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CONSTITUTIVE RELATIONS OF ORTHOTROPIC ELASTIC DAMAGE WITH DUAL PROPERTIES

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Abstract

A phenomenological constitutive framework of orthotropic elastic damage in initially-isotropic materials is presented. Focus is made on secant stress/strain relations that are derived through the application of the so-called damage-effect tensors, namely the fourth-order operators that define the linear transformations between nominal and effective stress and strain quantities. In the attempt to provide selected forms of anisotropic damage approaching general orthotropy, several proposals of damage-effect tensors are formulated. Such fourth-order operators are obtained from the general orthotropic representations as particular instances that satisfy a specific duality requirement between compliance- and stiffness-based derivations.

Key words: Orthotropic damage, 4^{th}-order damage-effect tensor, 2^{nd}-order damage tensor, tensor inverses, dual secant relations

1 Introduction

Starting from the original contributions by Kachanov [10] and Rabotnov [16], Continuum Damage Mechanics (CDM) has reached by now a considerable stage of development. This includes in particular the constitutive modeling of *anisotropic* elastic stiffness degradation in quasi-brittle materials such as e.g. concrete, rocks, composites. The CDM formulations are typically based on the introduction of damage variables of various tensor orders, e.g. scalars, vectors, second- and fourth-order tensors (see the comprehensive reference lists provided in both research articles, e.g. [23, 5, 6, 3], and specific monographs on the subject that are now available, e.g. [13, 11, 19].

The present authors have contributed to the topic with a proposal of a unified theoretical framework of elastic stiffness degradation and damage based on a loading surface [5], and with the formulation of constitutive models for anisotropic stiffness degradation in initially-isotropic materials [6, 7, 17]. The latter models are characterized by second-order symmetric damage tensor variables with evolution laws expressed in terms of a (non-holonomic) pseudo-logarithmic rate of damage. The resulting secant elastic relations correspond to the restricted form of orthotropic material behavior described by Valanis-type damage [21, 24].

During these investigations the request of deriving more general forms of orthotropic elastic degradation spontaneously arose, together with the desire of preserving at the same time a full *duality* between possible alternative compliance- and stiffness-based derivations of the constitutive relations. These dual properties can be read directly in the structure of the so-called *damage-effect tensors*, namely the fourth-order operators that, based on the underlying damage tensor variables, define the linear transformations between nominal and effective stress and strain quantities, and that may prescribe in practice the secant relations of elastic damage [15, 8, 1, 14]. Summaries of the different proposals of the damage-effect tensors in the literature are available e.g. in [12, 23, 22, 3].

Recently, the authors [18] have attempted a generalization of these previous propositions by providing a set of *dual orthotropic damage-effect tensors*, which are obtained from the general fourth-order orthotropic representations as specific instances that satisfy the duality requirement. The present note reconsiders some of these new proposals and provides additional instances of dual damage-effect tensors including those that complete a solution family based on a specific non-singular tensor generator.

The secant CDM relations of the elastic-damage model and the definition of the damage-effect tensors are provided in Section 2. There, the general orthotropic representations of fourth-order symmetric damage-effect tensors, and secant compliance and stiffness tensors, in terms of three 'shear-like' and six 'non-shear' coefficients and corresponding tensor addends are introduced, and the requirement of duality is precisely stated. To elucidate the type of representations that embed the sought dual structure, a few examples of both symmetric and non-symmetric dual damage-effect tensors are given first in Section 3, including a particular symmetric instance that lacks only two 'shear-like' coefficients and embeds all the 'non-shear' coefficients. Then, Section 4 outlines a complete family of symmetric solution instances based on a specific 'shear-like' generator, starting from the more general one that includes all the 'non-shear' coefficients, going through new solution instances that involve just subsets of the 'non-shear' coefficients (and work with or without constraints on the coefficients), to arrive finally at the sole 'shear-like' generator itself. All the solution instances of the family are expressed in complete invariant form and are resumed in synoptic Table 1 at the end of the section. A few final comments are also gathered in the closing section.

Notation. Compact or index tensor notation is used throughout. Second-order tensors are identified by boldface characters (e.g. $\mathbf{w}, \boldsymbol{\phi}, \boldsymbol{\epsilon}, \boldsymbol{\sigma}$), whereas fourth-order tensors are denoted by blackboard-bold fonts (e.g. $\mathbb{A}, \mathbb{C}, \mathbb{E}$). Symbols '$\cdot$' and ':' denote the inner products with single and double contraction. Superscript $^{\mathrm{T}}$ indicates the transpose operation applied either to second-order tensors, or to fourth-order tensors; componentwise $(w^{\mathrm{T}})_{ij}{=}w_{ji}$ and $(A^{\mathrm{T}})_{ijkl}{=}A_{klij}$. The dyadic product of second-order tensors is indicated with '$\otimes$' and defined as $(\mathbf{A}{\otimes}\mathbf{B}){:}\mathbf{C}{=}(\mathbf{B}{:}\mathbf{C})\,\mathbf{A}$, for any second-order tensors $\mathbf{A}, \mathbf{B}, \mathbf{C}$, whereas '$\underline{\overline{\otimes}}$' denotes the symmetrized dyadic product of second-order tensors defined as $(\mathbf{A}\,\underline{\overline{\otimes}}\,\mathbf{B}){:}\mathbf{C}{=}\mathbf{A}{\cdot}\mathbf{C}^s{\cdot}\mathbf{B}^{\mathrm{T}}$, for any second-order tensors $\mathbf{A}, \mathbf{B}, \mathbf{C}$, where $\mathbf{C}^s{=}(\mathbf{C}{+}\mathbf{C}^{\mathrm{T}})/2$ is the symmetric part of $\mathbf{C}$; componentwise $(\mathbf{A}{\otimes}\mathbf{B})_{ijkl}{=}A_{ij}\,B_{kl}$ and $(\mathbf{A}\,\underline{\overline{\otimes}}\,\mathbf{B})_{ijkl}{=}(A_{ik}\,B_{jl}{+}A_{il}\,B_{jk})/2$. $\mathbf{I}$ and $\mathbb{I}^s{=}\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}$ are respectively the second-order and symmetric (major and minor symmetries) fourth-order identity tensors; componentwise $I_{ij}{=}\delta_{ij}$ and $I^s_{ijkl}{=}(\delta_{ik}\delta_{jl}{+}\delta_{il}\delta_{jk})/2$, where δ_{ij} is the Kronecker delta ($\delta_{ij}{=}1$ if $i{=}j$, $\delta_{ij}{=}0$ if $i{\neq}j$). $\mathbb{I}^s$ maps any second-order tensor $\mathbf{A}$ into its symmetric part $\mathbf{A}^s$, i.e. $\mathbb{I}^s{:}\mathbf{A}{=}\mathbf{A}^s$, and any symmetric second-order tensor $\mathbf{B}{=}\mathbf{B}^{\mathrm{T}}$ into itself, i.e. $\mathbb{I}^s{:}\mathbf{B}{=}\mathbf{B}$. Symbol 'tr' denotes the trace operator applied to second-order tensors, i.e. $\operatorname{tr}\mathbf{A}{=}\mathbf{I}{:}\mathbf{A}{=}A_{ii}$. For more detailed definitions see e.g. [17,Appendix A].

2 Secant relations of orthotropic elastic damage

At any damage state the *nominal* (small) strain tensor $\boldsymbol{\epsilon}$ and stress tensor $\boldsymbol{\sigma}$ are related by the following secant elastic constitutive law:

$$\boldsymbol{\epsilon} = \mathbb{C}(\mathbb{C}_0, \boldsymbol{\mathcal{D}}) : \boldsymbol{\sigma}\ ; \qquad \boldsymbol{\sigma} = \mathbb{E}(\mathbb{E}_0, \bar{\boldsymbol{\mathcal{D}}}) : \boldsymbol{\epsilon}\ , \tag{1}$$

where $\mathbb{C}$ and $\mathbb{E}$ are the current positive-definite fourth-order compliance and stiffness tensors, inverse of each other (i.e. $\mathbb{C}{:}\mathbb{E}{=}\mathbb{E}{:}\mathbb{C}{=}\mathbb{I}^s$) and endowed with both major and minor symmetries. The current values of compliance $\mathbb{C}(\mathbb{C}_0, \boldsymbol{\mathcal{D}})$ and stiffness $\mathbb{E}(\mathbb{E}_0, \bar{\boldsymbol{\mathcal{D}}})$ start from their initial values $\mathbb{C}_0$, $\mathbb{E}_0$ in the undamaged state and evolve as functions of generally-defined damage tensor variables $\boldsymbol{\mathcal{D}}$, or of a dual damage tensor variables $\bar{\boldsymbol{\mathcal{D}}}$. Assuming that the undamaged behavior is isotropic, the initial compliance and stiffness are expressed by the classical relations

$$\mathbb{C}_0 = \frac{1+\nu_0}{E_0}\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I} - \frac{\nu_0}{E_0}\,\mathbf{I}\otimes\mathbf{I}\ ; \qquad \mathbb{E}_0 = 2\,G_0\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I} + \Lambda_0\,\mathbf{I}\otimes\mathbf{I}\ , \tag{2}$$

in terms of undamaged Poisson's ratio ν_0 and Young's modulus E_0, or undamaged shear modulus G_0 and Lamé's constant Λ_0. Alternatively, the undamaged bulk modulus K_0 could be employed instead of Λ_0, through the usual relations $3K_0{=}3\Lambda_0{+}2G_0{=}E_0/(1{-}2\nu_0)$, in a convenient volumetric/deviatoric representation of Eq. (2).

Through a purely phenomenological approach, the damage-state relations $\mathbb{C}{=}\mathbb{C}(\mathbb{C}_0, \boldsymbol{\mathcal{D}})$ and $\mathbb{E}{=}\mathbb{E}(\mathbb{E}_0, \bar{\boldsymbol{\mathcal{D}}})$ are derived here by following steps that are typical of the CDM framework (see e.g. the references quoted in the Introduction and the schemes provided in

[17, 7]): i) a constitutive law is introduced for the undamaged material relating *effective* strain and stress quantities, $\boldsymbol{\epsilon}_{\text{eff}}$ and $\boldsymbol{\sigma}_{\text{eff}}$, acting in the intact material between micro-cracks: $\boldsymbol{\epsilon}_{\text{eff}}=\mathbb{C}_0:\boldsymbol{\sigma}_{\text{eff}}$, $\boldsymbol{\sigma}_{\text{eff}}=\mathbb{E}_0:\boldsymbol{\epsilon}_{\text{eff}}$; ii) a relation between nominal and effective (stress or strain) quantities is assumed, in linear form, by introducing a non-singular fourth-order *damage-effect tensor* which is a function of the damage variables, e.g. $\mathbb{A}(\boldsymbol{\mathcal{D}})$ in the stress relation $\boldsymbol{\sigma}_{\text{eff}}=\mathbb{A}(\boldsymbol{\mathcal{D}}):\boldsymbol{\sigma}$; iii) a second link between nominal and effective states is postulated through a principle of *'energy equivalence'* [8], $\boldsymbol{\sigma}:\boldsymbol{\epsilon}/2=\boldsymbol{\sigma}_{\text{eff}}:\boldsymbol{\epsilon}_{\text{eff}}/2$, which automatically renders secant stiffness and compliance enjoying major symmetry. The following nominal/effective relations are then consistently assumed/obtained:

$$\boldsymbol{\sigma}_{\text{eff}} = \mathbb{A}\,(\boldsymbol{\mathcal{D}}) : \boldsymbol{\sigma}\ ,\quad \boldsymbol{\epsilon} = \mathbb{A}^{\text{T}}(\boldsymbol{\mathcal{D}}) : \boldsymbol{\epsilon}_{\text{eff}}\ ;\qquad \boldsymbol{\epsilon}_{\text{eff}} = \bar{\mathbb{A}}^{\text{T}}(\bar{\boldsymbol{\mathcal{D}}}) : \boldsymbol{\epsilon}\ ,\quad \boldsymbol{\sigma} = \bar{\mathbb{A}}\,(\bar{\boldsymbol{\mathcal{D}}}) : \boldsymbol{\sigma}_{\text{eff}}\ , \tag{3}$$

and compliance and stiffness are expressed as:

$$\mathbb{C}(\mathbb{C}_0,\boldsymbol{\mathcal{D}}) = \mathbb{A}^{\text{T}}(\boldsymbol{\mathcal{D}}) : \mathbb{C}_0 : \mathbb{A}\,(\boldsymbol{\mathcal{D}})\ ;\qquad \mathbb{E}(\mathbb{C}_0,\bar{\boldsymbol{\mathcal{D}}}) = \bar{\mathbb{A}}\,(\bar{\boldsymbol{\mathcal{D}}}) : \mathbb{E}_0 : \bar{\mathbb{A}}^{\text{T}}(\bar{\boldsymbol{\mathcal{D}}})\ , \tag{4}$$

where $\mathbb{A}\,(\boldsymbol{\mathcal{D}})=\bar{\mathbb{A}}^{-1}(\bar{\boldsymbol{\mathcal{D}}})$ and $\bar{\mathbb{A}}\,(\bar{\boldsymbol{\mathcal{D}}})=\mathbb{A}^{-1}(\boldsymbol{\mathcal{D}})$ are *dual* non-singular fourth-order damage-effect tensors, inverse of each other (i.e. $\mathbb{A}:\bar{\mathbb{A}}=\bar{\mathbb{A}}:\mathbb{A}=\mathbb{I}^s$) and endowed with minor symmetries (not necessarily major symmetry).

Concerning the dual underlying damage variables $\boldsymbol{\mathcal{D}}$ and $\bar{\boldsymbol{\mathcal{D}}}$ entering the dependence of the damage-effect tensors with the damage state, the model which is in the focus of the present paper makes use of positive-definite symmetric second-order tensor variables: the so-called integrity tensor $\bar{\boldsymbol{\phi}}$ of Valanis [21], varying between $\mathbf{I}$ and $\mathbf{0}$, or its inverse $\boldsymbol{\phi}=\bar{\boldsymbol{\phi}}^{-1}$, with complementary variation between $\mathbf{I}$ and $\boldsymbol{\infty}$. The square-root tensors $\mathbf{w}=\boldsymbol{\phi}^{1/2}$, $\bar{\mathbf{w}}=\bar{\boldsymbol{\phi}}^{1/2}$ are as well employed in notation to express explicitly the final functional dependence of $\mathbb{A}$ and $\bar{\mathbb{A}}$ on the damage variables.

Now, since either the damage-effect tensor $\mathbb{A}(\mathbf{w})$ or the damage-effect tensor $\bar{\mathbb{A}}(\bar{\mathbf{w}})$ could be postulated independently as the source ingredient of the constitutive formulation, we are interested in seeking particular instances of the general orthotropic representations of $\mathbb{A}(\mathbf{w})$ and $\bar{\mathbb{A}}(\bar{\mathbf{w}})$, with the property that their inverses display the structure of the transposes of the tensors obtained by replacing $\mathbf{w}$ with its dual inverse $\bar{\mathbf{w}}$ (or viceversa). Indeed, notice that inverse tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ play the same role in Eqs (3), (4), except for a transpose operation (which obviously matters only if the damage-effect tensors are not fully symmetric). The resulting damage-effect tensors are said then to possess *dual structures* [18].

General representations of orthotropic fourth-order tensors can be obtained either by algebraic decomposition, e.g. [25], or through representation theorems, e.g. [2, 4], and could be used for either the compliance and stiffness or the damage-effect tensors. The damage-effect tensors could be represented in both symmetric and non-symmetric forms. In the present paper focus is made mainly on symmetric representations, while non-symmetric expansions of the damage-effect tensors are treated in [18]. Then, the general representation of a symmetric damage-effect tensor $\mathbb{A}(\mathbf{w})$ representing orthotropic damage in initially-isotropic materials, e.g. [12], can be given as follows, according to the ordering proposed by Zysset and Curnier [24]:

$$\begin{aligned}\mathbb{A} = {} & a_1\, \mathbf{I} \otimes \mathbf{I} + a_2\, \mathbf{I} \underline{\overline{\otimes}} \mathbf{I} + a_3\, \mathbf{w} \otimes \mathbf{w} + a_4\, (\mathbf{w} \underline{\overline{\otimes}} \mathbf{I} + \mathbf{I} \underline{\overline{\otimes}} \mathbf{w}) + a_5\, \mathbf{w}^2 \otimes \mathbf{w}^2 + a_6\, \mathbf{w} \underline{\overline{\otimes}} \mathbf{w} \\ & + a_7\, (\mathbf{w} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{w}) + a_8\, (\mathbf{w}^2 \otimes \mathbf{w} + \mathbf{w} \otimes \mathbf{w}^2) + a_9\, (\mathbf{w}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{w}^2)\,,\end{aligned} \tag{5}$$

where the 9 scalar coefficients a_i, i=1-9, are any polynomial functions of the three principal invariants of $\mathbf{w}$ (which can be classically defined as ${}^{\mathrm{w}}I_1 = \mathrm{tr}\, \mathbf{w}$, ${}^{\mathrm{w}}I_2 = (\mathrm{tr}^2\, \mathbf{w} - \mathrm{tr}\, \mathbf{w}^2)/2$, ${}^{\mathrm{w}}I_3 = \det \mathbf{w} = \mathrm{tr}\, \mathbf{w}^3/3 + \mathrm{tr}^3\, \mathbf{w}/6 - \mathrm{tr}\, \mathbf{w}\;\; \mathrm{tr}\, \mathbf{w}^2/2$ and enter the Cayley-Hamilton theorem applied to $\mathbf{w}$, i.e. $\mathbf{w}^3 - {}^{\mathrm{w}}I_1\, \mathbf{w}^2 + {}^{\mathrm{w}}I_2\, \mathbf{w} - {}^{\mathrm{w}}I_3\, \mathbf{I} = \mathbf{0}$). In the non-symmetric case each of the three coefficients a_7, a_8, a_9 would split in two, namely a_{71}, a_{72}, a_{81}, a_{82}, a_{91}, a_{92} (e.g. a_{71} and a_{72} attached respectively to $\mathbf{w} \otimes \mathbf{I}$ and $\mathbf{I} \otimes \mathbf{w}$, and so on), for a total of 6+6=12 coefficients.

