 5\%$ for the first two modes, again in the hypothesis of Rayleigh damping. Kept mass ratio $\mu_1 = 0.0526$, the following optimal parameters are found by the minimax procedure: $f_{opt} = 0.910196$ and $(c/\tilde{c}_c)_{opt} = 0.147424$, leading to $k_{opt} \cong 1797326\ N/m$ and $c_{opt} \cong 122876\ N{\cdot}s/m$. Fig. 23 reports the responses of the top-floor displacement.

From the plots in Figs. 21–23 the effect of the TMD device can be appreciated at increasing structural damping. Obviously, the effectiveness of the device is better appreciated at decreasing values of structural damping. To further evaluate quantitatively the reduction in seismic response, further output is also provided in following Table 4. There, the vibration reduction in terms of top-floor displacement, velocity and acceleration is evaluated for the different damping ratios. Both the peak values of these parameters and the average values in time estimated by the Root Mean Square method are analysed, to evaluate as well a global performance in the whole time window. Higher reduction is observed on the RMS estimates, basically in decreasing order for displacement, velocity and acceleration.

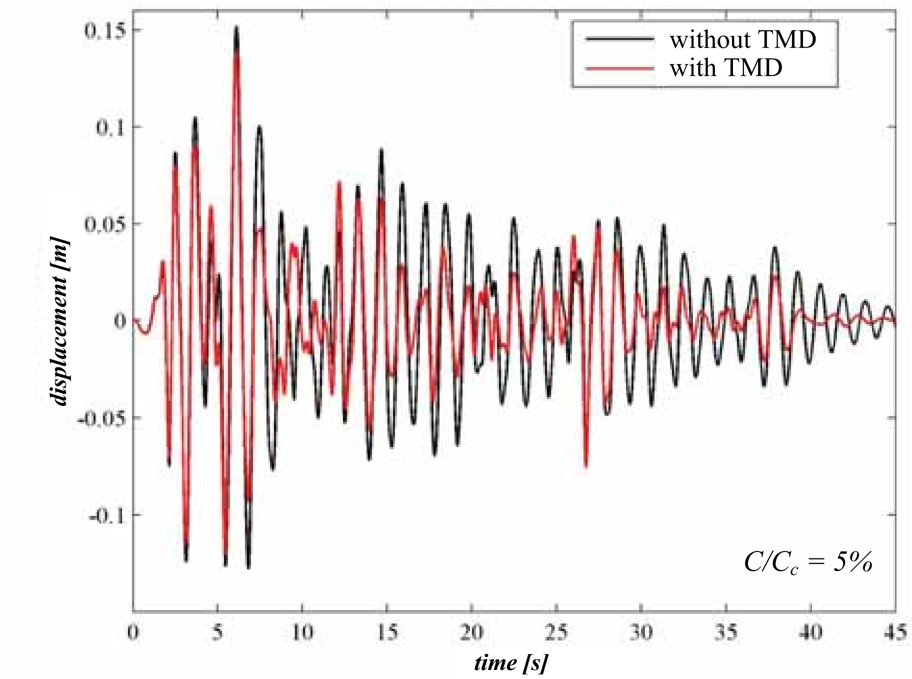


Figure 23: Top-floor displacement of the building with or without TMD. Case with 5% structural damping for the first two modes and classical Rayleigh damping.

Table 4: Evaluation of the structural performance of the TMD device for the different damping ratios assumed.

STRUCTURAL SYSTEM	x_{max} [m]	$\dot{x}_{max}$ [m/s]	$\ddot{x}_{max}$ [m/s^2]	x^{RMS} [m]	$\dot{x}^{RMS}$ [m/s]	$\ddot{x}^{RMS}$ [m/s^2]
Without TMD, undamped $(C/C_c = 0\%)$	0.2465	1.6824	19.7690	0.1293	0.6819	7.0168
With TMD, undamped $(C/C_c = 0\%)$	0.1597	1.0539	13.0962	0.0376	0.2565	3.6238
Vibration reduction	***35.2%***	***37.4%***	***33.8%***	***70.9%***	***62.4%***	***48.4%***
Without TMD, damped $(C/C_c = 2\%)$	0.1749	1.0023	9.2692	0.0557	0.2883	1.9599
With TMD, damped $(C/C_c = 2\%)$	0.1554	0.9110	8.9629	0.0329	0.1860	1.5661
Vibration reduction	***11.1%***	***9.1%***	***3.3%***	***40.9%***	***35.5%***	***20.1%***
Without TMD, damped $(C/C_c = 5\%)$	0.1518	0.8279	8.4166	0.0386	0.2033	1.3890
With TMD, damped $(C/C_c = 5\%)$	0.1383	0.7540	8.1659	0.0329	0.1860	1.5661
Vibration reduction	***8.9%***	***8.9%***	***3.0%***	***25.6%***	***22.7%***	***12.4%***

5 CONCLUSIONS

The present note considered the optimal tuning of TMD devices. After reviewing the classical theory by Den Hartog [1], a systematic analysis has revealed the role of structural damping and of the location of the harmonic excitation (at the primary mass or at the support). Tables 1–3 and Figures 1–19 summarised the achieved results.

On the optimal tuning by a minimax procedure, the estimate of the frequency ratio f_{opt} yields, at no structural damping, practically the same results as Den Hartog's ones. This constitutes a sort of numerical proof of those. Some divergence appears for the tuning of the damping ratio $(c/\tilde{c}_c)_{opt}$ when the mass ratio increases beyond high values of about 0.25–0.30. It increases slightly with respect to Den Hartog's estimate. Accordingly, the lowest estimate of $(x_1/x_{st})_{peak}$ is obtained by the minimax procedure, though is still very close to Den Hartog's prediction. Same trends are confirmed for the case of harmonic excitation at the support. Tuning has been then investigated systematically at increasing structural damping, for different mass ratios. Plots with the functional dependence of the optimal parameters and synoptic abacuses have been produced. This should allow easy access to the best tuning. This way of proceeding was inspired by the work of Rana and Soong [3]. With respect to their results, our estimate of $(x_1/x_{st})_{peak}$ takes slightly lower values, which should confirm the higher precision of the present analysis. A comparison to the alternative approach by Villaverde [6] and Sadek et al. [8], which do not consider the role of the external action, still needs to be made.

From the numerical analysis of the seismic response of a prototype ten-storey shear-type building (Figs. 21–23, Table 4) it might be concluded that: the reduction of structural vibration of the top floor in terms of maximum displacement may be quantified as 35.2% for the undamped case, 11.1% in case of 0.02 damping, 8.9% in case of 0.05 damping; in terms of RMS average of the top-floor displacement a higher reduction as 70.9% for no damping, 40.9% for 0.02 damping, 25.6% for 0.05 damping. As expected, at constant inserted mass of the TMD, the effectiveness of the device is less apparent at increasing structural damping. Also, lower reductions are most often obtained in terms of top-floor velocity and acceleration.

This contribution attempted a preliminary study on the usefulness of passive TMD devices in the field of seismic engineering, where the true validity of the use of a TMD element is still debated in the literature. Further results on the tuning at different seismic input should help in clearing if passive TMDs could work effectively for seismic isolation. The analysis of tuning for detailed practical cases is also left for further work.

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