# On the theory of the ellipse of elasticity as a natural discretisation method in the design of Paderno d'Adda Bridge (Italy)

R. Ferrari & E. Rizzi

Dip.to di Progettazione e Tecnologie, Facoltà di Ingegneria, Università di Bergamo, Dalmine (BG), Italy

ABSTRACT: The Paderno d'Adda Bridge, Lombardia, northern Italy, is one of the very first great iron constructions designed through the practical application of the theory of the ellipse of elasticity, a graphical-analytical method of structural analysis that was developed in the 19th century. It embeds a natural discretisation of the structure into a series of elastic elements, treated then with standard tools of geometry of masses. In this work, the application of such theory to the calculation of the parabolic arch of the bridge is inquired, attempting to breathe, at the same time, the beauty of the architectonic and structural conception directly linked to that; later, results are compared with much modern approaches that also consider now-available numerical discretisation methods. A further, definite aim of this work is also that of trying to promote interest on the bridge, on its actual state of conservation and future destinations. Not only it represents a true industrial monument and a living testimony of the scientific and technological developments of the time but also a beautiful, effective achievement of architecture and engineering through the methods of Strength of Materials.

#### 1 INTRODUCTION

#### 1.1 The Paderno d'Adda Bridge

The Paderno d'Adda Bridge, also called San Miche-le Bridge, is a metallic viaduct that crosses the Adda river between Paderno and Calusco d'Adda to a height of approximately 85 m from water, allowing to connect the two provinces of Lecco and Bergamo, near Milano, in the Lombardia region, northern Italy (SNOS 1889, Nascè et al. 1984). At that location the river flowsdown from the exit of Lecco's branch of Como's lake to the river Po through an impressive natural scenery that even seems to have inspired celebrated paintings by Leonardo (Fig. 1).

The main upper continuous beam, 5 m wide, is formed by a 266 m long metallic truss supported by



Figure 1. Front view of the Paderno d'Adda Bridge, 1887–1889 (from up-stream; left bank Calusco, right bank Paderno). A crossing train is visible inside the upper horizontal continuous beam. Automotive and pedestrian traffic runs on top.

nine bearings. Four of these supports are provided by a marvellous doubly built-in parabolic metallic arch of about 150 m of span and 37.5 m of rise. The bridge shares its architectural style with similar arch bridges built in Europe at the time (Timoshenko 1953, Benvenuto 1981, Nascè et al. 1984), like e.g. that of Garabit (1884, France, Eiffel and Boyer) and Maria Pia (1887, Oporto, Eiffel and Seyrig), both doubly hinged at the shoulders, and the Dom Luiz I (1886, Oporto, Seyrig), doubly built-in as that of Paderno. The viaduct was quickly constructed between 1887 and 1889 (thus practically at the same time of the most celebrated Tour Eiffel), to comply with the needs of the rapidly growing industrial activities in Lombardia. It was built by the Società Nazionale delle Officine di Savigliano (SNOS), Cuneo, Italy, under the technical direction of Swiss Engineer Giulio Röthlisberger (1851-1911), the man whom the design of the bridge is normally attributed to. He was formed at the Polytechnic of Zürich, graduated in 1872 and got later in charge of the Technical Office of the SNOS since 1885, for 25 years. The bridge is still in service, with alternated oneway automotive traffic, restricted to no heavy-weight vehicles, and trains crossing at slow speed.

The bridge is designed through the graphicalanalytical methods of structural analysis that were booming in the 19th century (Culmann 1880, Timoshenko 1953, Benvenuto 1981). Specifically, it is a remarkable application of the so-called *theory of the ellipse of elasticity* (Culmann 1880, Belluzzi 1942). This theory was originally conceived by Karl Culmann (1821–1881) and then systematically developed and applied by his pupil Wilhelm Ritter (1847-1906). It represents a very elegant method for the analysis of the flexural elastic response of a structure and is based on an intrinsic discretisation of a continuous beam in a series of elements, each with a proper *elastic weight*, directly proportional to its length and inversely proportional to its bending stiffness. The theory of the ellipse of elasticity is based on the concepts of projective geometry, which lead to a correspondence between the ellipse of elasticity of the structure and the central ellipse of inertia of the distribution of the elastic weight of the structure. This correspondence brings back the problem of the determination of the flexural elastic deformation of a beam to a problem of pure geometry of masses, of more convenient solution and direct interpretation in terms of the design of the structure.