Notice that the three terms embedding symmetrized dyadic products $\underline{\overline{\otimes}}$ in representation (5) are attached to the three *'shear-like'* coefficients a_2, a_4, a_6 and affect only the diagonal entries of a 6×6 matrix representation of the damage-effect tensor in the principal axes of damage. The six supplemental rank-one updates provided by the addends with standard dyadic products $\otimes$ are attached to the remaining six *'non-shear'* coefficients a_1, a_3, a_5, a_7, a_8, a_9 and affect only the upper-left 3×3 submatrix representation of $\mathbb{A}$.

Representations similar to (5) hold as well for the dual damage-effect tensor $\bar{\mathbb{A}}$, in terms of the dual square root integrity variable $\bar{\mathbf{w}}$ and dual coefficients with bars, $\bar{a}_i$, i=1-9 (generally functions of the three principal invariants of $\bar{\mathbf{w}}$, ${}^{\bar{\mathrm{w}}}I_1, {}^{\bar{\mathrm{w}}}I_2, {}^{\bar{\mathrm{w}}}I_3$), and also for the current compliance $\mathbb{C}$ and stiffness $\mathbb{E}$ in terms of damage variables $\boldsymbol{\phi}$, $\bar{\boldsymbol{\phi}}$ and analogous scalar coefficients c_i, e_i, i=1-9. The links between alternative representations of each fourth-order tensor in terms of either $\boldsymbol{\phi}$ or $\mathbf{w}$ (and of $\bar{\boldsymbol{\phi}}$ or $\bar{\mathbf{w}}$) can be obtained through the isotropic functions $\boldsymbol{\phi} = \mathbf{w}^2$ and $\mathbf{w} = \boldsymbol{\phi}^{1/2}$ (and $\bar{\boldsymbol{\phi}} = \bar{\mathbf{w}}^2$ and $\bar{\mathbf{w}} = \bar{\boldsymbol{\phi}}^{1/2}$) [20].

Notice that a natural constraint on representation (5) (and dual one for $\bar{\mathbb{A}}$) arises from Eq. (3) in the absence of damage: nominal and effective quantities come to coincide and the linear transformations must reduce to the identity. Then, for $\mathbf{w} = \mathbf{I}$, $\mathbb{A}(\mathbf{I}) = \mathbb{I}^s = \mathbf{I} \underline{\overline{\otimes}} \mathbf{I}$, that is, when all the scalar coefficients are evaluated in $\mathbf{I}$:

$$[a_2 + 2a_4 + a_6](\mathbf{I}) = 1\,; \quad [a_1 + a_3 + a_5 + a_7 + a_8 + a_9](\mathbf{I}) = 0\,. \tag{6}$$

3 Significative examples of dual damage-effect tensors

Considering representation (5) for $\mathbb{A}$ and dual one for $\bar{\mathbb{A}}$, the point under consideration here is precisely that of seeking particular instances of such general representations (possibly with a limited number of terms) that correspond to each other through an inversion operation spanning the same set of tensor terms. In other words, if a coefficient is lacking in $\mathbb{A}$, say e.g. a_1, the dual one $\bar{a}_1$ should also disappear in the dual representation of $\bar{\mathbb{A}}$, while the others should remain untouched.

The task of seeking instances that solve the problem at hand has been tackled in [18]. A set of solution instances has been derived, based on either a rigorous treatment, whenever possible, or on guessing procedures and guided searches, as well as on the use of tensor multiplication tables and on the repeated application of Sherman-Morrison's formula for the inversion of a rank-one update of a given tensor.

The rigorous analysis showed that solution sets including all the three 'shear-like' coefficients are possible and that the unknown dual 'shear-like' coefficients can be readily expressed in closed form. On the other hand, locating specific subsets of the 'non-shear' coefficients corresponding to each other in the dual structures is more involved. Particular solution instances missing one or two of the three 'shear-like' terms can also be conveniently identified, in particular those containing only either 'shear-like' coefficients a_2, $\bar{a}_2$ or a_6, $\bar{a}_6$. These two possibilities are further explored in the present paper, especially the second one concerning 'shear-like' coefficients $a_6, \bar{a}_6$, which originates the solution family presented in Section 4. A new general solution based on 'shear-like' coefficients $a_2, \bar{a}_2$ and containing all six 'non-shear' coefficients is given as well in the present section.

Before presenting the solution instances of the family characterized by 'shear-like' coefficients $a_6, \bar{a}_6$ (Section 4), a few particular instances of damage-effect tensors endowed with dual structures and based on both 'shear-like' generators $a_2\, \mathbf{I} \underline{\overline{\otimes}} \mathbf{I}$, $\bar{a}_2\, \mathbf{I} \underline{\overline{\otimes}} \mathbf{I}$ and $a_6\, \mathbf{w} \underline{\overline{\otimes}} \mathbf{w}$, $\bar{a}_6\, \bar{\mathbf{w}} \underline{\overline{\otimes}} \bar{\mathbf{w}}$ are given below as illustrative examples of the sought correspondence. All the presented cases work without constraints on the coefficients.

One first case is readily apparent:

Solution (2.1). The isotropic case in which only the two-coefficients sets (a_1, a_2) and $(\bar{a}_1, \bar{a}_2)$ are kept in the expansions of $\mathbb{A}$ and $\bar{\mathbb{A}}$:

$$\mathbb{A} = a_2\, \mathbf{I} \underline{\overline{\otimes}} \mathbf{I} + a_1\, \mathbf{I} \otimes \mathbf{I}\,; \quad \bar{\mathbb{A}} = \bar{a}_2\, \mathbf{I} \underline{\overline{\otimes}} \mathbf{I} + \bar{a}_1\, \mathbf{I} \otimes \mathbf{I}\,, \tag{7}$$

with

$$\bar{a}_2 = \frac{1}{a_2}\,; \qquad \bar{a}_1 = -\frac{a_1}{a_2\,(3a_1 + a_2)}\,. \tag{8}$$

This assumption may lead to a general form of isotropic damage based on two independent scalar damage variables (if the two coefficient functions a_1, a_2 are independent), or to a restricted form of isotropic damage based on a single scalar damage variable (if the two coefficient functions a_1, a_2 are linked to each other, as for instance in the classical scalar damage models of the $(1-D)$-type), see e.g. [9]. *Solution* instance *(2.1)* is based on 'shear-like' generators $a_2\, \mathbf{I} \underline{\overline{\otimes}} \mathbf{I}$, $\bar{a}_2\, \mathbf{I} \underline{\overline{\otimes}} \mathbf{I}$ (number 2 as the first label digit) and contains a single 'non-shear' coefficient (number 1 as the second label digit).

A new solution instance based on the same 'shear-like' generators $a_2\, \mathbf{I} \underline{\overline{\otimes}} \mathbf{I}$, $\bar{a}_2\, \mathbf{I} \underline{\overline{\otimes}} \mathbf{I}$ of *Solution (2.1)* and representing a full generalization of it that contains all six 'non-shear' coefficients $a_1, a_3, a_5, a_7, a_8, a_9$ (and thus without constraints on the coefficients) can be derived as follows in full invariant form:

Solution (2.6). Symmetric damage-effect tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ with the seven-coefficients sets $(a_1, a_2, a_3, a_5, a_7, a_8, a_9)$ and $(\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_5, \bar{a}_7, \bar{a}_8, \bar{a}_9)$ (lacking only the two 'shear-like' coefficients a_4, a_6 and $\bar{a}_4, \bar{a}_6$) form the symmetric dual inverse pair:

$$
\begin{aligned}
\mathbb{A} &= a_2\, \mathbf{I} \,\overline{\underline{\otimes}}\, \mathbf{I} + a_1\, \mathbf{I} \otimes \mathbf{I} + a_3\, \mathbf{w} \otimes \mathbf{w} + a_5\, \mathbf{w}^2 \otimes \mathbf{w}^2 \\
&\quad + a_7\, (\mathbf{w} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{w}) + a_8\, (\mathbf{w}^2 \otimes \mathbf{w} + \mathbf{w} \otimes \mathbf{w}^2) + a_9\, (\mathbf{w}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{w}^2)\, ; \\
\bar{\mathbb{A}} &= \bar{a}_2\, \mathbf{I} \,\overline{\underline{\otimes}}\, \mathbf{I} + \bar{a}_1\, \mathbf{I} \otimes \mathbf{I} + \bar{a}_3\, \bar{\mathbf{w}} \otimes \bar{\mathbf{w}} + \bar{a}_5\, \bar{\mathbf{w}}^2 \otimes \bar{\mathbf{w}}^2 \\
&\quad + \bar{a}_7\, (\bar{\mathbf{w}} \otimes \mathbf{I} + \mathbf{I} \otimes \bar{\mathbf{w}}) + \bar{a}_8\, (\bar{\mathbf{w}}^2 \otimes \bar{\mathbf{w}} + \bar{\mathbf{w}} \otimes \bar{\mathbf{w}}^2) + \bar{a}_9\, (\bar{\mathbf{w}}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \bar{\mathbf{w}}^2)\, ,
\end{aligned}
\tag{9}
$$

with

$$
\begin{aligned}
&\bar{a}_2 = \frac{1}{a_2}\, ; \quad \bar{a}_1 = \frac{\bar{n}_{21}}{a_2\, \bar{d}_2}\, , \quad \bar{a}_3 = \frac{\bar{n}_{23}}{a_2\, \bar{d}_2}\, , \quad \bar{a}_5 = \frac{\bar{n}_{25}\, {}^{\mathrm{w}}I_3{}^2}{a_2\, \bar{d}_2}\, , \\
&\bar{a}_7 = \frac{\bar{n}_{27}}{a_2\, \bar{d}_2}\, , \quad \bar{a}_8 = \frac{\bar{n}_{28}\, {}^{\mathrm{w}}I_3}{a_2\, \bar{d}_2}\, , \quad \bar{a}_9 = -\frac{\bar{n}_{29}\, {}^{\mathrm{w}}I_3}{a_2\, \bar{d}_2}\, ,
\end{aligned}
\tag{10}
$$

where

$$
\begin{aligned}
\bar{d}_2 &= a_2^3 + \Big(a_1 a_3 a_5 - a_1 a_8^2 - a_5 a_7^2 - a_9(a_3 a_9 - 2a_7 a_8) + a_2(a_3 a_5 - a_8^2)\Big)\cdot \\
&\qquad \cdot({}^{\mathrm{w}}I_1{}^2\, {}^{\mathrm{w}}I_2{}^2 - 4\,{}^{\mathrm{w}}I_2{}^3 - 4\,{}^{\mathrm{w}}I_1{}^3\, {}^{\mathrm{w}}I_3 + 18\,{}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3 - 27\,{}^{\mathrm{w}}I_3{}^2) \\
&\quad + a_2^2\Big(3a_1 + 2a_7\, {}^{\mathrm{w}}I_1 + (a_3 + 2a_9)({}^{\mathrm{w}}I_1{}^2 - 2\,{}^{\mathrm{w}}I_2) + 2a_8({}^{\mathrm{w}}I_1{}^3 - 3\,{}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2 + 3\,{}^{\mathrm{w}}I_3) \\
&\qquad + a_5({}^{\mathrm{w}}I_1{}^4 - 4\,{}^{\mathrm{w}}I_1{}^2\, {}^{\mathrm{w}}I_2 + 2\,{}^{\mathrm{w}}I_2{}^2 + 4\,{}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3)\Big) \\
&\quad + 2a_2\Big[(a_1 a_5 - a_9^2)({}^{\mathrm{w}}I_1{}^4 - 4\,{}^{\mathrm{w}}I_1{}^2\, {}^{\mathrm{w}}I_2 + {}^{\mathrm{w}}I_2{}^2 + 6\,{}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3) \\
&\qquad + \Big(a_1 a_3 - a_7^2 + (a_1 a_8 - a_7 a_9)\, {}^{\mathrm{w}}I_1 + (a_5 a_7 - a_8 a_9)\, {}^{\mathrm{w}}I_3\Big)({}^{\mathrm{w}}I_1{}^2 - 3\,{}^{\mathrm{w}}I_2) \\
&\qquad + \Big(a_1 a_8 - a_7 a_9 + (a_5 a_7 - a_8 a_9)\, {}^{\mathrm{w}}I_2 + (a_3 a_5 - a_8^2)\, {}^{\mathrm{w}}I_3\Big)({}^{\mathrm{w}}I_1{}^3 - 4\,{}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2 + 9\,{}^{\mathrm{w}}I_3) \\
&\qquad - (a_3 a_9 - a_7 a_8)({}^{\mathrm{w}}I_1{}^2\, {}^{\mathrm{w}}I_2 - 4\,{}^{\mathrm{w}}I_2{}^2 + 3\,{}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3)\Big]\, ,
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
\bar{n}_{21} &= \Big(a_5 a_7^2 - a_1(a_3 a_5 - a_8^2) + a_9(a_3 a_9 - 2a_7 a_8)\Big)\cdot \\
&\qquad \cdot({}^{\mathrm{w}}I_1{}^2\, {}^{\mathrm{w}}I_2{}^2 - 2\,{}^{\mathrm{w}}I_2{}^3 - 4\,{}^{\mathrm{w}}I_1{}^3\, {}^{\mathrm{w}}I_3 + 10\,{}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3 - 9\,{}^{\mathrm{w}}I_3{}^2) \\
&\quad - a_2^2\Big[a_1 + 2a_7\, {}^{\mathrm{w}}I_1 + a_3\, {}^{\mathrm{w}}I_1{}^2 + \Big(2a_9 + 2a_8\, {}^{\mathrm{w}}I_1 + a_5({}^{\mathrm{w}}I_1{}^2 - {}^{\mathrm{w}}I_2)\Big)({}^{\mathrm{w}}I_1{}^2 - {}^{\mathrm{w}}I_2)\Big] \\
&\quad - a_2\Big[2(a_1 a_3 - a_7^2)({}^{\mathrm{w}}I_1{}^2 - {}^{\mathrm{w}}I_2) + (a_1 a_5 - a_9^2)(2\,{}^{\mathrm{w}}I_1{}^4 - 4\,{}^{\mathrm{w}}I_1{}^2\, {}^{\mathrm{w}}I_2 + {}^{\mathrm{w}}I_2{}^2 + 4\,{}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3) \\
&\qquad + 2(a_1 a_8 - a_7 a_9)(2\,{}^{\mathrm{w}}I_1{}^3 - 3\,{}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2 + 3\,{}^{\mathrm{w}}I_3) \\
&\qquad - 2(a_3 a_9 - a_7 a_8)({}^{\mathrm{w}}I_1{}^2\, {}^{\mathrm{w}}I_2 - 2\,{}^{\mathrm{w}}I_2{}^2 + 3\,{}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3) \\
&\qquad + 2(a_5 a_7 - a_8 a_9)({}^{\mathrm{w}}I_1{}^3\, {}^{\mathrm{w}}I_2 - 2\,{}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2{}^2 + {}^{\mathrm{w}}I_1{}^2\, {}^{\mathrm{w}}I_3 + 3\,{}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3) \\
&\qquad + \Big(a_3 a_5\, {}^{\mathrm{w}}I_1{}^2 - a_8^2({}^{\mathrm{w}}I_1{}^2 - 2\,{}^{\mathrm{w}}I_2)\Big)({}^{\mathrm{w}}I_2{}^2 - 2\,{}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3) \\
&\qquad - 2\Big(a_8^2\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3 + a_3 a_5({}^{\mathrm{w}}I_2{}^2 - 3\,{}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3)\Big)\, {}^{\mathrm{w}}I_2\Big]\, ,
\end{aligned}
\tag{12}
$$

$$\begin{aligned}\bar{n}_{23} = &\Big(2a_5a_7^2 - (2a_1+a_2)(a_3a_5-a_8^2) + 2a_9(a_3a_9-2a_7a_8)\Big)\cdot\\ &\qquad\cdot({}^{\mathrm{w}}I_2{}^4 - 4{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2{}^{2\mathrm{w}}I_3 + {}^{\mathrm{w}}I_1{}^{2\mathrm{w}}I_3{}^2 + 6{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3{}^2)\\ &+a_2\Big\{\Big(3a_7^{2\mathrm{w}}I_2 - (a_3a_5-a_8^2)({}^{\mathrm{w}}I_2{}^3 - 2{}^{\mathrm{w}}I_3{}^2) + 2(a_3a_9-a_7a_8)(2{}^{\mathrm{w}}I_2{}^2 - {}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)\Big){}^{\mathrm{w}}I_2\\ &\quad-(3a_1+a_2)\Big[a_3{}^{\mathrm{w}}I_2{}^2 + \Big(a_5({}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2 - {}^{\mathrm{w}}I_3) + 2a_8{}^{\mathrm{w}}I_2\Big)({}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2 - {}^{\mathrm{w}}I_3)\Big]\\ &\quad+\Big(6a_7a_9{}^{\mathrm{w}}I_2 - 2(a_5a_7-a_8a_9)(2{}^{\mathrm{w}}I_2{}^2 - {}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3) + 3a_9^2({}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2 - {}^{\mathrm{w}}I_3)\Big)({}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2 - {}^{\mathrm{w}}I_3)\Big\},\end{aligned} \tag{13}$$