# 1.2 Main technical features of the bridge

The main technical features of the bridge are reported in details in Nascè et al. (1984), which is, to our knowledge, the most comprehensive publication, and one of the very few, concerning the bridge. We rely very much on this very valuable contribution and on the Technical Report (SNOS 1889) that was originally issued at the time of the first try-out. Here, the essential characteristics are reported.

The 266 m long upper flyover is made by a continuous box girder with nine equally-distributed supports, at 33.25 m distance from each other. Four of the supports are sustained by a big parabolic metallic arch; two of them bear directly on the same arch's masonry abutments (made with Moltrasio masonry, with Baveno granite coverings); a seventh, on the Calusco bank, rests on a smaller masonry foundation placed between the arch shoulder and the higher bridge supports; the last two, in masonry work as well, are the two direct beam bearings at its two ends, on top of the two river banks. The four piers resting on the arch are placed symmetrically, in between keystone, haunches and shoulders of the arch. The inner side of the beam girder, on which the railway is located, runs at about 255.00 m on the sea level (osl); the rails are placed at 255.45 m osl, the upper road at 261.75 m osl. The main vertical longitudinal trussed beams of the upper continuous girder are 6.25 m high and placed at a respective transverse distance of 5.00 m, leaving a free passage for the trains of 4.60 m. They are composed of two main T-ribs connected by a metallic truss. The upper-level road is 5.00 m wide and includes also two additional cantilever sidewalks, each 1 m long, with iron parapets 1.50 m high.

The big arch is composed by two couples of secondary inclined arches. Each couple is formed by two arches posed at a respective distance of 1 m and laying symmetrically to a mean plane inclined of about

 $\pm 8.63^{\circ}$  to the vertical. The parabolic axis of the arch has a span of 150.00 m and rise of 37.50 m in the inclined plane; the transverse arch's cross section is 4.00 m high at the keystone and 8.00 m high at the abutments (i.e. in the same 1:2 ratio between rise and half span). The two mean inclined planes of the arches are located at a distance of 5.096 m at the keystone and 16.346 m at the shoulders. The wall of each composing arch is also a truss structure with two main T-ribs connected by vertical and inclined bars. The two couples of twin arches are gathered together by two transverse brace systems located at the extrados and at the intrados of the arch's body. In essence, the resulting cross section of the main parabolic arch supporting the horizontal beam is trapezoidal, with variable, increasing cross section from the crown to the shoulders. This, and specifically the inclination of the twin arches, is a beautiful key feature of Röthlisberger's conception of the bridge, in view of counteracting effectively wind and transverse horizontal actions in spite of the considerable slenderness of the structure. The arch cross section at the impost is inclined of 45° to the horizontal, so as the local tangent to the parabolic axis of the arch to the vertical. The vertical bridge piers that sustain the upper continuous beam are made by eight T-section columns, linked to each other by a brace system with horizontal bars and St. Andrew's crosses and, on top, by transverse beams that directly serve as supports for the bearing devices of the upper beam. For inspection and maintenance purposes a 1 m large boardwalk is provided into the body of the arch and, inside the bridge piers, a system of ladders along their height. The bridge is a riveted wrought iron structure of about 2600 t of metals, with near 100000 rivets just in the arch.

# 1.3 Aim of this work

In this work, which in its main part largely refers to a study developed in a Laurea Thesis (Ferrari 2006), a detailed analysis of the SNOS Report (1889) is presented. The point of view here is the following: inquire the application of the theory of the ellipse of elasticity to the calculation of the bridge, breathe the beauty of the architectonic and structural conception directly linked to that, compare results with modern structural approaches that also consider now-available numerical discretisation methods.

After a careful review of the Report by the SNOS, a full 3D truss Finite Element model of the arch of the bridge has been elaborated, based also on direct inspections of the bridge and on the screening of the marvellous original drawings that are guarded at the Archivio Storico Nazionale di Torino. Different loading conditions have been considered and results compared with those reported in the SNOS Report, showing the remarkable accuracy of the adopted graphical-analytical methods and allowing to experience the unrepeated beauty of the original analysis with respect to rather impersonal computer structural analysis. Moreover, the model that has been put in place shows promise for possible further analyses that could inspect other behaviours of the bridge, as for example dynamical and inelastic, also connected to the present and future state of conservation of the structure. These aspects are left for further developments of the present study.

The paper is organized as follows: Section 2 provides a short account on the theory of the ellipse of elasticity; Section 3 reports its application to the structural analysis of the arch of the bridge; Section 4 presents an independent validation of the original design results with present analytical-numerical methods.