$$\begin{aligned}\bar{n}_{25} = &\ 2\Big(a_5a_7^2 + a_9(a_3a_9-2a_7a_8) - (a_1+a_2)(a_3a_5-a_8^2)\Big)({}^{\mathrm{w}}I_2{}^2 - 3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)\\ &+ a_2\Big[3a_7^2 + 3a_9(2a_7 + a_9{}^{\mathrm{w}}I_1){}^{\mathrm{w}}I_1 - (3a_1+a_2)\Big(a_3 + (2a_8 + a_5{}^{\mathrm{w}}I_1){}^{\mathrm{w}}I_1\Big)\\ &\qquad -4\Big(a_7(a_8 + a_5{}^{\mathrm{w}}I_1) - a_9(a_3 + a_8{}^{\mathrm{w}}I_1)\Big){}^{\mathrm{w}}I_2 - 4(a_3a_5-a_8^2){}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3\Big],\end{aligned} \tag{14}$$

$$\begin{aligned}\bar{n}_{27} = &\Big((a_1+a_2)(a_3a_5-a_8^2) - a_5a_7^2 - a_9(a_3a_9-2a_7a_8)\Big)\cdot\\ &\qquad\qquad\cdot\Big({}^{\mathrm{w}}I_1({}^{\mathrm{w}}I_2{}^3 + 6{}^{\mathrm{w}}I_3{}^2) - {}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3(4{}^{\mathrm{w}}I_1{}^2 - {}^{\mathrm{w}}I_2)\Big)\\ &+a_2^2\Big[(a_7 + a_3{}^{\mathrm{w}}I_1){}^{\mathrm{w}}I_2 + \Big(a_9 + a_5({}^{\mathrm{w}}I_1{}^2 - {}^{\mathrm{w}}I_2)\Big)({}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2 - {}^{\mathrm{w}}I_3)\\ &\quad -a_8({}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3 - 2{}^{\mathrm{w}}I_1{}^{2\mathrm{w}}I_2 + {}^{\mathrm{w}}I_2{}^2)\Big]\\ &+a_2\Big[2\Big(a_1a_3 - a_7^2 + (a_5a_7-a_8a_9){}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2\Big){}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2 + (a_1a_5-a_9^2)({}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2 - {}^{\mathrm{w}}I_3)(2{}^{\mathrm{w}}I_1{}^2 - {}^{\mathrm{w}}I_2)\\ &\quad -\Big(a_5a_7 - a_8a_9 - (a_3a_5-a_8^2){}^{\mathrm{w}}I_1\Big)(2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2 - 3{}^{\mathrm{w}}I_3){}^{\mathrm{w}}I_3\\ &\quad -(a_1a_8-a_7a_9)(2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3 - 4{}^{\mathrm{w}}I_1{}^{2\mathrm{w}}I_2 + {}^{\mathrm{w}}I_2{}^2) - (a_3a_9-a_7a_8)(2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2 + {}^{\mathrm{w}}I_3){}^{\mathrm{w}}I_2\Big],\end{aligned} \tag{15}$$

$$\begin{aligned}\bar{n}_{28} = &\ \Big((a_1+a_2)(a_3a_5-a_8^2) - a_3a_9^2 - a_7(a_5a_7 - 2a_8a_9)\Big)(2{}^{\mathrm{w}}I_2{}^3 - 7{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3 + 9{}^{\mathrm{w}}I_3{}^2)\\ &+ a_2\Big[(3a_1+a_2)\Big(a_3{}^{\mathrm{w}}I_2 + a_5{}^{\mathrm{w}}I_1({}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2 - {}^{\mathrm{w}}I_3)\Big)\\ &\quad -3(a_7^2 - a_9^{2\mathrm{w}}I_1{}^2){}^{\mathrm{w}}I_2 - 2(a_5a_7-a_8a_9){}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3 - 4(a_3a_5-a_8^2){}^{\mathrm{w}}I_3{}^2\\ &\quad -\Big(3a_7a_9 - a_8(3a_1+a_2) + 3a_9^{2\mathrm{w}}I_1 - 2(a_3a_5-a_8^2){}^{\mathrm{w}}I_3\Big)(2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2 - {}^{\mathrm{w}}I_3)\\ &\quad -\Big((a_3a_9-a_7a_8) - (a_5a_7-a_8a_9){}^{\mathrm{w}}I_1\Big)(4{}^{\mathrm{w}}I_2{}^2 - {}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)\Big],\end{aligned} \tag{16}$$

$$\begin{aligned}\bar{n}_{29} = &\Big((a_1+a_2)(a_3a_5-a_8^2) - a_5a_7^2 - a_9(a_3a_9-2a_7a_8)\Big)({}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2{}^2 - 4{}^{\mathrm{w}}I_1{}^{2\mathrm{w}}I_3 + 3{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3)\\ &+a_2^2\Big(a_7 + (a_3+a_9){}^{\mathrm{w}}I_1 + a_5({}^{\mathrm{w}}I_1{}^2 - {}^{\mathrm{w}}I_2){}^{\mathrm{w}}I_1 + a_8(2{}^{\mathrm{w}}I_1{}^2 - {}^{\mathrm{w}}I_2)\Big)\\ &+a_2\Big[\Big(2(a_1a_3-a_7^2) + 2(a_3a_5-a_8^2){}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3 + (a_1a_5-a_9^2)(2{}^{\mathrm{w}}I_1{}^2 - {}^{\mathrm{w}}I_2)\\ &\qquad +(a_5a_7-a_8a_9)(2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2 + {}^{\mathrm{w}}I_3)\Big){}^{\mathrm{w}}I_1\\ &\quad +(a_1a_8-a_7a_9)(4{}^{\mathrm{w}}I_1{}^2 - {}^{\mathrm{w}}I_2) - (a_3a_9-a_7a_8)(2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2 + 3{}^{\mathrm{w}}I_3)\Big].\end{aligned} \tag{17}$$

Reduced particular cases of this solution based on five and four 'non-shear' coefficients (including also non-symmetric instances that are not comprised in the general relations provided here) are given in full details in [18]. There are no particular cases of this general solution that work without constraints on the coefficients (except of course for isotropic *Solution (2.1)* and for the degenerate instance containing only the 'shear-like' tensor generators $a_2\,\mathbf{I}\,\underline{\otimes}\,\mathbf{I}$, $\bar{a}_2\,\mathbf{I}\,\underline{\otimes}\,\mathbf{I}$ alone).

Four additional significative cases based on 'shear-like' generators $a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}$, $\bar{a}_6\,\bar{\mathbf{w}}\,\overline{\underline{\otimes}}\,\bar{\mathbf{w}}$ can be reported (including two non-symmetric ones):

Solution (6.0). The 'shear-like' generators attached to a_6 and $\bar{a}_6$ taken alone (no 'non-shear' coefficients):

$$\mathbb{A} = a_6\;\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}\;; \qquad \bar{\mathbb{A}} = \bar{a}_6\;\bar{\mathbf{w}}\,\overline{\underline{\otimes}}\,\bar{\mathbf{w}}\;, \tag{18}$$

with

$$\bar{a}_6 = \frac{1}{a_6}\;. \tag{19}$$

This case is remarkable since it renders, through Eqs (2), (4), the Valanis-type compliance and stiffness [21, 24] in which the inverse integrity tensor $\boldsymbol{\phi}$ and integrity tensor $\bar{\boldsymbol{\phi}}$ just replace the identity $\mathbf{I}$ in the original isotropic compliance and stiffness (2). Indeed, taking a_6=$\bar{a}_6$=1, i.e. by assuming the 'basic' damage-effect tensors $\mathbb{A}_{\rm bas}$=$\boldsymbol{\phi}^{1/2}\,\overline{\underline{\otimes}}\,\boldsymbol{\phi}^{1/2}$ and $\bar{\mathbb{A}}_{\rm bas}$=$\bar{\boldsymbol{\phi}}^{1/2}\,\overline{\underline{\otimes}}\,\bar{\boldsymbol{\phi}}^{1/2}$, Valanis-type secant compliance and stiffness are recovered [6]: $\mathbb{C}$=$(1+\nu_0)/E_0\;\boldsymbol{\phi}\,\overline{\underline{\otimes}}\,\boldsymbol{\phi}-\nu_0/E_0\;\boldsymbol{\phi}\otimes\boldsymbol{\phi}$, $\mathbb{E}$=$2G_0\;\bar{\boldsymbol{\phi}}\,\overline{\underline{\otimes}}\,\bar{\boldsymbol{\phi}}+\Lambda_0\;\bar{\boldsymbol{\phi}}\otimes\bar{\boldsymbol{\phi}}$.

Solution (6.1). The implications of *Solution (6.0)* suggest that the symmetric Valanis-type structure of compliance and stiffness could be taken as well for the damage-effect tensors themselves, i.e. by keeping only the dual two-coefficients sets (a_6, a_3) and $(\bar{a}_6, \bar{a}_3)$ in the representations of $\mathbb{A}$ and $\bar{\mathbb{A}}$:

$$\mathbb{A} = a_6\;\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w} + a_3\;\mathbf{w}\otimes\mathbf{w}\;; \qquad \bar{\mathbb{A}} = \bar{a}_6\;\bar{\mathbf{w}}\,\overline{\underline{\otimes}}\,\bar{\mathbf{w}} + \bar{a}_3\;\bar{\mathbf{w}}\otimes\bar{\mathbf{w}}\;, \tag{20}$$

with

$$\bar{a}_6 = \frac{1}{a_6}\;; \qquad \bar{a}_3 = -\frac{a_3}{a_6\,(3a_3 + a_6)}\;. \tag{21}$$

Clearly, the arising secant compliance and stiffness are no longer of the Valanis-type. This solution represents the first symmetric generalization of *Solution (6.0)* based on single additional 'non-shear' coefficients $a_3, \bar{a}_3$. Further generalizations containing more 'non-shear' coefficients are pursued in Section 4, where all particular symmetric solutions of the family based on 'shear-like' generators $a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}$, $\bar{a}_6\,\bar{\mathbf{w}}\,\overline{\underline{\otimes}}\,\bar{\mathbf{w}}$ are determined. Two supplemental non-symmetric instances are given instead below.

Solution (6.1ns). A non-symmetric solution instance still based on the same 'shear-like' generators attached to $a_6, \bar{a}_6$, and in a sense similar to *Solution (6.1)*, is given by the dual non-symmetric damage-effect tensors embedding just two-coefficients sets (a_6, a_{92}) and $(\bar{a}_6, \bar{a}_{91})$ (in a 12-coefficients non-symmetric counterpart of representation (5)):

$$\mathbb{A} = a_6 \ \mathbf{w} \underline{\overline{\otimes}} \mathbf{w} + a_{92} \ \mathbf{I} \otimes \mathbf{w}^2 \ ; \quad \bar{\mathbb{A}} = \bar{a}_6 \ \bar{\mathbf{w}} \underline{\overline{\otimes}} \bar{\mathbf{w}} + \bar{a}_{91} \ \bar{\mathbf{w}}^2 \otimes \mathbf{I} \ , \tag{22}$$

with

$$\bar{a}_6 = \frac{1}{a_6} \ ; \quad \bar{a}_{91} = -\frac{a_{92}}{a_6 \left(3a_{92} + a_6\right)} \ . \tag{23}$$

The arising secant compliance and stiffness still belong to the above-mentioned Valanis-type structure but include enhanced elastic parameters replacing undamaged ones and embed a convenient volumetric/deviatoric decomposition of the damage properties that allows to assign different weights to bulk and shear damage components ('extended' formulation, see [7] for the details).

An additional non-symmetric case still based on the same 'shear-like' coefficients and generalizing the 'extended' model in *Solution (6.1ns)* can also be derived as reported in [18]. This solution case includes four 'non-shear' coefficients; it is remarkable because it comprises previous *Solutions (6.0), (6.1), (6.1ns)* and works as well without constraint on the coefficients:

Solution (6.4ns). Five-coefficients sets $(a_6, a_3, a_{72}, a_{82}, a_{92})$ and $(\bar{a}_6, \bar{a}_3, \bar{a}_{71}, \bar{a}_{81}, \bar{a}_{91})$ give rise to the non-symmetric dual inverse pair (based on four 'non-shear' coefficients):

$$\begin{aligned} \mathbb{A} &= a_6 \ \mathbf{w} \underline{\overline{\otimes}} \mathbf{w} + a_3 \ \mathbf{w} \otimes \mathbf{w} + a_{72} \ \mathbf{I} \otimes \mathbf{w} + a_{82} \ \mathbf{w} \otimes \mathbf{w}^2 + a_{92} \ \mathbf{I} \otimes \mathbf{w}^2 \ ; \\ \bar{\mathbb{A}} &= \bar{a}_6 \ \bar{\mathbf{w}} \underline{\overline{\otimes}} \bar{\mathbf{w}} + \bar{a}_3 \ \bar{\mathbf{w}} \otimes \bar{\mathbf{w}} + \bar{a}_{71} \ \bar{\mathbf{w}} \otimes \mathbf{I} + \bar{a}_{81} \ \bar{\mathbf{w}}^2 \otimes \bar{\mathbf{w}} + \bar{a}_{91} \ \bar{\mathbf{w}}^2 \otimes \mathbf{I} \ , \end{aligned} \tag{24}$$

with

$$\begin{aligned} \bar{a}_6 = \frac{1}{a_6}, \quad \bar{a}_3 &= -\frac{a_6 a_3 + 3(a_3 a_{92} - a_{72} a_{82})}{a_6 \ \bar{d}_{ns}}, \quad \bar{a}_{71} = -\frac{a_6 a_{82} - {}^{\mathrm{w}}I_2/{}^{\mathrm{w}}I_3 (a_3 a_{92} - a_{72} a_{82})}{a_6 \ \bar{d}_{ns}}, \\ \bar{a}_{91} &= -\frac{a_6 a_{92} + 3(a_3 a_{92} - a_{72} a_{82})}{a_6 \ \bar{d}_{ns}}, \quad \bar{a}_{81} = -\frac{a_6 a_{72} - \ {}^{\mathrm{w}}I_1 \ (a_3 a_{92} - a_{72} a_{82})}{a_6 \ \bar{d}_{ns}}, \end{aligned} \tag{25}$$

where

$$\bar{d}_{ns} = a_6 (a_6 + 3a_3 + 3a_{92} + a_{72} {}^{\mathrm{w}}I_2/{}^{\mathrm{w}}I_3 + a_{82} {}^{\mathrm{w}}I_1) + (a_3 a_{92} - a_{72} a_{82})(9 - {}^{\mathrm{w}}I_1 \ {}^{\mathrm{w}}I_2/{}^{\mathrm{w}}I_3) \ . \tag{26}$$

Solution (6.1ns) is recovered from *Solution (6.4ns)* by setting consistently a_3=a_{72}=a_{82}=0, $\bar{a}_3$=$\bar{a}_{72}$=$\bar{a}_{82}$=0 in Eqs (24)-(26). Notice that *'twins'* of *Solutions (6.1ns)* and *(6.4ns)* can also be obtained just by inverting the roles between coefficients with and without bars in Eqs (22) and (24).