# 2 ON THE THEORY OF THE ELLIPSE OF ELASTICITY

## 2.1 Fundamentals

The theory of the ellipse of elasticity can be considered as a main icon of the so-called Graphical Statics, the discipline which often characterised the resolving approach of practical design problems during the 2nd half of the 19th century. It represents a very elegant and practical method for the analysis of the flexural response of an elastic structure. It is based on an intrinsic discretisation of a continuous elastic problem. This theory is basically associated to the two outstanding figures of Culmann and Ritter, but also of people, like Giulio Röthlisberger, that were formed at the time at the Polytechnical Schools in Europe and that became later structural engineers and designers and largely contributed in the practical and effective application of the method.

The theory is based on the following main hypotheses (we refer here to the Italian text by Belluzzi 1942, which reports results from the technical literature of the time, basically ascribed to the two names of Culmann and Ritter): (a) linear elastic behavior of the material and the structure, which leads to the proportionality between acting forces and (reversible) displacements provoked by them (property that in turn implies the validity of the principle of superposition of effects); (b) existence of the ellipse of elasticity, referred to a section of a structure; (c) correspondence between the latter and the central ellipse of inertia of the distribution of the so-called elastic weight of the structure. This correspondence transforms the problem of the determination of the elastic response of a continuous structure to a task of pure geometry of masses. The latter can be feasibly handled by taking advantage of the assumed discrete character of the distribution of the elastic weight and is endowed with a visible interpretation of the elastic performance of the structure,



Figure 2. Representation of a force R applied along line of action r to a beam section A and the centre of rotation C of A. C' is the symmetric of C with respect to the centre S of the ellipse of elasticity of beam AB referred to section A.

in view of its conception and design. Furthermore, the ellipse itself may actually play the role of an hidden, underlying, graphical construct. Indeed, the properties of projective geometry that are attached to that allow the elastic solution of the structure even without the explicit drawing of the ellipse itself. The methods are also said graphical-analytical because, in practice, main technical steps that are framed on the graphical constructions may be carried-out analytically, by working-out formulas that arise from the inspection of the drawings (Belluzzi 1942).

The concept of the ellipse of elasticity referred to a section of an elastic structure is achieved by inspecting the correspondence existing between the line of action r of a force R applied to a section A of a general, curvilinear, elastic beam (with little curvature and continuously-varying cross section) and the centre of rotation C of the same section (Fig. 2).

In particular, refer to a planar beam acted upon by forces laying in the same plane and cross sections of the beam that, during the beam's deformation, are assumed to remain plane and perpendicular to the geometric axis, also deforming in its original plane. The theory states that there exists an involutory relationship between the line of action r and the centre of rotation C of the section. Moreover, the ellipse of elasticity is the fundamental real conic of the polarity existing between the line of action r and the point C', which is the symmetric of C with respect to the centre S of the ellipse. In other words, the ellipse of elasticity can be defined as the fundamental conic with respect to which the lines of action r and the respective centres of rotation C correspond to each other through an antipolarity relationship.

The determination of the central ellipse of inertia of the distribution of the elastic weight of the structure, which coincides with the ellipse of elasticity above, is linked first to the definition of the general concept of elastic weight and then to the quantification of its distribution for the structure under consideration. The concept of elastic weight goes as follows. If on a section A of a beam, a moment M acts in the plane which contains the geometric axis, it causes a rotation  $\phi$  of A, around the centre S of the ellipse of elasticity. The rotation is proportional to the applied moment M as:

$$\varphi = M \cdot G$$
, or  $G = \frac{\varphi}{M}$ , (1)

where  $\mathcal{G}$  represents the so-called *elastic weight* of the beam. Thus,  $\mathcal{G}$  can be defined as the angle of rotation  $\phi$  that is caused by the application of a unitary moment M = 1; it depends on the beam's geometrical and physical properties; it gives a global measure of the beam's aptitude to deform. In case of a straight cantilever beam of length *l* loaded by a moment *M* at its free end, composed by a linear elastic material with Young's modulus *E* and endowed with a constant cross section with moment of inertia *J* with respect to the axis perpendicular to the beam's plane, it turns out that, referring to the case of flexure of de Saint Venant, the rotation of the free end is:

$$\varphi = \frac{Ml}{EJ} \quad \Rightarrow \quad G = \frac{\varphi}{M} = \frac{l}{EJ}.$$
 (2)

This relation clearly illustrates the physical meaning of  $\mathcal{G}$  as the global parameter that expresses the flexural elastic deformability of the structure.

Now, the idea arises of thinking at the structure as the assembly of a series of discrete elastic elements of length  $\Delta s$ , each with a proper elastic weight

$$\Delta G = \frac{\Delta s}{EJ},\tag{3}$$

such that the total elastic weight of the structure is represented by the discrete distribution of these elastic weights. It is possible to demonstrate that such a sought distribution of  $\mathcal{G}$  is univocally known only for a statically-determined structure, whereas for a statically-undetermined structure the distribution of elastic weights is not univocally defined. This is not surprising, due to the redundancy of equilibrium in an hyperstatic system. Despite this, an hyperstatic structure can still be solved, via the Forces Method (with hyperstatic quantities as unknown), through the superposition of effects on underlying isostatic structures and imposition of the corresponding compatibility conditions. As the underlying isostatic structure can also be analysed with a univocally-defined distribution of elastic weights, such distribution can also be used to solve the original hyperstatic structure. Thus, indirectly, its ellipse of elasticity can in essence be determined, so the corresponding ellipse of inertia of the distribution of elastic weights.

This allows one to write the so-called "theorems of the theory of the ellipse of elasticity" (Belluzzi 1942), as a function of the properties of the distribution of elastic weights. For example, the rotation  $\phi$ 



Figure 3. Representation of the quantities reported in Eq. (4) for the calculation of the rotation and displacements of terminal section A caused by a force Q applied to the same section.

and horizontal and vertical displacements  $d_x$ ,  $d_y$  of a terminal beam section A caused by a force Q applied to the same section along line of action q (Fig. 3), can be nominally written as follows:

$$\begin{cases} \varphi = Q \cdot S_q = Q \cdot u_s \cdot G, \\ d_x = Q \cdot J_{xq} = Q \cdot y_s \cdot u_x \cdot G, \\ d_y = Q \cdot J_{yq} = Q \cdot x_s \cdot u_y \cdot G, \end{cases}$$
(4)

where  $\mathcal{G} = \sum \Delta \mathcal{G}$  represents the total elastic weight of the structure;  $S_q$ ,  $J_{xq}$ ,  $J_{yq}$  the static moment of  $\mathcal{G}$  with respect to q and the centrifugal moments of inertia of  $\mathcal{G}$  with respect to q and axis x, and q and axis y. These parameters depend only on the distribution of elastic weights and on the position of the applied load Q, and can be expressed as a function of the quantities  $x_S$ ,  $y_S$ ,  $u_S$ ,  $u_X$ ,  $u_Y$  depicted in Fig. 3.

The point S in Fig. 3 represents the centre of gravity of the elastic weights of the structure; the points X,Y represent the antipoles of the reference system axes x, y with respect to the central ellipse of inertia of the elastic weights. It is then apparent that, once the position of points S, X, Y and total elastic weight G are found, the elastic response of the structure is determined. The coordinates  $x_S$ ,  $y_S$ ,  $x_Y$ ,  $y_X$  defining the position of these points can be evaluated by standard calculations of geometry of masses, once given the discrete distribution of elastic weights.

#### 2.2 Application to a doubly built-in parabolic arch

In the SNOS Report (1889), the remarkable application of the theory of the ellipse of elasticity to the analysis of the arch of the bridge refers to the determination of the elastic response of a parabolic arch, built-in at the two extremities, which is subjected to a vertical load P (that can be put equal to 1) and acting in an arbitrary position along the arch, at a horizontal distance a from left extreme A (Fig. 4).

First, the position of the line of action of the reaction *A* needs to be determined. This can be solved by a



Figure 4. Calculation of line of action LO of left reaction A; the segments  $\overline{FL}$  and  $\overline{VO}$  that the left reaction A locates on the vertical lines from A and on load P, below and above the horizontal line from centre S are determined.



Figure 5. Calculation of line of action HK of left reaction A; segments a, b, c to be drawn at points S,X,Y are determined analytically. Points H,K are then located, so the direction of A.

graphical-analytical procedure. Figs. 4 and 5 represent two ways to solve this problem (SNOS 1889, Ferrari 2006). Underlying to these constructions lay the compatibility conditions  $\phi = 0$ ,  $d_x = 0$ ,  $d_y = 0$  for the left built-in constraint at terminal section A. These can be worked-out from Eq. (4), leading to:

$$\begin{cases} P \cdot u'_S + A \cdot u_S = 0\\ P \cdot u'_X + A \cdot u_X = 0\\ P \cdot u'_Y + A \cdot u_Y = 0 \end{cases} \implies \frac{u_S}{u'_S} = \frac{u_X}{u'_X} = \frac{u_Y}{u'_Y} = -\frac{P}{A}, \quad (5)$$