4 A family of symmetric dual damage-effect tensors

Solution (6.6). A full solution instance based on 'shear-like' generators $a_6\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w}$, $\bar{a}_6\bar{\mathbf{w}}\,\underline{\overline{\otimes}}\,\bar{\mathbf{w}}$ generalizing *Solutions (6.0)*, *(6.1)* and containing all six 'non-shear' terms (thus without constraints on the coefficients) can be obtained by taking symmetric damage-effect tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ with the seven-coefficients sets $(a_1, a_3, a_5, a_6, a_7, a_8, a_9)$ and $(\bar{a}_1, \bar{a}_3, \bar{a}_5, \bar{a}_6, \bar{a}_7, \bar{a}_8, \bar{a}_9)$ (lacking only the two 'shear-like' coefficients a_2, a_4 and $\bar{a}_2, \bar{a}_4$):

$$\begin{aligned}
\mathbb{A} &= a_6\ \mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w} + a_1\ \mathbf{I}\otimes\mathbf{I} + a_3\ \mathbf{w}\otimes\mathbf{w} + a_5\ \mathbf{w}^2\otimes\mathbf{w}^2 \\
&\quad + a_7\ (\mathbf{w}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w}) + a_8\ (\mathbf{w}^2\otimes\mathbf{w}+\mathbf{w}\otimes\mathbf{w}^2) + a_9\ (\mathbf{w}^2\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w}^2)\,; \\
\bar{\mathbb{A}} &= \bar{a}_6\ \bar{\mathbf{w}}\,\underline{\overline{\otimes}}\,\bar{\mathbf{w}} + \bar{a}_1\ \mathbf{I}\otimes\mathbf{I} + \bar{a}_3\ \bar{\mathbf{w}}\otimes\bar{\mathbf{w}} + \bar{a}_5\ \bar{\mathbf{w}}^2\otimes\bar{\mathbf{w}}^2 \\
&\quad + \bar{a}_7\ (\bar{\mathbf{w}}\otimes\mathbf{I}+\mathbf{I}\otimes\bar{\mathbf{w}}) + \bar{a}_8\ (\bar{\mathbf{w}}^2\otimes\bar{\mathbf{w}}+\bar{\mathbf{w}}\otimes\bar{\mathbf{w}}^2) + \bar{a}_9\ (\bar{\mathbf{w}}^2\otimes\mathbf{I}+\mathbf{I}\otimes\bar{\mathbf{w}}^2)\,,
\end{aligned} \tag{27}$$

with

$$\begin{aligned}
\bar{a}_6 = \frac{1}{a_6}\,; \quad \bar{a}_1 = -\frac{\bar{n}_{61}}{a_6\ \bar{d}_6}\,, \quad \bar{a}_3 = -\frac{\bar{n}_{63}}{a_6\ \bar{d}_6}\,, \quad \bar{a}_5 = \frac{\bar{n}_{65}\ {}^{\mathrm{w}}I_3{}^2}{a_6\ \bar{d}_6}\,, \\
\bar{a}_7 = -\frac{\bar{n}_{67}}{a_6\ \bar{d}_6}\,, \quad \bar{a}_8 = \frac{\bar{n}_{68}\ {}^{\mathrm{w}}I_3}{a_6\ \bar{d}_6}\,, \quad \bar{a}_9 = \frac{\bar{n}_{69}\ {}^{\mathrm{w}}I_3}{a_6\ \bar{d}_6}\,,
\end{aligned} \tag{28}$$

where

$$\begin{aligned}
\bar{d}_6 &= a_6\Big(-9a_1a_5 + 3a_3a_6 + (a_6+3a_9)^2 + 2a_6a_8{}^{\mathrm{w}}I_1 \\
&\qquad + a_5a_6({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2) + 2(a_3a_5 - a_8^2)({}^{\mathrm{w}}I_1{}^2 - 3{}^{\mathrm{w}}I_2)\Big){}^{\mathrm{w}}I_3{}^2 \\
&\quad - a_6\Big[2(a_3a_9 - a_7a_8)({}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2 - 9{}^{\mathrm{w}}I_3) - 2\Big(a_6a_7 - 3(a_1a_8 - a_7a_9)\Big){}^{\mathrm{w}}I_2 \\
&\qquad + 2(a_1a_3 - a_7^2){}^{\mathrm{w}}I_1 - 2(a_5a_7 - a_8a_9)\Big({}^{\mathrm{w}}I_2({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2) - 3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3\Big)\Big]{}^{\mathrm{w}}I_3 \\
&\quad + a_6\Big(a_1a_6 + 2(a_1a_3 - a_7^2) + 2(a_1a_8 - a_7a_9){}^{\mathrm{w}}I_1 \\
&\qquad + (a_1a_5 - a_9^2)({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2)\Big)\ ({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3) \\
&\quad - \Big(a_1a_8^2 - a_5(a_1a_3 - a_7^2) + a_9(a_3a_9 - 2a_7a_8)\Big)\cdot \\
&\qquad\qquad \cdot({}^{\mathrm{w}}I_1{}^{2\mathrm{w}}I_2{}^2 - 4{}^{\mathrm{w}}I_2{}^3 - 4{}^{\mathrm{w}}I_1{}^{3\mathrm{w}}I_3 + 18{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3 - 27{}^{\mathrm{w}}I_3{}^2)\,,
\end{aligned} \tag{29}$$

$$\begin{aligned}
\bar{n}_{61} &= a_6\Big[(a_1a_5 - a_9^2){}^{\mathrm{w}}I_1 + 2(a_5a_7 - a_8a_9){}^{\mathrm{w}}I_2 + \Big(a_5(3a_3+a_6) - 3a_8^2\Big){}^{\mathrm{w}}I_3\Big]{}^{\mathrm{w}}I_3 \\
&\quad - \Big[2a_1a_8^2 - a_5\Big(a_1(2a_3+a_6) - 2a_7^2\Big) + a_9\Big(a_9(2a_3+a_6) - 4a_7a_8\Big)\Big]\cdot \\
&\qquad\qquad \cdot({}^{\mathrm{w}}I_2{}^2 - 3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)\ ,
\end{aligned} \tag{30}$$

$$
\begin{aligned}
\bar{n}_{63} = {} & a_6\Big(a_3 a_6 + 6(a_3 a_9 - a_7 a_8) + (a_3 a_5 - a_8^2)({}^{\mathrm{w}}I_1{}^2 - 2\,{}^{\mathrm{w}}I_2)\Big)\,{}^{\mathrm{w}}I_3{}^2 \\
& - \Big(a_1 a_8^2 - a_5(a_1 a_3 - a_7^2) + a_9(a_3 a_9 - 2a_7 a_8)\Big)\cdot \\
& \qquad \cdot\Big(({}^{\mathrm{w}}I_1{}^2 - 2\,{}^{\mathrm{w}}I_2)({}^{\mathrm{w}}I_2{}^2 - 2\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3) - 9\,{}^{\mathrm{w}}I_3{}^2\Big) \\
& + a_6(a_1 a_3 - a_7^2)({}^{\mathrm{w}}I_2{}^2 - 2\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3)\ ,
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
\bar{n}_{65} = {} & a_6\Big(3a_7^2 - a_1(3a_3 + a_6) - 2(a_1 a_8 - a_7 a_9)\,{}^{\mathrm{w}}I_1 - (a_1 a_5 - a_9^2)({}^{\mathrm{w}}I_1{}^2 - 2\,{}^{\mathrm{w}}I_2)\Big) \\
& + 2\Big(a_1 a_8^2 - a_5(a_1 a_3 - a_7^2) + a_9(a_3 a_9 - 2a_7 a_8)\Big)({}^{\mathrm{w}}I_1{}^2 - 3\,{}^{\mathrm{w}}I_2)\ ,
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
\bar{n}_{67} = {} & a_6 a_8(a_6 + 3a_9 + a_8\,{}^{\mathrm{w}}I_1)\,{}^{\mathrm{w}}I_3{}^2 + a_6(a_1 a_8 - a_7 a_9)({}^{\mathrm{w}}I_2{}^2 - 2\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3) \\
& + \Big(a_5 a_7^2 + a_9(a_3 a_9 - 2a_7 a_8) - a_1(a_3 a_5 - a_8^2)\Big)\Big({}^{\mathrm{w}}I_1({}^{\mathrm{w}}I_2{}^2 - 2\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3) - 3\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3\Big) \\
& - a_6\,{}^{\mathrm{w}}I_3\Big(a_7(3a_5\,{}^{\mathrm{w}}I_3 - a_8\,{}^{\mathrm{w}}I_2) + a_3(a_5\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3 + a_9\,{}^{\mathrm{w}}I_2)\Big)\ ,
\end{aligned} \tag{33}
$$

$$
\begin{aligned}
\bar{n}_{68} = {} & a_6(a_1 a_3 - a_7^2)\,{}^{\mathrm{w}}I_2 + a_6\Big[3a_1 a_8 - a_7(a_6 + 3a_9) - a_7\Big(a_8\,{}^{\mathrm{w}}I_1 + a_5({}^{\mathrm{w}}I_1{}^2 - 2\,{}^{\mathrm{w}}I_2)\Big) \\
& \qquad + a_9\Big(a_3\,{}^{\mathrm{w}}I_1 + a_8({}^{\mathrm{w}}I_1{}^2 - 2\,{}^{\mathrm{w}}I_2)\Big)\Big]\,{}^{\mathrm{w}}I_3 \\
& - \Big(a_1 a_8^2 - a_5(a_1 a_3 - a_7^2) + a_9(a_3 a_9 - 2a_7 a_8)\Big)\Big({}^{\mathrm{w}}I_2({}^{\mathrm{w}}I_1{}^2 - 2\,{}^{\mathrm{w}}I_2) - 3\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3\Big)\ ,
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
\bar{n}_{69} = {} & a_6(a_1 a_8 - a_7 a_9)\,{}^{\mathrm{w}}I_2 + a_6\Big(3(a_1 a_5 + a_7 a_8) - a_9(3a_3 + a_6 + 3a_9) + (a_5 a_7 - a_8 a_9)\,{}^{\mathrm{w}}I_1\Big)\,{}^{\mathrm{w}}I_3 \\
& + \Big(a_1 a_8^2 - a_5(a_1 a_3 - a_7^2) + a_9(a_3 a_9 - 2a_7 a_8)\Big)({}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2 - 9\,{}^{\mathrm{w}}I_3)\ .
\end{aligned} \tag{35}
$$

Although still given by quite lengthy expressions, *Solution (6.6)* looks much simpler than its counterpart *Solution (2.6)* based on 'shear-like' generators $a_2\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}, \bar{a}_2\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}$ (Section 3) and originates further interesting particular cases (including one that works without constraints on the coefficients).

We start listing now below the six solution cases that contain five of the 'non-shear' coefficients and are obtained by eliminating in turn one of the 'non-shear' coefficients from *Solution (6.6)*. The missing 'non-shear' coefficient is indicated by the third label digit.

Solution (6.5.1). Symmetric damage-effect tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ with the six-coefficients sets $(a_3, a_5, a_6, a_7, a_8, a_9)$ and $(\bar{a}_3, \bar{a}_5, \bar{a}_6, \bar{a}_7, \bar{a}_8, \bar{a}_9)$ (lacking only 'shear-like' coefficients a_2, a_4 and $\bar{a}_2, \bar{a}_4$, and 'non-shear' coefficients a_1 and $\bar{a}_1$) form the dual inverse pair:

$$
\begin{aligned}
\mathbb{A} = {} & a_6\,\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w} + a_3\,\mathbf{w}\otimes\mathbf{w} + a_5\,\mathbf{w}^2\otimes\mathbf{w}^2 \\
& + a_7\,(\mathbf{w}\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{w}) + a_8\,(\mathbf{w}^2\otimes\mathbf{w} + \mathbf{w}\otimes\mathbf{w}^2) + a_9\,(\mathbf{w}^2\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{w}^2)\,; \\
\bar{\mathbb{A}} = {} & \bar{a}_6\,\bar{\mathbf{w}}\,\underline{\overline{\otimes}}\,\bar{\mathbf{w}} + \bar{a}_3\,\bar{\mathbf{w}}\otimes\bar{\mathbf{w}} + \bar{a}_5\,\bar{\mathbf{w}}^2\otimes\bar{\mathbf{w}}^2 \\
& + \bar{a}_7\,(\bar{\mathbf{w}}\otimes\mathbf{I} + \mathbf{I}\otimes\bar{\mathbf{w}}) + \bar{a}_8\,(\bar{\mathbf{w}}^2\otimes\bar{\mathbf{w}} + \bar{\mathbf{w}}\otimes\bar{\mathbf{w}}^2) + \bar{a}_9\,(\bar{\mathbf{w}}^2\otimes\mathbf{I} + \mathbf{I}\otimes\bar{\mathbf{w}}^2)\,,
\end{aligned} \tag{36}
$$

provided that

$$a_3 = \frac{n^\star_{13}}{3a_5a_6{}^{\mathrm{w}}I_3{}^2 - 2a_9^2({}^{\mathrm{w}}I_2{}^2 - 3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)}; \qquad \bar{a}_3 = \frac{\bar{n}^\star_{13}}{3\bar{a}_5\bar{a}_6{}^{\bar{\mathrm{w}}}I_3{}^2 - 2\bar{a}_9^2({}^{\bar{\mathrm{w}}}I_2{}^2 - 3{}^{\bar{\mathrm{w}}}I_1{}^{\bar{\mathrm{w}}}I_3)}, \tag{37}$$

where

$$\begin{aligned} n^\star_{13} &= a_6{}^{\mathrm{w}}I_3\Big(a_9^2{}^{\mathrm{w}}I_1 - 2(a_5a_7 - a_8a_9){}^{\mathrm{w}}I_2 - (a_5a_6 - 3a_8^2){}^{\mathrm{w}}I_3\Big) \\ &\quad + \Big(2a_5a_7^2 + a_9(a_6a_9 - 4a_7a_8)\Big)\,({}^{\mathrm{w}}I_2{}^2 - 3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)\ ; \\ \bar{n}^\star_{13} &= \bar{a}_6{}^{\bar{\mathrm{w}}}I_3\Big(\bar{a}_9^2{}^{\bar{\mathrm{w}}}I_1 - 2(\bar{a}_5\bar{a}_7 - \bar{a}_8\bar{a}_9){}^{\bar{\mathrm{w}}}I_2 - (\bar{a}_5\bar{a}_6 - 3\bar{a}_8^2){}^{\bar{\mathrm{w}}}I_3\Big) \\ &\quad + \Big(2\bar{a}_5\bar{a}_7^2 + \bar{a}_9(\bar{a}_6\bar{a}_9 - 4\bar{a}_7\bar{a}_8)\Big)\,({}^{\bar{\mathrm{w}}}I_2{}^2 - 3{}^{\bar{\mathrm{w}}}I_1{}^{\bar{\mathrm{w}}}I_3)\ , \end{aligned} \tag{38}$$

with

$$\begin{aligned} \bar{a}_6 &= \frac{1}{a_6}\ ; \quad \bar{a}_3 = \frac{\bar{n}_{13}}{a_6\ \bar{d}_1^2}\ , \quad \bar{a}_5 = \frac{\bar{n}_{15}\ {}^{\mathrm{w}}I_3{}^2}{a_6\ \bar{d}_1^2}\ , \\ \bar{a}_7 &= \frac{a_9^2({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3) - (a_5a_6{}^{\mathrm{w}}I_3 + (a_5a_7 - a_8a_9){}^{\mathrm{w}}I_2){}^{\mathrm{w}}I_3}{a_6\ \bar{d}_1}\ , \\ \bar{a}_8 &= \frac{\bar{n}_{18}\ {}^{\mathrm{w}}I_3}{a_6\ \bar{d}_1^2}\ , \quad \bar{a}_9 = \frac{3a_5a_7{}^{\mathrm{w}}I_3 - a_9(a_9{}^{\mathrm{w}}I_2 + 3a_8{}^{\mathrm{w}}I_3)}{a_6\ \bar{d}_1}\ {}^{\mathrm{w}}I_3\ , \end{aligned} \tag{39}$$

where

$$\begin{aligned} \bar{d}_1 &= a_6(3a_8 + a_5{}^{\mathrm{w}}I_1){}^{\mathrm{w}}I_3{}^2 + a_9{}^{\mathrm{w}}I_3(a_6{}^{\mathrm{w}}I_2 - a_8{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2 + 9a_8{}^{\mathrm{w}}I_3) \\ &\quad - a_9^2({}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{2\mathrm{w}}I_3 - 3{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3) + a_7\Big(a_5({}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2 - 9{}^{\mathrm{w}}I_3){}^{\mathrm{w}}I_3 - 2a_9({}^{\mathrm{w}}I_2{}^2 - 3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)\Big), \end{aligned} \tag{40}$$

$$\begin{aligned} \bar{n}_{13} &= 2a_7{}^{\mathrm{w}}I_3\Big[a_5^2a_6{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3{}^2({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2) + 2a_8a_9(a_6 + 3a_9){}^{\mathrm{w}}I_3({}^{\mathrm{w}}I_2{}^2 - 3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3) \\ &\qquad + a_5\Big(a_6{}^{\mathrm{w}}I_3{}^2(a_6{}^{\mathrm{w}}I_2 + 9a_8{}^{\mathrm{w}}I_3) \\ &\qquad\qquad - a_9{}^{\mathrm{w}}I_3(a_8{}^{\mathrm{w}}I_1{}^{2\mathrm{w}}I_2{}^2 - 2a_8{}^{\mathrm{w}}I_2{}^3 - 6a_6{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3 - 27a_8{}^{\mathrm{w}}I_3{}^2)\Big)\Big] \\ &\quad + a_9^2\Big(2a_8a_9{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3 - 2a_5{}^{\mathrm{w}}I_3(a_7{}^{\mathrm{w}}I_2 + a_6{}^{\mathrm{w}}I_3) + a_9^2({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)\Big)\cdot \\ &\qquad\qquad \cdot({}^{\mathrm{w}}I_1{}^{2\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_2{}^3 - 2{}^{\mathrm{w}}I_1{}^{3\mathrm{w}}I_3 + 4{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3 - 9{}^{\mathrm{w}}I_3{}^2) \\ &\quad + a_6^2\Big(a_5a_6 - 3a_8^2 + a_5^2({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2)\Big){}^{\mathrm{w}}I_3{}^4 \\ &\quad - 2a_6a_9{}^{\mathrm{w}}I_3{}^3\Big[a_6(a_8{}^{\mathrm{w}}I_2 - 3a_5{}^{\mathrm{w}}I_3) + a_8\Big(a_5({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2){}^{\mathrm{w}}I_2 + 9a_8{}^{\mathrm{w}}I_3\Big)\Big] \\ &\quad + (a_5^2a_7^2 + a_8^2a_9^2)({}^{\mathrm{w}}I_1{}^{2\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_2{}^3 - 27{}^{\mathrm{w}}I_3{}^2){}^{\mathrm{w}}I_3{}^2 \\ &\quad - a_7^2\Big(12a_5a_9{}^{\mathrm{w}}I_3{}^2({}^{\mathrm{w}}I_2{}^2 - 3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3) + 2a_9^2({}^{\mathrm{w}}I_2{}^2 - 3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3) - a_5a_6{}^{\mathrm{w}}I_2{}^{2\mathrm{w}}I_3{}^2\Big) \\ &\quad - a_6a_9^2{}^{\mathrm{w}}I_3{}^2\Big((a_6 + 6a_9)({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3) + 3{}^{\mathrm{w}}I_3(4a_8{}^{\mathrm{w}}I_2 + 3a_5{}^{\mathrm{w}}I_3)\Big), \end{aligned} \tag{41}$$