where A is now taking the role that force Q had in (4) and the superposition of effects is considered with load P, so that  $u'_S = a$ ,  $u'_X = b$ ,  $u'_Y = c$  (a, b, c denote segments used below in Fig. 5) are quantities similar to those entering Eq. (4), but related to load P at position a from A. They can be calculated analytically as follows, given the distribution of elastic weights:

$$u'_{S} = \frac{1}{G} \cdot \sum_{a}^{l} (x-a) \cdot \Delta G; \quad u'_{X} = \frac{1}{y_{S} \cdot G} \cdot \sum_{a}^{l} y \cdot (x-a) \cdot \Delta G; \\ u'_{Y} = \frac{1}{x_{S} \cdot G} \cdot \sum_{a}^{l} x \cdot (x-a) \cdot \Delta G.$$
(6)

Once scalars  $u'_{S} = a$ ,  $u'_{X} = b$ ,  $u'_{Y} = c$  are evaluated, the construction in Fig. 5 determines the position of A.



Figure 6. Equilibrium requires that the left and right reactions A and B form with load P a closed polygon of forces. Segments  $V = \overline{OQ}$  and  $V' = \overline{QP}$  represent the vertical components of A and B, with V + V' = P; segment  $H = \overline{TQ}$  their horizontal component.

Alternatively, one may also proceed as sketched in Fig. 4, by calculating parameters  $\mu$ ,  $\nu$  and segments  $\overline{FL}$ ,  $\overline{VO}$ , with same results. The last relations in (5) also give the magnitude of reaction *A*, given *P*.

Once the position of the lines of action of left builtin reaction A, and, by equilibrium, of right reaction B are known, the value of their vertical components V, V' and (common) horizontal component H can be found as follows:

$$V = P \cdot \frac{a'f}{a'f + af'}, V' = P \cdot \frac{af'}{a'f + af'}; H = P \cdot \frac{aa'}{a'f + af'}, \quad (7)$$

where quantities a, a', f, f' are represented in Fig. 6.

Following similar arguments, it is possible to derive the in-plane arch's deflections  $d_y$  at any position x of the arch, for the different locations a of the load P acting on the arch (SNOS 1989). The equation that gives the mean parabolic line at the arch is taken as:

$$y = \frac{4f}{l^2} \cdot x \cdot (l - x), \qquad (8)$$

where *f* and *l* represent now the rise and span of the arch. The profile of the arch is symmetric with respect to the vertical axis x = l/2 at half span. The formulas for  $d_y$  obtained by the SNOS with this graphical-analytical procedure correspond indeed to those that may be obtained by the application of the Virtual Works Principle (VWP):

$$d_{y} = \frac{x^{2}}{6C} \cdot \left( V \cdot x - H \cdot \frac{2f}{l^{2}} \cdot x \cdot (2l - x) + 3M + -\frac{1}{x^{2}} \cdot \sum_{0}^{x} P \cdot (x - a)^{3} \right),$$
(9)

where H, V, M represent the "components" of the reaction force A (built-in moment M is positive clockwise) and C = EJdx/ds = cost is a quantity related to



Figure 7. Discretisation of the half arch in 14 elements with  $\Delta x = cost$ , with indication of the geometrical quantities useful to determine the corresponding elastic weights (quotes in m).

the bending stiffness and local inclination of the arch. C is assumed by the designer constant along the arch, since while J decreases from the shoulder to the keystone, the ratio dx/ds between horizontal projection dx of infinitesimal element length ds of the arch and ds itself increases (SNOS 1889).

# 3 STRUCTURAL ANALYSIS OF THE BRIDGE

## 3.1 Explicit application of the theory of the ellipse of elasticity to the analysis of the arch of the bridge

Going to the explicit definition of the distribution of elastic weights made by the SNOS (1989), all the calculations reported there were developed considering an arch which is the projection on their mean inclined plane of one of the two couples of inclined parabolic arches that are placed symmetrically to the vertical longitudinal median plane of the bridge. Such arch is placed on a plane inclined of about  $\alpha = 8.63^{\circ}$  to the vertical (such that  $\sin \alpha = 0.15$ ) and consists of a truss beam with parabolic axis of 150 m of span and 37.5 m of rise in such plane, having extrados and intrados lines both described by parabolic functions so as to determine a cross-high of the arch of 4 m at the keystone and of 8 m at the abutment. On the basis of this model, the elastic weights of the structure have been calculated according to a symmetric structural discretisation with 28 elements of different  $\Delta s$  extensions as reported in Fig. 7.

In the Report, the procedure adopted to calculate the elastic weights  $\Delta \mathcal{G} = \Delta s/EJ$  is not really apparent. An attempt of careful analysis is provided in Ferrari (2006). Also, the elastic weights are actually

determined without the constant proportionality factor  $E = 17000000 \text{ t/m}^2$ , Young's modulus of the iron, i.e.  $\Delta G' = \Delta s/J$ . The coordinates  $x_S$ ,  $y_S$ ,  $x_Y$ ,  $y_X$  and total elastic weight G' = EG are finally found as:

$$x_s = 75 m, y_s = 28.787 m;$$
  
 $x_y = 92.43 m, y_x = 31.747 m; G' = 6.5536 m^{-3}.$  (10)

The constant C in Eq. (9) is also evaluated in  $55539000 t \cdot m^2$ .

Anyway, tables are presented in which the left reaction and the deflections of the arch are determined for a unitary load located at the various elastic elements. In practice, these influence coefficients, which are later used for the design of the truss members, are determined by a true application of the theory of the ellipse of elasticity as applied to the arch.

#### 3.2 Evaluation of the loads

The SNOS Report analyses independently, one by one, the various loadings on the arch, for subsequent superposition of effects:

- permanent weight of the arch;
- permanent weight of the upper girder beam, of the bridge piers and vertical actions induced by the wind acting on the girder beam;
- accidental vertical load on the upper girder beam;
- temperature effects and compression on the arch due to the horizontal thrust *H*;
- direct horizontal wind action on the arch.

For each listed item, the SNOS reports the calculation of the stresses in the various arch's elements, as well as the final value that arises by the superposition of effects.

#### 3.3 Dimensioning of the structural elements

The Report by the SNOS provides the calculation of the stresses in the various bar elements of the arch and at the stone abutments as compared to the target admissible values that are summarised below. This is done for the final geometries of the structural members. On the other end, the Report does not provide specific information about the design procedure that has lead, through pre-dimensioning, to such final structural dimensions. As the overall architectural and structural conception of the entire viaduct, these phases seem to be linked to the engineering practice and experience of the designer with the standards of metallic carpentry in use at the time.

The material employed in the structural members of the bridge is a wrought iron, with very low carbon percentage of about 0.01% (Nascè et al. 1984). The admissible stresses are taken differently for each structural component of the bridge: for the main upper and lower arch's ribs  $6.0 \text{ kg/mm}^2$ ; for the vertical and



Figure 8. Scheme with four test loading configurations, with indication of the four piers resting symmetrically on the arch (view from down-stream; Paderno left side, Calusco right side). Pier III is at half length of the upper continuous beam.

diagonal bars linking the inferior and superior ribs of the arch  $6.0 \text{ kg/mm}^2$ ; for the elements that compose the transverse bridge's brace system counteracting wind  $4.2 \text{ kg/mm}^2$ . For each arch's stone abutment, the allowable compression stress is assumed in nearly  $31 \text{ kg/cm}^2$ . Stresses are all found safely below these target values.

The executive drawings, including the various cross sections of the structural elements, the details of the riveted joints and part of the graphical/analytical calculations are reported in 147 marvellous drawing tables. They are still in an excellent state of conservation and show the meticulous design of the complex structure of the bridge. Often, the calculus is integrated in the drawing itself, which testimonies the beautiful, intimate link between conception, structural analysis and executive design.

#### 3.4 Viaduct's tests

The first viaduct's tests took place from 12th to 19th May 1889. The measured deflections for loading conditions conforming to those considered at design stage were compared to the corresponding theoretical values. To this purpose, the tests were carried out in two moments: first, the different road loads were obtained by deposition of gravel on the upper deck; second, with a uniformly distributed gravel load of 3.9 t/m all over the road, 6 locomotives with tender, each of 83 t of weight, corresponding to a distributed load of 5.1 t/m, were displaced on the railway track following four different loading configurations (Fig. 8). Tables 1-2 below list the calculated and measured vertical deflections. The good overall agreement doubtlessly shows the validity of the structural approach used by the SNOS at design stage, in species the effectiveness of the theory of the ellipse of elasticity as applied to the explicit calculation of the arch of the bridge.

A final test was also run with 3 locomotives and 30 wagons, loaded by gravel, for a total weight of the convoy of 600 t. The train was let running 3 times,

Table 1. Arch's vertical deflections calculated at design stage for four different loading tests (SNOS 1889, p. 71). Negative values indicate downward displacements.