$$
\begin{aligned}
\bar{n}_{15} = {} & 3a_7a_5{}^{\mathrm{w}}{I_3}^2\Big(a_6(3a_7+2a_9{}^{\mathrm{w}}I_1)+2(a_5a_7-2a_8a_9)({}^{\mathrm{w}}{I_1}^2-3{}^{\mathrm{w}}I_2)\Big)\\
& + a_9^2\Big[a_5a_6{}^{\mathrm{w}}{I_1}^2{}^{\mathrm{w}}{I_3}^2-2a_7(3a_7+2a_9{}^{\mathrm{w}}I_1)({}^{\mathrm{w}}{I_2}^2-3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)-2a_9^2({}^{\mathrm{w}}{I_2}^3-{}^{\mathrm{w}}{I_1}^3{}^{\mathrm{w}}I_3)\\
& \qquad -2\Big(2a_5a_7{}^{\mathrm{w}}I_2-a_8(2a_9{}^{\mathrm{w}}I_2+3a_8{}^{\mathrm{w}}I_3)\Big)({}^{\mathrm{w}}{I_1}^2-3{}^{\mathrm{w}}I_2){}^{\mathrm{w}}I_3\Big]\,,
\end{aligned}
\tag{42}
$$

$$
\begin{aligned}
\bar{n}_{18} = {} & a_5(a_6a_9^2-3a_5a_7^2)({}^{\mathrm{w}}{I_1}^2{}^{\mathrm{w}}I_2-2{}^{\mathrm{w}}{I_2}^2-3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3){}^{\mathrm{w}}{I_3}^2\\
& - a_7^2\Big(3a_5a_6{}^{\mathrm{w}}I_2{}^{\mathrm{w}}{I_3}^2-2a_5a_9{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3({}^{\mathrm{w}}{I_2}^2-3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)-2a_9^2({}^{\mathrm{w}}{I_2}^3-3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3)\Big)\\
& - 2a_9^2{}^{\mathrm{w}}I_3(a_9a_8-a_5a_7)(2{}^{\mathrm{w}}{I_1}^2{}^{\mathrm{w}}{I_2}^2-4{}^{\mathrm{w}}{I_2}^3-3{}^{\mathrm{w}}{I_1}^3{}^{\mathrm{w}}I_3+3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3)\\
& + a_9\Big(3a_8(2a_5a_7-a_8a_9){}^{\mathrm{w}}{I_3}^2-a_9^3({}^{\mathrm{w}}{I_2}^2-2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)\Big)({}^{\mathrm{w}}{I_1}^2{}^{\mathrm{w}}I_2-2{}^{\mathrm{w}}{I_2}^2-3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)\\
& + a_6a_9\Big[\Big(3a_8{}^{\mathrm{w}}I_1(a_8+a_5{}^{\mathrm{w}}I_1)-a_5(a_6{}^{\mathrm{w}}I_1+6a_8{}^{\mathrm{w}}I_2)\Big){}^{\mathrm{w}}{I_3}^3\\
& \qquad +a_9{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3\Big(2a_8{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3+a_9({}^{\mathrm{w}}{I_2}^2-2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)\Big)\Big]\\
& - a_7{}^{\mathrm{w}}I_3\Big[a_6\Big(3a_5(a_6+a_8{}^{\mathrm{w}}I_1+a_5{}^{\mathrm{w}}{I_1}^2-2a_5{}^{\mathrm{w}}I_2){}^{\mathrm{w}}{I_3}^2+a_5a_9{}^{\mathrm{w}}I_3(2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2+9{}^{\mathrm{w}}I_3)\Big)\\
& \qquad -2a_9^2(a_6+3a_9-a_8{}^{\mathrm{w}}I_1)({}^{\mathrm{w}}{I_2}^2-3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)\Big]\,.
\end{aligned}
\tag{43}
$$

Solution (6.5.1) is obtained as a particular case of *Solution (6.6)* by setting a_1=0, $\bar{a}_1$=0, which leads to the constraint (37).

Solution (6.5.3). Symmetric damage-effect tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ with the six-coefficients sets $(a_1, a_5, a_6, a_7, a_8, a_9)$ and $(\bar{a}_1, \bar{a}_5, \bar{a}_6, \bar{a}_7, \bar{a}_8, \bar{a}_9)$ (lacking only 'shear-like' coefficients a_2, a_4 and $\bar{a}_2, \bar{a}_4$, and 'non-shear' coefficients a_3 and $\bar{a}_3$) form the dual inverse pair:

$$
\begin{aligned}
\mathbb{A} = {} & a_6\;\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}+a_1\,\mathbf{I}\otimes\mathbf{I}+a_5\,\mathbf{w}^2\otimes\mathbf{w}^2\\
& + a_7\,(\mathbf{w}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w})+a_8\,(\mathbf{w}^2\otimes\mathbf{w}+\mathbf{w}\otimes\mathbf{w}^2)+a_9\,(\mathbf{w}^2\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w}^2)\,;\\
\bar{\mathbb{A}} = {} & \bar{a}_6\;\bar{\mathbf{w}}\,\overline{\underline{\otimes}}\,\bar{\mathbf{w}}+\bar{a}_1\,\mathbf{I}\otimes\mathbf{I}+\bar{a}_5\,\bar{\mathbf{w}}^2\otimes\bar{\mathbf{w}}^2\\
& + \bar{a}_7\,(\bar{\mathbf{w}}\otimes\mathbf{I}+\mathbf{I}\otimes\bar{\mathbf{w}})+\bar{a}_8\,(\bar{\mathbf{w}}^2\otimes\bar{\mathbf{w}}+\bar{\mathbf{w}}\otimes\bar{\mathbf{w}}^2)+\bar{a}_9\,(\bar{\mathbf{w}}^2\otimes\mathbf{I}+\mathbf{I}\otimes\bar{\mathbf{w}}^2)\,,
\end{aligned}
\tag{44}
$$

provided that

$$
a_1=\frac{1}{a_8^2}\,\frac{n_{31}^\star}{9{}^{\mathrm{w}}{I_3}^2-({}^{\mathrm{w}}{I_1}^2-2{}^{\mathrm{w}}I_2)({}^{\mathrm{w}}{I_2}^2-2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)};\;\;
\bar{a}_1=\frac{1}{\bar{a}_8^2}\,\frac{\bar{n}_{31}^\star}{9{}^{\bar{\mathrm{w}}}{I_3}^2-({}^{\bar{\mathrm{w}}}{I_1}^2-2{}^{\bar{\mathrm{w}}}I_2)({}^{\bar{\mathrm{w}}}{I_2}^2-2{}^{\bar{\mathrm{w}}}I_1{}^{\bar{\mathrm{w}}}I_3)},
\tag{45}
$$

where

$$\begin{aligned} n^\star_{31} &= a_6 a_8\Big(6a_7 + a_8({}^{\mathrm{w}}\!I_1{}^2 - 2{}^{\mathrm{w}}\!I_2)\Big){}^{\mathrm{w}}\!I_3{}^2 + a_6 a_7^2({}^{\mathrm{w}}\!I_2{}^2 - 2{}^{\mathrm{w}}\!I_1{}^{\mathrm{w}}\!I_3) \\ &\quad - a_7(a_5 a_7 - 2a_8 a_9)\Big(9{}^{\mathrm{w}}\!I_3{}^2 - ({}^{\mathrm{w}}\!I_1{}^2 - 2{}^{\mathrm{w}}\!I_2)({}^{\mathrm{w}}\!I_2{}^2 - 2{}^{\mathrm{w}}\!I_1{}^{\mathrm{w}}\!I_3)\Big)\ ; \\ \bar n^\star_{31} &= \bar a_6 \bar a_8\Big(6\bar a_7 + \bar a_8({}^{\bar{\mathrm{w}}}\!I_1{}^2 - 2{}^{\bar{\mathrm{w}}}\!I_2)\Big){}^{\bar{\mathrm{w}}}\!I_3{}^2 + \bar a_6 \bar a_7^2({}^{\bar{\mathrm{w}}}\!I_2{}^2 - 2{}^{\bar{\mathrm{w}}}\!I_1{}^{\bar{\mathrm{w}}}\!I_3) \\ &\quad - \bar a_7(\bar a_5 \bar a_7 - 2\bar a_8 \bar a_9)\Big(9{}^{\bar{\mathrm{w}}}\!I_3{}^2 - ({}^{\bar{\mathrm{w}}}\!I_1{}^2 - 2{}^{\bar{\mathrm{w}}}\!I_2)({}^{\bar{\mathrm{w}}}\!I_2{}^2 - 2{}^{\bar{\mathrm{w}}}\!I_1{}^{\bar{\mathrm{w}}}\!I_3)\Big)\ , \end{aligned} \tag{46}$$

with

$$\begin{aligned} \bar a_6 &= \frac{1}{a_6}\ ; \quad \bar a_1 = \frac{\bar n_{31}}{a_6\ \bar d_3^2}\ , \quad \bar a_5 = -\frac{\bar n_{35}\ {}^{\mathrm{w}}\!I_3{}^2}{a_6\ \bar d_3^2}\ , \\ \bar a_7 &= -a_8\ \frac{a_7({}^{\mathrm{w}}\!I_2{}^2 - 2{}^{\mathrm{w}}\!I_1{}^{\mathrm{w}}\!I_3) + 3a_8{}^{\mathrm{w}}\!I_3{}^2}{a_6\ \bar d_3}\ , \\ \bar a_8 &= a_8\ \frac{3a_7 + a_8({}^{\mathrm{w}}\!I_1{}^2 - 2{}^{\mathrm{w}}\!I_2)}{a_6\ \bar d_3}\ {}^{\mathrm{w}}\!I_3{}^2\ , \quad \bar a_9 = \frac{\bar n_{39}\ {}^{\mathrm{w}}\!I_3}{a_6\ \bar d_3^2}\ , \end{aligned} \tag{47}$$

where

$$\begin{aligned} \bar d_3 &= a_6\Big(a_7({}^{\mathrm{w}}\!I_2{}^2 - 2{}^{\mathrm{w}}\!I_1{}^{\mathrm{w}}\!I_3) + 3a_8{}^{\mathrm{w}}\!I_3{}^2\Big) \\ &\quad + a_8\Big(a_7({}^{\mathrm{w}}\!I_1{}^{\mathrm{w}}\!I_2{}^2 - 2{}^{\mathrm{w}}\!I_1{}^2{}^{\mathrm{w}}\!I_3 - 3{}^{\mathrm{w}}\!I_2{}^{\mathrm{w}}\!I_3) - a_8{}^{\mathrm{w}}\!I_3({}^{\mathrm{w}}\!I_1{}^2{}^{\mathrm{w}}\!I_2 - 2{}^{\mathrm{w}}\!I_2{}^2 - 3{}^{\mathrm{w}}\!I_1{}^{\mathrm{w}}\!I_3)\Big) \\ &\quad + (a_5 a_7 - a_8 a_9)({}^{\mathrm{w}}\!I_1{}^2{}^{\mathrm{w}}\!I_2{}^2 - 2{}^{\mathrm{w}}\!I_2{}^3 - 2{}^{\mathrm{w}}\!I_1{}^3{}^{\mathrm{w}}\!I_3 + 4{}^{\mathrm{w}}\!I_1{}^{\mathrm{w}}\!I_2{}^{\mathrm{w}}\!I_3 - 9{}^{\mathrm{w}}\!I_3{}^2)\ , \end{aligned} \tag{48}$$

$$\begin{aligned} \bar n_{31} &= (a_5 a_7 - a_8 a_9)\Big[a_8\Big(2a_8{}^{\mathrm{w}}\!I_2{}^{\mathrm{w}}\!I_3 + a_9({}^{\mathrm{w}}\!I_2{}^2 - 2{}^{\mathrm{w}}\!I_1{}^{\mathrm{w}}\!I_3)\Big) - a_5 a_7({}^{\mathrm{w}}\!I_2{}^2 - 2{}^{\mathrm{w}}\!I_1{}^{\mathrm{w}}\!I_3)\Big]\cdot \\ &\qquad \cdot({}^{\mathrm{w}}\!I_1{}^2{}^{\mathrm{w}}\!I_2{}^2 - 2{}^{\mathrm{w}}\!I_2{}^3 - 2{}^{\mathrm{w}}\!I_1{}^3{}^{\mathrm{w}}\!I_3 + 4{}^{\mathrm{w}}\!I_1{}^{\mathrm{w}}\!I_2{}^{\mathrm{w}}\!I_3 - 9{}^{\mathrm{w}}\!I_3{}^2) \\ &\quad + a_8^2\Big[2a_7\Big(6a_8{}^{\mathrm{w}}\!I_3{}^2 + a_7({}^{\mathrm{w}}\!I_2{}^2 - 2{}^{\mathrm{w}}\!I_1{}^{\mathrm{w}}\!I_3)\Big)({}^{\mathrm{w}}\!I_2{}^2 - 3{}^{\mathrm{w}}\!I_1{}^{\mathrm{w}}\!I_3) \\ &\qquad - a_8^2{}^{\mathrm{w}}\!I_3{}^2({}^{\mathrm{w}}\!I_1{}^2{}^{\mathrm{w}}\!I_2{}^2 - 2{}^{\mathrm{w}}\!I_2{}^3 - 27{}^{\mathrm{w}}\!I_3{}^2)\Big] \\ &\quad - a_5 a_6\Big(3a_8{}^{\mathrm{w}}\!I_3{}^2 + a_7({}^{\mathrm{w}}\!I_2{}^2 - 2{}^{\mathrm{w}}\!I_1{}^{\mathrm{w}}\!I_3)\Big)^2, \end{aligned} \tag{49}$$

$$\begin{aligned} \bar n_{35} &= (a_5 a_7 - a_8 a_9)\Big[a_7\Big(2(a_6 + a_8{}^{\mathrm{w}}\!I_1) + a_5({}^{\mathrm{w}}\!I_1{}^2 - 2{}^{\mathrm{w}}\!I_2)\Big) - a_8 a_9({}^{\mathrm{w}}\!I_1{}^2 - 2{}^{\mathrm{w}}\!I_2)\Big]\cdot \\ &\qquad \cdot({}^{\mathrm{w}}\!I_1{}^2{}^{\mathrm{w}}\!I_2{}^2 - 2{}^{\mathrm{w}}\!I_2{}^3 - 2{}^{\mathrm{w}}\!I_1{}^3{}^{\mathrm{w}}\!I_3 + 4{}^{\mathrm{w}}\!I_1{}^{\mathrm{w}}\!I_2{}^{\mathrm{w}}\!I_3 - 9{}^{\mathrm{w}}\!I_3{}^2) \\ &\quad + 3a_7\Big[a_5 a_6\Big(3a_7 + 2a_8({}^{\mathrm{w}}\!I_1{}^2 - 2{}^{\mathrm{w}}\!I_2)\Big) + 4a_8^2\Big({}^{\mathrm{w}}\!I_1(a_6 - a_8{}^{\mathrm{w}}\!I_1) + 3a_8{}^{\mathrm{w}}\!I_2\Big)\Big]{}^{\mathrm{w}}\!I_3{}^2 \\ &\quad + a_7^2 a_8\Big(2a_6{}^{\mathrm{w}}\!I_1({}^{\mathrm{w}}\!I_2{}^2 - 2{}^{\mathrm{w}}\!I_1{}^{\mathrm{w}}\!I_3) + a_8({}^{\mathrm{w}}\!I_1{}^2{}^{\mathrm{w}}\!I_2{}^2 - 2{}^{\mathrm{w}}\!I_1{}^3{}^{\mathrm{w}}\!I_3 - 27{}^{\mathrm{w}}\!I_3{}^2)\Big) \\ &\quad + a_8^2\Big[a_6\Big(2a_8{}^{\mathrm{w}}\!I_1 + a_5({}^{\mathrm{w}}\!I_1{}^2 - 2{}^{\mathrm{w}}\!I_2)\Big) - 2a_8^2({}^{\mathrm{w}}\!I_1{}^2 - 3{}^{\mathrm{w}}\!I_2)\Big]({}^{\mathrm{w}}\!I_1{}^2 - 2{}^{\mathrm{w}}\!I_2){}^{\mathrm{w}}\!I_3{}^2 \\ &\quad + a_6^2\Big(6a_7 a_8{}^{\mathrm{w}}\!I_3{}^2 + a_8^2({}^{\mathrm{w}}\!I_1{}^2 - 2{}^{\mathrm{w}}\!I_2){}^{\mathrm{w}}\!I_3{}^2 + a_7^2({}^{\mathrm{w}}\!I_2{}^2 - 2{}^{\mathrm{w}}\!I_1{}^{\mathrm{w}}\!I_3)\Big)\ , \end{aligned} \tag{50}$$

$$\bar{n}_{39} = \Big[3a_5^2a_7^2\,{}^{\mathrm{w}}I_3 + a_5a_7a_8\Big(a_7\,{}^{\mathrm{w}}I_2 - (6a_9 + a_8\,{}^{\mathrm{w}}I_1)\,{}^{\mathrm{w}}I_3\Big) - a_8^2\Big(a_7(a_9\,{}^{\mathrm{w}}I_2 + 3a_8\,{}^{\mathrm{w}}I_3) - a_9(a_6 + 3a_9 + a_8\,{}^{\mathrm{w}}I_1)\,{}^{\mathrm{w}}I_3\Big)\Big]\cdot \cdot({}^{\mathrm{w}}I_1{}^2\,{}^{\mathrm{w}}I_2{}^2 - 2\,{}^{\mathrm{w}}I_2{}^3 - 2\,{}^{\mathrm{w}}I_1{}^3\,{}^{\mathrm{w}}I_3 + 4\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3 - 9\,{}^{\mathrm{w}}I_3{}^2) + \Big(a_8^2({}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2 - 9\,{}^{\mathrm{w}}I_3) + a_6(a_8\,{}^{\mathrm{w}}I_2 + 3a_5\,{}^{\mathrm{w}}I_3)\Big)\cdot \cdot\Big(6a_7a_8\,{}^{\mathrm{w}}I_3{}^2 + a_8^2({}^{\mathrm{w}}I_1{}^2 - 2\,{}^{\mathrm{w}}I_2)\,{}^{\mathrm{w}}I_3{}^2 + a_7^2({}^{\mathrm{w}}I_2{}^2 - 2\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3)\Big)\,. \quad (51)$$

Solution (6.5.3) is obtained as a particular case of *Solution (6.6)* by setting a_3=0, $\bar{a}_3$=0, which leads to the constraint (45).

Solution (6.5.5). Symmetric damage-effect tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ with the six-coefficients sets $(a_1, a_3, a_6, a_7, a_8, a_9)$ and $(\bar{a}_1, \bar{a}_3, \bar{a}_6, \bar{a}_7, \bar{a}_8, \bar{a}_9)$ (lacking only 'shear-like' coefficients a_2, a_4 and $\bar{a}_2, \bar{a}_4$, and 'non-shear' coefficients a_5 and $\bar{a}_5$) form the dual inverse pair:

$$\begin{aligned}\mathbb{A} &= a_6\,\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w} + a_1\,\mathbf{I}\otimes\mathbf{I} + a_3\,\mathbf{w}\otimes\mathbf{w}\\ &+ a_7\,(\mathbf{w}\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{w}) + a_8\,(\mathbf{w}^2\otimes\mathbf{w} + \mathbf{w}\otimes\mathbf{w}^2) + a_9\,(\mathbf{w}^2\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{w}^2)\,;\\ \bar{\mathbb{A}} &= \bar{a}_6\,\bar{\mathbf{w}}\,\underline{\overline{\otimes}}\,\bar{\mathbf{w}} + \bar{a}_1\,\mathbf{I}\otimes\mathbf{I} + \bar{a}_3\,\bar{\mathbf{w}}\otimes\bar{\mathbf{w}}\\ &+ \bar{a}_7\,(\bar{\mathbf{w}}\otimes\mathbf{I} + \mathbf{I}\otimes\bar{\mathbf{w}}) + \bar{a}_8\,(\bar{\mathbf{w}}^2\otimes\bar{\mathbf{w}} + \bar{\mathbf{w}}\otimes\bar{\mathbf{w}}^2) + \bar{a}_9\,(\bar{\mathbf{w}}^2\otimes\mathbf{I} + \mathbf{I}\otimes\bar{\mathbf{w}}^2)\,,\end{aligned} \quad (52)$$

provided that

$$\begin{aligned} a_3 &= -\frac{a_6\Big(a_1a_6 - 3a_7^2 + 2(a_1a_8 - a_7a_9)\,{}^{\mathrm{w}}I_1 - a_9^2({}^{\mathrm{w}}I_1{}^2 - 2\,{}^{\mathrm{w}}I_2)\Big) - 2a_8(a_1a_8 - 2a_7a_9)({}^{\mathrm{w}}I_1{}^2 - 3\,{}^{\mathrm{w}}I_2)}{3a_1a_6 - 2a_9^2({}^{\mathrm{w}}I_1{}^2 - 3\,{}^{\mathrm{w}}I_2)}\,;\\ \bar{a}_3 &= -\frac{\bar{a}_6\Big(\bar{a}_1\bar{a}_6 - 3\bar{a}_7^2 + 2(\bar{a}_1\bar{a}_8 - \bar{a}_7\bar{a}_9)\,{}^{\bar{\mathrm{w}}}I_1 - \bar{a}_9^2({}^{\bar{\mathrm{w}}}I_1{}^2 - 2\,{}^{\bar{\mathrm{w}}}I_2)\Big) - 2\bar{a}_8(\bar{a}_1\bar{a}_8 - 2\bar{a}_7\bar{a}_9)({}^{\bar{\mathrm{w}}}I_1{}^2 - 3\,{}^{\bar{\mathrm{w}}}I_2)}{3\bar{a}_1\bar{a}_6 - 2\bar{a}_9^2({}^{\bar{\mathrm{w}}}I_1{}^2 - 3\,{}^{\bar{\mathrm{w}}}I_2)}\,,\end{aligned} \quad (53)$$

with

$$\begin{aligned} &\bar{a}_6 = \frac{1}{a_6}\,; \quad \bar{a}_1 = \frac{\bar{n}_{51}}{a_6\,\bar{d}_5^2}\,, \quad \bar{a}_3 = \frac{\bar{n}_{53}}{a_6\,\bar{d}_5^2}\,,\\ &\bar{a}_7 = -\frac{\bar{n}_{57}}{a_6\,\bar{d}_5^2}\,, \quad \bar{a}_8 = -\frac{a_1\,(a_6 + a_8\,{}^{\mathrm{w}}I_1) - a_9\,\Big(a_7\,{}^{\mathrm{w}}I_1 + a_9\,({}^{\mathrm{w}}I_1{}^2 - 2\,{}^{\mathrm{w}}I_2)\Big)}{a_6\,\bar{d}_5}\,{}^{\mathrm{w}}I_3\,,\\ &\bar{a}_9 = \frac{3\,a_1\,a_8 - a_9\,(3\,a_7 + a_9\,{}^{\mathrm{w}}I_1)}{a_6\,\bar{d}_5}\,{}^{\mathrm{w}}I_3\,,\end{aligned} \quad (54)$$

where

$$\begin{aligned}
\bar{d}_5 = {} & a_1\Big(a_6{}^{\mathrm{w}}I_2 + a_8({}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2 - 9{}^{\mathrm{w}}I_3)\Big) + a_6(3a_7 + a_9{}^{\mathrm{w}}I_1){}^{\mathrm{w}}I_3 - a_7a_9({}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2 - 9{}^{\mathrm{w}}I_3) \\
& - 2a_8a_9({}^{\mathrm{w}}I_1{}^2 - 3{}^{\mathrm{w}}I_2){}^{\mathrm{w}}I_3 - a_9^2({}^{\mathrm{w}}I_1{}^{2\,\mathrm{w}}I_2 - 2{}^{\mathrm{w}}I_2{}^2 - 3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)\,,
\end{aligned} \tag{55}$$

$$\begin{aligned}
\bar{n}_{51} = {} & a_1\Big(a_6(3a_8{}^{\mathrm{w}}I_3 + a_9{}^{\mathrm{w}}I_2)^2 + 2a_8(3a_1a_8 - 6a_7a_9 - 2a_9^{2\,\mathrm{w}}I_1)({}^{\mathrm{w}}I_2{}^2 - 3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)\Big) \\
& + 2a_9^2\Big(a_7(3a_7 + 2a_9{}^{\mathrm{w}}I_1)({}^{\mathrm{w}}I_2{}^2 - 3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3) - a_8(3a_8{}^{\mathrm{w}}I_3 + 2a_9{}^{\mathrm{w}}I_2)({}^{\mathrm{w}}I_1{}^2 - 3{}^{\mathrm{w}}I_2){}^{\mathrm{w}}I_3 \\
& \qquad + a_9^2({}^{\mathrm{w}}I_2{}^3 - {}^{\mathrm{w}}I_1{}^{3\,\mathrm{w}}I_3)\Big)\,,
\end{aligned} \tag{56}$$

$$\begin{aligned}
\bar{n}_{53} = {} & \Big[a_1a_8(a_1a_8 - 2a_7a_9) - a_9^2\Big(2a_1a_6 - a_7^2 + 2(a_1a_8 - a_7a_9){}^{\mathrm{w}}I_1 - a_9^2({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2)\Big)\Big]\cdot \\
& \qquad \cdot({}^{\mathrm{w}}I_1{}^{2\,\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{3\,\mathrm{w}}I_3 - 27{}^{\mathrm{w}}I_3{}^2) \\
& + a_1\Big\{a_6^2(a_6 + 6a_9 + 2a_8{}^{\mathrm{w}}I_1){}^{\mathrm{w}}I_3{}^2 - 12a_8^2a_9({}^{\mathrm{w}}I_1{}^2 - 3{}^{\mathrm{w}}I_2){}^{\mathrm{w}}I_3{}^2 \\
& \qquad + a_6\Big[a_8(12a_9 + a_8{}^{\mathrm{w}}I_1){}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3{}^2 + 9(2a_7a_8 - a_9^2){}^{\mathrm{w}}I_3{}^2 \\
& \qquad\qquad + \Big(a_1a_6 + 2(a_1a_8 - a_7a_9){}^{\mathrm{w}}I_1\Big)({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)\Big]\Big\} \\
& - a_7\Big[(a_6 + 6a_9)\Big(3a_6a_7 + 2a_9(a_6{}^{\mathrm{w}}I_1 - a_8{}^{\mathrm{w}}I_1{}^2 + 3a_8{}^{\mathrm{w}}I_2)\Big) - 2a_6a_8a_9({}^{\mathrm{w}}I_1{}^2 - 3{}^{\mathrm{w}}I_2)\Big]{}^{\mathrm{w}}I_3{}^2 \\
& - a_9^2\Big(6a_6a_9 + a_6^2 + 2a_8^2({}^{\mathrm{w}}I_1{}^2 - 3{}^{\mathrm{w}}I_2)\Big)({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2){}^{\mathrm{w}}I_3{}^2 \\
& - 2a_9^2\Big(2a_1a_6 + 2(a_1a_8 - a_7a_9){}^{\mathrm{w}}I_1 - a_9^2({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2)\Big)(9{}^{\mathrm{w}}I_3{}^2 - {}^{\mathrm{w}}I_2{}^3 + 2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3)\,,
\end{aligned} \tag{57}$$

$$\begin{aligned}
\bar{n}_{57} = {} & a_1\Big[a_6^2(3a_8{}^{\mathrm{w}}I_3 + a_9{}^{\mathrm{w}}I_2){}^{\mathrm{w}}I_3 - 2a_8^2a_9({}^{\mathrm{w}}I_1{}^2 - 3{}^{\mathrm{w}}I_2){}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3 \\
& \qquad + a_6\Big(a_8(3a_8{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3 + 2a_9{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2 + 9a_9{}^{\mathrm{w}}I_3){}^{\mathrm{w}}I_3 - 3a_7(a_9{}^{\mathrm{w}}I_2{}^2 - a_8{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3 - 2a_9{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)\Big)\Big] \\
& + 3a_1^2a_6a_8({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3) \\
& - 2a_9^2(a_1a_8 - a_7a_9)(2{}^{\mathrm{w}}I_1{}^{2\,\mathrm{w}}I_2{}^2 - 3{}^{\mathrm{w}}I_2{}^3 - 4{}^{\mathrm{w}}I_1{}^{3\,\mathrm{w}}I_3 + 3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3) \\
& + \Big(3a_1^2a_8^2 + 3a_7^2a_9^2 - a_1a_9(6a_7a_8 + a_6a_9) + a_9^4({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2)\Big)({}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{2\,\mathrm{w}}I_3 - 3{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3) \\
& + a_9{}^{\mathrm{w}}I_3\Big\{2a_8a_9({}^{\mathrm{w}}I_1{}^2 - 3{}^{\mathrm{w}}I_2)(a_7{}^{\mathrm{w}}I_2 - a_8{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3 - 3a_9{}^{\mathrm{w}}I_3) \\
& \qquad - a_6\Big[a_7(3a_7 + 2a_9{}^{\mathrm{w}}I_1){}^{\mathrm{w}}I_2 + a_9\Big(2a_8({}^{\mathrm{w}}I_1{}^2 - 3{}^{\mathrm{w}}I_2){}^{\mathrm{w}}I_3 + a_9({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2){}^{\mathrm{w}}I_2\Big)\Big]\Big\}\,.
\end{aligned} \tag{58}$$

Solution (6.5.5) is obtained as a particular case of *Solution (6.6)* by setting a_5=0, $\bar{a}_5$=0, which leads to the constraint (53).

Solution (6.5.7). Symmetric damage-effect tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ with the six-coefficients sets $(a_1, a_3, a_5, a_6, a_8, a_9)$ and $(\bar{a}_1, \bar{a}_3, \bar{a}_5, \bar{a}_6, \bar{a}_8, \bar{a}_9)$ (lacking only 'shear-like' coefficients a_2, a_4 and $\bar{a}_2, \bar{a}_4$, and 'non-shear' coefficients a_7 and $\bar{a}_7$) form the dual inverse pair:

$$\begin{aligned}\mathbb{A} &= a_6\, \mathbf{w} \,\underline{\overline{\otimes}}\, \mathbf{w} + a_1\, \mathbf{I} \otimes \mathbf{I} + a_3\, \mathbf{w} \otimes \mathbf{w} + a_5\, \mathbf{w}^2 \otimes \mathbf{w}^2 \\ &\quad + a_8\, (\mathbf{w}^2 \otimes \mathbf{w} + \mathbf{w} \otimes \mathbf{w}^2) + a_9\, (\mathbf{w}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{w}^2)\,; \\ \bar{\mathbb{A}} &= \bar{a}_6\, \bar{\mathbf{w}} \,\underline{\overline{\otimes}}\, \bar{\mathbf{w}} + \bar{a}_1\, \mathbf{I} \otimes \mathbf{I} + \bar{a}_3\, \bar{\mathbf{w}} \otimes \bar{\mathbf{w}} + \bar{a}_5\, \bar{\mathbf{w}}^2 \otimes \bar{\mathbf{w}}^2 \\ &\quad + \bar{a}_8\, (\bar{\mathbf{w}}^2 \otimes \bar{\mathbf{w}} + \bar{\mathbf{w}} \otimes \bar{\mathbf{w}}^2) + \bar{a}_9\, (\bar{\mathbf{w}}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \bar{\mathbf{w}}^2)\,,\end{aligned} \tag{59}$$

provided that

$$\begin{aligned}a_5 &= \frac{1}{a_3}\, \frac{n^\star_{75}}{a_6 {}^{\mathrm{w}}I_1 {}^{\mathrm{w}}I_3{}^2 + a_1({}^{\mathrm{w}}I_1 {}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^2 {}^{\mathrm{w}}I_3 - 3{}^{\mathrm{w}}I_2 {}^{\mathrm{w}}I_3)}\,; \\ \bar{a}_5 &= \frac{1}{\bar{a}_3}\, \frac{\bar{n}^\star_{75}}{\bar{a}_6 {}^{\bar{\mathrm{w}}}I_1 {}^{\bar{\mathrm{w}}}I_3{}^2 + \bar{a}_1({}^{\bar{\mathrm{w}}}I_1 {}^{\bar{\mathrm{w}}}I_2{}^2 - 2{}^{\bar{\mathrm{w}}}I_1{}^2 {}^{\bar{\mathrm{w}}}I_3 - 3{}^{\bar{\mathrm{w}}}I_2 {}^{\bar{\mathrm{w}}}I_3)}\,,\end{aligned} \tag{60}$$

where

$$\begin{aligned}n^\star_{75} &= a_1 a_6 a_8 ({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1 {}^{\mathrm{w}}I_3) - a_3 a_6 a_9 {}^{\mathrm{w}}I_2 {}^{\mathrm{w}}I_3 + a_6 a_8 (a_6 + 3a_9 + a_8 {}^{\mathrm{w}}I_1) {}^{\mathrm{w}}I_3{}^2 \\ &\quad + (a_1 a_8^2 + a_3 a_9^2)({}^{\mathrm{w}}I_1 {}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^2 {}^{\mathrm{w}}I_3 - 3{}^{\mathrm{w}}I_2 {}^{\mathrm{w}}I_3)\,; \\ \bar{n}^\star_{75} &= \bar{a}_1 \bar{a}_6 \bar{a}_8 ({}^{\bar{\mathrm{w}}}I_2{}^2 - 2{}^{\bar{\mathrm{w}}}I_1 {}^{\bar{\mathrm{w}}}I_3) - \bar{a}_3 \bar{a}_6 \bar{a}_9 {}^{\bar{\mathrm{w}}}I_2 {}^{\bar{\mathrm{w}}}I_3 + \bar{a}_6 \bar{a}_8 (\bar{a}_6 + 3\bar{a}_9 + \bar{a}_8 {}^{\bar{\mathrm{w}}}I_1) {}^{\bar{\mathrm{w}}}I_3{}^2 \\ &\quad + (\bar{a}_1 \bar{a}_8^2 + \bar{a}_3 \bar{a}_9^2)({}^{\bar{\mathrm{w}}}I_1 {}^{\bar{\mathrm{w}}}I_2{}^2 - 2{}^{\bar{\mathrm{w}}}I_1{}^2 {}^{\bar{\mathrm{w}}}I_3 - 3{}^{\bar{\mathrm{w}}}I_2 {}^{\bar{\mathrm{w}}}I_3)\,,\end{aligned} \tag{61}$$