Deflections	Pier I	Pier II	Vertex	Pier III	Pier IV
	(m)	(m)	(m)	(m)	(m)
Test I	+0.0033	-0.0010	-0.0045	$-0.0108 \\ -0.0080 \\ -0.0022 \\ +0.0035$	-0.0066
Test II	-0.0016	-0.0080	-0.0097		-0.0016
Test III	-0.0040	-0.0101	-0.0094		+0.0027
Test IV	-0.0068	-0.0064	-0.0010		+0.0031

Table 2. Arch's vertical deflections measured in situ at the try-out for the corresponding loading tests (SNOS 1889, p. 71).

Deflections	Pier I (m)	Pier II (m)	Vertex (m)	Pier III (m)	Pier IV (m)
Test I	+0.0038	+0.0001	-0.0044	-0.0106	-0.0056
Test II	0	-0.0079	-0.0080	-0.0102	-0.0012
Test III	-0.0016	-0.0102	-0.0073	-0.0014	+0.0025
Test IV	-0.0063	-0.0058	-0.0007	+0.0042	+0.0026

with speed up to 45 km/h. Transverse oscillations at the keystone were recorded in less than 3.142 mm and vertical deflections in less than 10 mm. According to Nascè et al. (1984) a 2nd, final, try-out took place in June 1892, with different modalities and applied loads but with similar results.

## 4 VALIDATIONS OF THE REPORTS' RESULTS BY ANALYTICAL-NUMERICAL METHODS

## 4.1 Analysis of the elastic arch by the VWP

In order to compare the results presented by the SNOS, the hyperstatic scheme of the doubly built-in arch was solved by the Virtual Works Principle. Such analysis allowed to notice some little inconsistencies in the SNOS results. First, the reactions A and B due to P = 1at distance a from left support A were evaluated and compared to the Report's results. The match was not perfect, particularly for the bending moments at the extremes of the arch. The differences did not seem to be due to transcription errors, since the values reported by the SNOS look coherent with formulas and tables presented within the text. Second, further validation concerning the arch's deflections was attempted, since they also did not seem to be calculated through the reactions that are listed in the Report (Ferrari 2006). However, the final values reported by the SNOS correspond, to a good degree of accuracy, to the present results obtained by the VWP (Tables 3-4). The values reported in Tables 3-4 refer to a precise load configuration (1st distribution, see Section 4.2.2 and Table 5) and

Table 3. Bending deflections listed in SNOS (1889, p. 59).

Load	Pier I	Pier II	Pier III	Pier IV
Point	(m)	(m)	(m)	(m)
Pier I	-0.0089	-0.0029	+0.0071	+0.0042
Pier II	-0.0011	-0.0042	+0.0026	+0.0030
Pier III	-0.0004	-0.0003	+0.0005	+0.0001
Pier IV	+0.0001	+0.0001	-0.0000	-0.0001
Total	-0.0103	-0.0073	+0.0102	+0.0071

Table 4. Bending deflections calculated by the VWP.

Load Point	Pier I (m)	Pier II (m)	Pier III (m)	Pier IV (m)
Pier I Pier II Pier III Pier IV	$\begin{array}{r} -0.00893 \\ -0.00126 \\ -0.00038 \\ +0.00006 \end{array}$	$\begin{array}{r} -0.00298 \\ -0.00425 \\ -0.00033 \\ +0.00010 \end{array}$	+0.00703 +0.00255 +0.00055 -0.00004	+0.00422 +0.00297 +0.00016 -0.00013
Total	-0.01051	-0.00746	+0.01009	+0.00722

Table 5. Total vertical deflections (1st distribution) reported by SNOS (1889, p. 60) and present results by the FE model.

Point	Vertical loads (t)	SNOS deflections (m)	FE deflections (m)
Pier I	+340.6	-0.0120	-0.015903
Pier II	+144.0	-0.0113	-0.013847
Pier III	-18.5	+0.0062	+0.010322
Pier IV	+4.9	+0.0054	+0.007222

regard the vertical deflections (due to pure bending) at the four piers due to forces acting at their locations.

The reason of the discrepancies above may be due to the fact that the SNOS might have used in the final Report also data concerning a preliminary project. Indeed, the bridge's layout was slightly modified in the executive project, due to new requirements on the railway trace that were posed by the Strade Ferrate Meridionali, after checks on the Adda's banks (Nascè et al. 1984). It is therefore possible that some specific data were referring to a previous project, while important global quantities, such as the deflections caused by external actions, were indeed corresponding to the final one. As a matter of fact, even if this Report has been probably conceived to present to a general audience the main steps of the calculations, including a very valuable account on the theory of the ellipse of elasticity and on its explicit application to the analysis of the arch, it is doubtlessly very concise and it obviously presents concepts and practical considerations that may not be directly apparent to the contemporary reader.