with

$$\begin{aligned}\bar{a}_6 &= \frac{1}{a_6}\,; \quad \bar{a}_1 = \frac{(a_3 a_9 {}^{\mathrm{w}}I_2 - a_6 a_8 {}^{\mathrm{w}}I_3){}^{\mathrm{w}}I_3 - a_1 a_8 ({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1 {}^{\mathrm{w}}I_3)}{a_6\, \bar{d}_{71}}\,, \\ \bar{a}_3 &= -a_3\, \frac{a_1({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1 {}^{\mathrm{w}}I_3) + (a_6 + 3a_9){}^{\mathrm{w}}I_3{}^2}{a_6\, \bar{d}_{73}}\,, \quad \bar{a}_5 = \frac{\bar{n}_{75}\ {}^{\mathrm{w}}I_3{}^2}{a_6\, \bar{d}_{71}\, \bar{d}_{73}}\,, \\ \bar{a}_8 &= a_3\, \frac{a_1 {}^{\mathrm{w}}I_2 + a_9 {}^{\mathrm{w}}I_1 {}^{\mathrm{w}}I_3}{a_6\, \bar{d}_{73}}\ {}^{\mathrm{w}}I_3\,, \quad \bar{a}_9 = \frac{3a_1 a_8 - a_3 a_9 {}^{\mathrm{w}}I_1}{a_6\, \bar{d}_{71}}\ {}^{\mathrm{w}}I_3{}^2\,,\end{aligned} \tag{62}$$

where

$$\begin{aligned}\bar{d}_{71} &= \Big(a_3(a_6 + 3a_9){}^{\mathrm{w}}I_1 {}^{\mathrm{w}}I_3 - (a_3 a_9 {}^{\mathrm{w}}I_2 - a_6 a_8 {}^{\mathrm{w}}I_3)({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2)\Big){}^{\mathrm{w}}I_3 \\ &\quad + a_1\Big(a_3({}^{\mathrm{w}}I_1 {}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^2 {}^{\mathrm{w}}I_3 - 3{}^{\mathrm{w}}I_2 {}^{\mathrm{w}}I_3) + \\ &\qquad a_8({}^{\mathrm{w}}I_1{}^2 {}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_2{}^3 - 2{}^{\mathrm{w}}I_1{}^3 {}^{\mathrm{w}}I_3 + 4{}^{\mathrm{w}}I_1 {}^{\mathrm{w}}I_2 {}^{\mathrm{w}}I_3 - 9{}^{\mathrm{w}}I_3{}^2)\Big),\end{aligned} \tag{63}$$

$$\begin{aligned}\bar{d}_{73} = &\Big[a_6(a_6+3a_9+a_8{}^{\mathrm{w}}I_1){}^{\mathrm{w}}I_3 + a_3\Big(3(a_6+3a_9){}^{\mathrm{w}}I_3 - a_9{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2\Big)\Big]{}^{\mathrm{w}}I_3 \\ &+ a_1\Big(2a_3({}^{\mathrm{w}}I_2{}^2 - 3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3) + a_6({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3) + a_8({}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{2\mathrm{w}}I_3 - 3{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3)\Big),\end{aligned} \tag{64}$$

$$\begin{aligned}\bar{n}_{75} = &\, a_3a_9^2\Big(2a_3({}^{\mathrm{w}}I_1{}^2 - 3{}^{\mathrm{w}}I_2) + a_6({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2)\Big){}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3{}^2 \\ &- a_1{}^{\mathrm{w}}I_3\Big\{a_6a_8(a_6+3a_9+a_8{}^{\mathrm{w}}I_1)({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2){}^{\mathrm{w}}I_3 + a_3^2\Big(3a_6{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3 - 2a_9({}^{\mathrm{w}}I_1{}^2 - 3{}^{\mathrm{w}}I_2){}^{\mathrm{w}}I_2\Big) \\ &\quad + a_3\Big[a_6^{2\mathrm{w}}I_1{}^{\mathrm{w}}I_3 + 6a_8a_9({}^{\mathrm{w}}I_1{}^2 - 3{}^{\mathrm{w}}I_2){}^{\mathrm{w}}I_3 \\ &\qquad + a_6\Big(2a_8(2{}^{\mathrm{w}}I_1{}^{2\mathrm{w}}I_3 - 3{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3) - a_9({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2){}^{\mathrm{w}}I_2\Big)\Big]\Big\} \\ &- a_1^2\Big[a_6a_8({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2)({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3) + 2a_3a_8(2{}^{\mathrm{w}}I_1{}^{2\mathrm{w}}I_2{}^2 - 3{}^{\mathrm{w}}I_2{}^3 - 4{}^{\mathrm{w}}I_1{}^{3\mathrm{w}}I_3 + 3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3) \\ &\quad + \Big(3a_3^2 + a_3a_6 + a_8^2({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2)\Big)({}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{2\mathrm{w}}I_3 - 3{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3)\Big].\end{aligned} \tag{65}$$

Solution (6.5.7) is obtained as a particular case of *Solution (6.6)* by setting $a_7{=}0$, $\bar{a}_7{=}0$, which leads to the constraint (60).

Solution (6.5.8). Symmetric damage-effect tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ with the six-coefficients sets $(a_1, a_3, a_5, a_6, a_7, a_9)$ and $(\bar{a}_1, \bar{a}_3, \bar{a}_5, \bar{a}_6, \bar{a}_7, \bar{a}_9)$ (lacking only 'shear-like' coefficients a_2, a_4 and $\bar{a}_2, \bar{a}_4$, and 'non-shear' coefficients a_8 and $\bar{a}_8$) form the dual inverse pair:

$$\begin{aligned}\mathbb{A} &= a_6\, \mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w} + a_1\, \mathbf{I}\otimes\mathbf{I} + a_3\, \mathbf{w}\otimes\mathbf{w} + a_5\, \mathbf{w}^2\otimes\mathbf{w}^2 \\ &\quad + a_7\,(\mathbf{w}\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{w}) + a_9\,(\mathbf{w}^2\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{w}^2)\,; \\ \bar{\mathbb{A}} &= \bar{a}_6\, \bar{\mathbf{w}}\,\overline{\underline{\otimes}}\,\bar{\mathbf{w}} + \bar{a}_1\, \mathbf{I}\otimes\mathbf{I} + \bar{a}_3\, \bar{\mathbf{w}}\otimes\bar{\mathbf{w}} + \bar{a}_5\, \bar{\mathbf{w}}^2\otimes\bar{\mathbf{w}}^2 \\ &\quad + \bar{a}_7\,(\bar{\mathbf{w}}\otimes\mathbf{I} + \mathbf{I}\otimes\bar{\mathbf{w}}) + \bar{a}_9\,(\bar{\mathbf{w}}^2\otimes\mathbf{I} + \mathbf{I}\otimes\bar{\mathbf{w}}^2)\,,\end{aligned} \tag{66}$$

provided that

$$\begin{aligned}a_1 &= \frac{1}{a_3}\;\frac{n^\star_{81}}{a_6{}^{\mathrm{w}}I_2 + a_5({}^{\mathrm{w}}I_1{}^{2\mathrm{w}}I_2 - 2{}^{\mathrm{w}}I_2{}^2 - 3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)}\;; \\ \bar{a}_1 &= \frac{1}{\bar{a}_3}\;\frac{\bar{n}^\star_{81}}{\bar{a}_6{}^{\bar{\mathrm{w}}}I_2 + \bar{a}_5({}^{\bar{\mathrm{w}}}I_1{}^{2\bar{\mathrm{w}}}I_2 - 2{}^{\bar{\mathrm{w}}}I_2{}^2 - 3{}^{\bar{\mathrm{w}}}I_1{}^{\bar{\mathrm{w}}}I_3)}\;,\end{aligned} \tag{67}$$

where

$$\begin{aligned}n^\star_{81} &= a_6^2a_7{}^{\mathrm{w}}I_3 + (a_3a_9^2 + a_5a_7^2)({}^{\mathrm{w}}I_1{}^{2\mathrm{w}}I_2 - 2{}^{\mathrm{w}}I_2{}^2 - 3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3) \\ &\quad + a_6\Big[a_7^{2\mathrm{w}}I_2 - a_3a_9{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3 + a_7\Big(3a_9 + a_5({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2)\Big){}^{\mathrm{w}}I_3\Big]\;; \\ \bar{n}^\star_{81} &= \bar{a}_6^2\bar{a}_7{}^{\bar{\mathrm{w}}}I_3 + (\bar{a}_3\bar{a}_9^2 + \bar{a}_5\bar{a}_7^2)({}^{\bar{\mathrm{w}}}I_1{}^{2\bar{\mathrm{w}}}I_2 - 2{}^{\bar{\mathrm{w}}}I_2{}^2 - 3{}^{\bar{\mathrm{w}}}I_1{}^{\bar{\mathrm{w}}}I_3) \\ &\quad + \bar{a}_6\Big[\bar{a}_7^{2\bar{\mathrm{w}}}I_2 - \bar{a}_3\bar{a}_9{}^{\bar{\mathrm{w}}}I_1{}^{\bar{\mathrm{w}}}I_3 + \bar{a}_7\Big(3\bar{a}_9 + \bar{a}_5({}^{\bar{\mathrm{w}}}I_1{}^2 - 2{}^{\bar{\mathrm{w}}}I_2)\Big){}^{\bar{\mathrm{w}}}I_3\Big]\;,\end{aligned} \tag{68}$$

with

$$\bar a_6 = \frac{1}{a_6}\;;\quad \bar a_1 = \frac{\bar n_{81}}{a_6\,\bar d_{83}\,\bar d_{85}}\;,\quad \bar a_3 = -a_3\,\frac{a_6+3a_9+a_5({}^{\mathrm w}I_1{}^2-2\,{}^{\mathrm w}I_2)}{a_6\,\bar d_{83}}\,{}^{\mathrm w}I_3\;,$$
$$\bar a_5 = -\frac{a_6a_7-a_3a_9\,{}^{\mathrm w}I_1+a_5a_7({}^{\mathrm w}I_1{}^2-2\,{}^{\mathrm w}I_2)}{a_6\,\bar d_{85}}\,{}^{\mathrm w}I_3{}^2\;, \tag{69}$$
$$\bar a_7 = a_3\,\frac{a_5\,{}^{\mathrm w}I_1\,{}^{\mathrm w}I_3+a_9\,{}^{\mathrm w}I_2}{a_6\,\bar d_{83}}\;,\quad \bar a_9 = -\frac{a_3a_9\,{}^{\mathrm w}I_2-3a_5a_7\,{}^{\mathrm w}I_3}{a_6\,\bar d_{85}}\,{}^{\mathrm w}I_3\,,$$

where

$$\begin{aligned}\bar d_{83} = {} & a_6^2\,{}^{\mathrm w}I_3+a_5a_7({}^{\mathrm w}I_1{}^2\,{}^{\mathrm w}I_2-2\,{}^{\mathrm w}I_2{}^2-3\,{}^{\mathrm w}I_1\,{}^{\mathrm w}I_3)+a_3\Big(2a_5({}^{\mathrm w}I_1{}^2-3\,{}^{\mathrm w}I_2)\,{}^{\mathrm w}I_3-a_9({}^{\mathrm w}I_1\,{}^{\mathrm w}I_2-9\,{}^{\mathrm w}I_3)\Big)\\ & +a_6\Big[a_7\,{}^{\mathrm w}I_2+\Big(3a_3+3a_9+a_5({}^{\mathrm w}I_1{}^2-2\,{}^{\mathrm w}I_2)\Big)\,{}^{\mathrm w}I_3\Big]\;,\end{aligned} \tag{70}$$

$$\begin{aligned}\bar d_{85} = {} & a_5a_7({}^{\mathrm w}I_1{}^2\,{}^{\mathrm w}I_2{}^2-2\,{}^{\mathrm w}I_2{}^3-2\,{}^{\mathrm w}I_1{}^3\,{}^{\mathrm w}I_3+4\,{}^{\mathrm w}I_1\,{}^{\mathrm w}I_2\,{}^{\mathrm w}I_3-9\,{}^{\mathrm w}I_3{}^2)+a_6\Big(a_3\,{}^{\mathrm w}I_2\,{}^{\mathrm w}I_3+a_7({}^{\mathrm w}I_2{}^2-2\,{}^{\mathrm w}I_1\,{}^{\mathrm w}I_3)\Big)\\ & +a_3\Big(a_5\,{}^{\mathrm w}I_3({}^{\mathrm w}I_1{}^2\,{}^{\mathrm w}I_2-2\,{}^{\mathrm w}I_2{}^2-3\,{}^{\mathrm w}I_1\,{}^{\mathrm w}I_3)-a_9({}^{\mathrm w}I_1\,{}^{\mathrm w}I_2{}^2-2\,{}^{\mathrm w}I_1{}^2\,{}^{\mathrm w}I_3-3\,{}^{\mathrm w}I_2\,{}^{\mathrm w}I_3)\Big)\;,\end{aligned} \tag{71}$$

$$\begin{aligned}\bar n_{81} = {} & a_3a_9^2\,{}^{\mathrm w}I_2\Big(2a_3({}^{\mathrm w}I_2{}^2-3\,{}^{\mathrm w}I_1\,{}^{\mathrm w}I_3)+a_6({}^{\mathrm w}I_2{}^2-2\,{}^{\mathrm w}I_1\,{}^{\mathrm w}I_3)\Big)\\ & -a_5^2\Big\{a_7\,{}^{\mathrm w}I_3\Big[a_6({}^{\mathrm w}I_1{}^2-2\,{}^{\mathrm w}I_2)({}^{\mathrm w}I_2{}^2-2\,{}^{\mathrm w}I_1\,{}^{\mathrm w}I_3)+2a_3\Big(2\,{}^{\mathrm w}I_2{}^2({}^{\mathrm w}I_1{}^2-2\,{}^{\mathrm w}I_2)-3\,{}^{\mathrm w}I_1\,{}^{\mathrm w}I_3({}^{\mathrm w}I_1{}^2-{}^{\mathrm w}I_2)\Big)\Big]\\ & \quad+\Big(a_3(3a_3+a_6)\,{}^{\mathrm w}I_3{}^2+a_7^2({}^{\mathrm w}I_2{}^2-2\,{}^{\mathrm w}I_1\,{}^{\mathrm w}I_3)\Big)({}^{\mathrm w}I_1{}^2\,{}^{\mathrm w}I_2-2\,{}^{\mathrm w}I_2{}^2-3\,{}^{\mathrm w}I_1\,{}^{\mathrm w}I_3)\Big\}\\ & -a_5\Big\{2a_3a_9(3a_7-a_3\,{}^{\mathrm w}I_1)({}^{\mathrm w}I_2{}^2-3\,{}^{\mathrm w}I_1\,{}^{\mathrm w}I_3)\,{}^{\mathrm w}I_3+a_6^2\,{}^{\mathrm w}I_3\Big(a_3\,{}^{\mathrm w}I_2\,{}^{\mathrm w}I_3+a_7({}^{\mathrm w}I_2{}^2-2\,{}^{\mathrm w}I_1\,{}^{\mathrm w}I_3)\Big)\\ & \quad+a_6\Big[a_7\,{}^{\mathrm w}I_3\Big((4a_3+3a_9)\,{}^{\mathrm w}I_2{}^2-6(a_3+a_9)\,{}^{\mathrm w}I_1\,{}^{\mathrm w}I_3\Big)+a_7^2\,{}^{\mathrm w}I_2({}^{\mathrm w}I_2{}^2-2\,{}^{\mathrm w}I_1\,{}^{\mathrm w}I_3)\\ & \qquad+a_3\,{}^{\mathrm w}I_3\Big(3a_3\,{}^{\mathrm w}I_2\,{}^{\mathrm w}I_3-a_9\,{}^{\mathrm w}I_1({}^{\mathrm w}I_2{}^2-2\,{}^{\mathrm w}I_1\,{}^{\mathrm w}I_3)\Big)\Big]\Big\}\;.\end{aligned} \tag{72}$$

Solution (6.5.8) is obtained as a particular case of *Solution (6.6)* by setting a_8=0, $\bar a_8$=0, which leads to the constraint (67).