#### 4.2 Structural analysis of the arch by the FEM

Currently, an attempt is made of building a full FE model of the bridge. So far, a truss mesh of the arch of the bridge has been developed and appropriate loading conditions has been considered for validation of the previous results. The FE analysis has been run with the commercial code ABAQUS<sup>®</sup>.

# 4.2.1 Structural model

The FE model consists of a 3D truss frame, reproducing as much as possible the actual arch geometry. It consists of two planar parabolic trusses referring to the in-plane geometry of the arch (see Fig. 7), placed in two inclined planes (of  $\pm 8.63^{\circ}$  to the vertical). The inclined planes are placed at a distance of 5.096 m from each other at the axis of the arches at the keystone. The truss nodes are linked to each other through a reticular system that corresponds to the actual bracing of the arch. To each bar of the model, a cross section with equivalent geometrical characteristics is attributed (area, principal moments of inertia, torsional stiffness). Some approximations were made as regards to the section's attribution to the superior and inferior arch ribs, which are made with a variable number of longitudinal plates. Also, at the node junctions, there are additional reinforcing plates, for local stiffening. In spite of this, the model has been simplified with bars of constant average cross section. The model is comprised of 752 beam elements and 266 nodes. Builtin constraints are imposed at the nodes of the arch shoulders.

#### 4.2.2 *Obtained results*

First trial loading cases considered a unitary load (1 t) applied at the piers and at the keystone, in view of verifying the order-of-magnitude agreement with the deflections that were listed in the Report and calculated here by the VWP. After these preliminary checks, the deflections were evaluated for five given load distributions with vertical loads acting at the four piers (SNOS 1889, p. 58–62).

A sample of these outcomes is given in Figure 9 for the 1st load distribution already considered in Tables 3–4, with main results summarized in Table 5. Such distribution considers spans 2–3 charged by a uniformly distributed load of 9 t/m, leads to a maximum pressure on Pier I and is somehow similar to Test IV considered in the try-outs (Fig. 8). The maximum compressive axial force at the intrados of the left shoulder is found in 377.0 t, in good agreement with the value 751.6/2 = 375.8 t calculated by the SNOS (1889, p. 62). Agreement is also found for the horizontal thrust *H* and vertical reactions *V*, *V'* at the shoulders, with H = 185.8 t, V = 417.2 t, V' = 53.8 t. The FE outcomes confirm, to a quite good degree of accuracy, the SNOS results, showing the true potential



Figure 9. FE deformed configuration of the arch for the 1st load distribution reported in Table 5 (amplification factor = 500).

of the theory of the ellipse of elasticity as applied to the elastic analysis of the arch of the bridge.

## 5 CLOSING REMARKS

An attempt to scrutinise in details the SNOS Report (1889) has been made in view of breathing the original conception of the design of the bridge and its specific calculation through the elegant method of the theory of the ellipse of elasticity. The intrinsic discretisation in elastic elements is remarkable and makes a natural connection with discretisations that can now be provided by FE codes. The beauty of the executive project can be totally appreciated only by the complementary screening of the 147 drawing tables of the bridge. All this shows the ingenuous, effective and beautiful approach to the design of the bridge. One might think at this once contemplating the giant still standing there silently, serving since almost 120 years of duty, with an actual state of conservation that actually poses serious questions about its future survival. We should take care of it.

## REFERENCES

- Belluzzi, O. 1942. Scienza delle Costruzioni. Bologna: Zanichelli.
- Benvenuto, E. 1981. La Scienza delle Costruzioni e il suo Sviluppo Storico. Firenze: Sansoni.
- Culmann, K. 1880. Traitè de Statique Graphique. Paris: Dunod.
- Ferrari, R. 2006. Sulla concezione strutturale ottocentesca del ponte in ferro di Paderno d'Adda secondo la teoria dell'ellisse d'elasticità. Laurea Thesis in Building Engineering. Advisor E. Rizzi, Università di Bergamo, 228 pages.
- Nascè, V., Zorgno, A.M., Bertolini, C., Carbone, V. I., Pistone, G., Roccati, R. 1984. Il ponte di Paderno: storia e struttura – Conservazione dell'architettura in ferro. Restauro, Anno XIII, n. 73–74, 215 pages.
- Società Nazionale delle Officine di Savigliano 1889. Viadotto di Paderno sull'Adda (Ferrovia Ponte S. Pietro-Seregno). Torino: Tip. e Lit. Camilla e Bertolero.
- Timoshenko, S.P. 1953. History of Strength of Materials. New York: McGraw-Hill.