Solution (6.5.9). Symmetric damage-effect tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ with the six-coefficients sets $(a_1, a_3, a_5, a_6, a_7, a_8)$ and $(\bar a_1, \bar a_3, \bar a_5, \bar a_6, \bar a_7, \bar a_8)$ (lacking only 'shear-like' coefficients a_2, a_4 and $\bar a_2, \bar a_4$, and 'non-shear' coefficients a_9 and $\bar a_9$) form the dual inverse pair:

$$\begin{aligned}\mathbb{A} = {} & a_6\,\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w}+a_1\,\mathbf{I}\otimes\mathbf{I}+a_3\,\mathbf{w}\otimes\mathbf{w}+a_5\,\mathbf{w}^2\otimes\mathbf{w}^2\\ & +a_7\,(\mathbf{w}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w})+a_8\,(\mathbf{w}^2\otimes\mathbf{w}+\mathbf{w}\otimes\mathbf{w}^2)\,;\\ \bar{\mathbb{A}} = {} & \bar a_6\,\bar{\mathbf{w}}\,\underline{\overline{\otimes}}\,\bar{\mathbf{w}}+\bar a_1\,\mathbf{I}\otimes\mathbf{I}+\bar a_3\,\bar{\mathbf{w}}\otimes\bar{\mathbf{w}}+\bar a_5\,\bar{\mathbf{w}}^2\otimes\bar{\mathbf{w}}^2\\ & +\bar a_7\,(\bar{\mathbf{w}}\otimes\mathbf{I}+\mathbf{I}\otimes\bar{\mathbf{w}})+\bar a_8\,(\bar{\mathbf{w}}^2\otimes\bar{\mathbf{w}}+\bar{\mathbf{w}}\otimes\bar{\mathbf{w}}^2)\,,\end{aligned} \tag{73}$$

provided that

$$
\begin{aligned}
a_3 &= \frac{a_1 a_6 a_8 {}^{\mathrm{w}}\!I_2 + (a_1 a_8^2 + a_5 a_7^2)({}^{\mathrm{w}}\!I_1 {}^{\mathrm{w}}\!I_2 - 9 {}^{\mathrm{w}}\!I_3) + a_6(3a_1 a_5 + 3a_7 a_8 + a_5 a_7 {}^{\mathrm{w}}\!I_1) {}^{\mathrm{w}}\!I_3}{a_1 a_5({}^{\mathrm{w}}\!I_1 {}^{\mathrm{w}}\!I_2 - 9 {}^{\mathrm{w}}\!I_3)} \; ; \\
\bar{a}_3 &= \frac{\bar{a}_1 \bar{a}_6 \bar{a}_8 {}^{\bar{\mathrm{w}}}\!I_2 + (\bar{a}_1 \bar{a}_8^2 + \bar{a}_5 \bar{a}_7^2)({}^{\bar{\mathrm{w}}}\!I_1 {}^{\bar{\mathrm{w}}}\!I_2 - 9 {}^{\bar{\mathrm{w}}}\!I_3) + \bar{a}_6(3\bar{a}_1 \bar{a}_5 + 3\bar{a}_7 \bar{a}_8 + \bar{a}_5 \bar{a}_7 {}^{\bar{\mathrm{w}}}\!I_1) {}^{\bar{\mathrm{w}}}\!I_3}{\bar{a}_1 \bar{a}_5({}^{\bar{\mathrm{w}}}\!I_1 {}^{\bar{\mathrm{w}}}\!I_2 - 9 {}^{\bar{\mathrm{w}}}\!I_3)} \; ,
\end{aligned}
\tag{74}
$$

with

$$
\begin{aligned}
&\bar{a}_6 = \frac{1}{a_6} \; ; \quad \bar{a}_1 = -a_5 \, \frac{a_1 {}^{\mathrm{w}}\!I_2 + 3a_7 {}^{\mathrm{w}}\!I_3}{a_6 \, \bar{d}_{91}} \, , \quad \bar{a}_3 = -\frac{\bar{n}_{93}}{a_6 \, \bar{d}_{91} \, \bar{d}_{95}} \, , \quad \bar{a}_5 = -a_1 \, \frac{3a_8 + a_5 {}^{\mathrm{w}}\!I_1}{a_6 \, \bar{d}_{95}} \, {}^{\mathrm{w}}\!I_3^{\,2} , \\
&\qquad\qquad \bar{a}_7 = a_5 \, \frac{3a_1 + a_7 {}^{\mathrm{w}}\!I_1}{a_6 \, \bar{d}_{91}} \, {}^{\mathrm{w}}\!I_3 \, , \qquad \bar{a}_8 = a_1 \, \frac{a_8 {}^{\mathrm{w}}\!I_2 + 3a_5 {}^{\mathrm{w}}\!I_3}{a_6 \, \bar{d}_{95}} \, {}^{\mathrm{w}}\!I_3 \, ,
\end{aligned}
\tag{75}
$$

where

$$
\begin{aligned}
\bar{d}_{91} = {} & a_7 \Big(3a_6 + 2a_5({}^{\mathrm{w}}\!I_1^{\,2} - 3 {}^{\mathrm{w}}\!I_2)\Big) {}^{\mathrm{w}}\!I_3 \\
& + a_1 \Big[\Big(a_6 + {}^{\mathrm{w}}\!I_1(a_8 + a_5 {}^{\mathrm{w}}\!I_1) - 2a_5 {}^{\mathrm{w}}\!I_2\Big) {}^{\mathrm{w}}\!I_2 - 3(3a_8 + a_5 {}^{\mathrm{w}}\!I_1) {}^{\mathrm{w}}\!I_3\Big] \, ,
\end{aligned}
\tag{76}
$$

$$
\begin{aligned}
\bar{d}_{95} = {} & \Big(3a_6 a_8 {}^{\mathrm{w}}\!I_3 + a_5(a_7 {}^{\mathrm{w}}\!I_1 {}^{\mathrm{w}}\!I_2 - 9a_7 {}^{\mathrm{w}}\!I_3 + a_6 {}^{\mathrm{w}}\!I_1 {}^{\mathrm{w}}\!I_3)\Big) {}^{\mathrm{w}}\!I_3 \\
& + a_1 \Big(2a_8({}^{\mathrm{w}}\!I_2^{\,2} - 3 {}^{\mathrm{w}}\!I_1 {}^{\mathrm{w}}\!I_3) + a_5({}^{\mathrm{w}}\!I_1 {}^{\mathrm{w}}\!I_2^{\,2} - 2 {}^{\mathrm{w}}\!I_1^{\,2} {}^{\mathrm{w}}\!I_3 - 3 {}^{\mathrm{w}}\!I_2 {}^{\mathrm{w}}\!I_3)\Big) \, ,
\end{aligned}
\tag{77}
$$

$$
\begin{aligned}
\bar{n}_{93} = {} & a_1 {}^{\mathrm{w}}\!I_3 \Big(a_7(3a_8 + a_5 {}^{\mathrm{w}}\!I_1)\Big(a_6 + a_5({}^{\mathrm{w}}\!I_1^{\,2} - 2 {}^{\mathrm{w}}\!I_2)\Big) {}^{\mathrm{w}}\!I_2^{\,2} \\
& \qquad - \Big\{ 2a_7(3a_8 + a_5 {}^{\mathrm{w}}\!I_1)(a_6 + a_5 {}^{\mathrm{w}}\!I_1^{\,2}) {}^{\mathrm{w}}\!I_1 \\
& \qquad\quad - \Big[2a_5 a_7(3a_8 + 2a_5 {}^{\mathrm{w}}\!I_1) {}^{\mathrm{w}}\!I_1 + a_6 a_8\Big(a_6 + (a_8 + a_5 {}^{\mathrm{w}}\!I_1) {}^{\mathrm{w}}\!I_1\Big)\Big] {}^{\mathrm{w}}\!I_2 + 2a_5 a_6 a_8 {}^{\mathrm{w}}\!I_2^{\,2} \Big\} {}^{\mathrm{w}}\!I_3 \\
& \qquad - 3\Big[3a_6 a_8^2 - a_5(a_6^2 + 9a_7 a_8) + a_5^2\Big(3a_7 {}^{\mathrm{w}}\!I_1 - a_6({}^{\mathrm{w}}\!I_1^{\,2} - 2 {}^{\mathrm{w}}\!I_2)\Big)\Big] {}^{\mathrm{w}}\!I_3^{\,2} \Big) \\
& + a_7\Big(a_6 + a_5({}^{\mathrm{w}}\!I_1^{\,2} - 2 {}^{\mathrm{w}}\!I_2)\Big)\Big[3a_6 a_8 {}^{\mathrm{w}}\!I_3 + a_5\Big(a_6 {}^{\mathrm{w}}\!I_1 {}^{\mathrm{w}}\!I_3 + a_7({}^{\mathrm{w}}\!I_1 {}^{\mathrm{w}}\!I_2 - 9 {}^{\mathrm{w}}\!I_3)\Big)\Big] {}^{\mathrm{w}}\!I_3^{\,2} \\
& + a_1^2\Big[\Big(a_6(a_8 {}^{\mathrm{w}}\!I_2 + 3a_5 {}^{\mathrm{w}}\!I_3) + a_8^2({}^{\mathrm{w}}\!I_1 {}^{\mathrm{w}}\!I_2 - 9 {}^{\mathrm{w}}\!I_3)\Big)({}^{\mathrm{w}}\!I_2^{\,2} - 2 {}^{\mathrm{w}}\!I_1 {}^{\mathrm{w}}\!I_3) \\
& \qquad + a_5(a_8 {}^{\mathrm{w}}\!I_2 + 3a_5 {}^{\mathrm{w}}\!I_3)({}^{\mathrm{w}}\!I_1^{\,2} {}^{\mathrm{w}}\!I_2^{\,2} - 2 {}^{\mathrm{w}}\!I_2^{\,3} - 2 {}^{\mathrm{w}}\!I_1^{\,3} {}^{\mathrm{w}}\!I_3 + 4 {}^{\mathrm{w}}\!I_1 {}^{\mathrm{w}}\!I_2 {}^{\mathrm{w}}\!I_3 - 9 {}^{\mathrm{w}}\!I_3^{\,2})\Big] \, .
\end{aligned}
\tag{78}
$$

Solution (6.5.9) is obtained as a particular case of *Solution (6.6)* by setting a_9=0, $\bar{a}_9$=0, which leads to the constraint (74). *Solutions (6.5.7), (6.5.8), (6.5.9)* display a similar degree of complexity, which is much lower than that shown by *Solutions (6.5.1), (6.5.3), (6.5.5)*. These, in turn, appear to be even more involved than source *Solution (6.6)*.

We consider now three solutions based on four 'non-shear' coefficients. The two missing coefficients are indicated by the last two label digits.

Solution (6.4.17). Symmetric damage-effect tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ with the five-coefficients sets $(a_3, a_5, a_6, a_8, a_9)$ and $(\bar{a}_3, \bar{a}_5, \bar{a}_6, \bar{a}_8, \bar{a}_9)$ (lacking only 'shear-like' coefficients a_2, a_4 and $\bar{a}_2, \bar{a}_4$, and 'non-shear' coefficients a_1, a_7 and $\bar{a}_1, \bar{a}_7$) form the dual inverse pair:

$$\begin{aligned}\mathbb{A} &= a_6\ \mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w} + a_3\ \mathbf{w}\otimes\mathbf{w} + a_5\ \mathbf{w}^2\otimes\mathbf{w}^2 + a_8\ (\mathbf{w}^2\otimes\mathbf{w}+\mathbf{w}\otimes\mathbf{w}^2) + a_9\ (\mathbf{w}^2\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w}^2);\\ \bar{\mathbb{A}} &= \bar{a}_6\ \bar{\mathbf{w}}\,\underline{\overline{\otimes}}\,\bar{\mathbf{w}} + \bar{a}_3\ \bar{\mathbf{w}}\otimes\bar{\mathbf{w}} + \bar{a}_5\ \bar{\mathbf{w}}^2\otimes\bar{\mathbf{w}}^2 + \bar{a}_8\ (\bar{\mathbf{w}}^2\otimes\bar{\mathbf{w}}+\bar{\mathbf{w}}\otimes\bar{\mathbf{w}}^2) + \bar{a}_9\ (\bar{\mathbf{w}}^2\otimes\mathbf{I}+\mathbf{I}\otimes\bar{\mathbf{w}}^2),\end{aligned} \tag{79}$$

provided that

$$\begin{aligned} a_3 &= \frac{a_6 a_8}{a_9}\,\frac{{}^{\mathrm{w}}I_3}{{}^{\mathrm{w}}I_2}\,, \qquad a_5 = a_9\,\frac{a_8\,{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3 + a_9\,({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)}{a_6\,{}^{\mathrm{w}}I_3{}^2}\,;\\ \bar{a}_3 &= \frac{\bar{a}_6 \bar{a}_8}{\bar{a}_9}\,\frac{{}^{\bar{\mathrm{w}}}I_3}{{}^{\bar{\mathrm{w}}}I_2}\,, \qquad \bar{a}_5 = \bar{a}_9\,\frac{\bar{a}_8\,{}^{\bar{\mathrm{w}}}I_2{}^{\bar{\mathrm{w}}}I_3 + \bar{a}_9\,({}^{\bar{\mathrm{w}}}I_2{}^2 - 2{}^{\bar{\mathrm{w}}}I_1{}^{\bar{\mathrm{w}}}I_3)}{\bar{a}_6\,{}^{\bar{\mathrm{w}}}I_3{}^2}\,,\end{aligned} \tag{80}$$

with

$$\begin{aligned} &\bar{a}_6 = \frac{1}{a_6};\quad \bar{a}_3 = -\frac{1}{a_6}\,\frac{a_8\,{}^{\mathrm{w}}I_3}{3a_8{}^{\mathrm{w}}I_3 + a_9{}^{\mathrm{w}}I_2},\quad \bar{a}_5 = a_9^2\,\frac{2a_8({}^{\mathrm{w}}I_1{}^2 - 3{}^{\mathrm{w}}I_2){}^{\mathrm{w}}I_3 + a_9({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2){}^{\mathrm{w}}I_2}{a_6\,(3a_9+a_6)^2\,(3a_8{}^{\mathrm{w}}I_3 + a_9{}^{\mathrm{w}}I_2)},\\ &\bar{a}_8 = \frac{a_8\,a_9{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3}{a_6\,(3a_9+a_6)\,(3a_8{}^{\mathrm{w}}I_3 + a_9{}^{\mathrm{w}}I_2)},\qquad \bar{a}_9 = -\frac{a_9}{a_6\,(3a_9+a_6)}.\end{aligned} \tag{81}$$

Solution (6.4.17) is obtained as a particular case of *Solution (6.6)* by setting $a_1{=}a_7{=}0$, $\bar{a}_1{=}\bar{a}_7{=}0$, which leads to the two constraints (80). If constraint (80a) is placed on a_9 instead on a_3, relation (81b) for $\bar{a}_3$ transforms to the typical inversion relation $\bar{a}_3{=}{-}a_3/[a_6(3a_3{+}a_6)]$ displayed by *Solution (6.1)*, Eq. (21b). Notice also the similar dual relation between a_9 and $\bar{a}_9$ in Eq. (81e).

Solution (6.4.58). Symmetric damage-effect tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ with the five-coefficients sets $(a_1, a_3, a_6, a_7, a_9)$ and $(\bar{a}_1, \bar{a}_3, \bar{a}_6, \bar{a}_7, \bar{a}_9)$ (lacking only 'shear-like' coefficients a_2, a_4 and $\bar{a}_2, \bar{a}_4$, and 'non-shear' coefficients a_5, a_8 and $\bar{a}_5, \bar{a}_8$) form the dual inverse pair:

$$\begin{aligned}\mathbb{A} &= a_6\ \mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w} + a_1\ \mathbf{I}\otimes\mathbf{I} + a_3\ \mathbf{w}\otimes\mathbf{w} + a_7\ (\mathbf{w}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w}) + a_9\ (\mathbf{w}^2\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w}^2)\,;\\ \bar{\mathbb{A}} &= \bar{a}_6\ \bar{\mathbf{w}}\,\underline{\overline{\otimes}}\,\bar{\mathbf{w}} + \bar{a}_1\ \mathbf{I}\otimes\mathbf{I} + \bar{a}_3\ \bar{\mathbf{w}}\otimes\bar{\mathbf{w}} + \bar{a}_7\ (\bar{\mathbf{w}}\otimes\mathbf{I}+\mathbf{I}\otimes\bar{\mathbf{w}}) + \bar{a}_9\ (\bar{\mathbf{w}}^2\otimes\mathbf{I}+\mathbf{I}\otimes\bar{\mathbf{w}}^2)\,,\end{aligned} \tag{82}$$

provided that

$$\begin{aligned} a_1 &= a_9\,\frac{a_7{}^{\mathrm{w}}I_1 + a_9({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2)}{a_6}\,, \qquad a_3 = \frac{a_6\,a_7}{a_9\,{}^{\mathrm{w}}I_1}\,;\\ \bar{a}_1 &= \bar{a}_9\,\frac{\bar{a}_7{}^{\bar{\mathrm{w}}}I_1 + \bar{a}_9({}^{\bar{\mathrm{w}}}I_1{}^2 - 2{}^{\bar{\mathrm{w}}}I_2)}{\bar{a}_6}\,, \qquad \bar{a}_3 = \frac{\bar{a}_6\,\bar{a}_7}{\bar{a}_9\,{}^{\bar{\mathrm{w}}}I_1}\,,\end{aligned} \tag{83}$$

with

$$\bar{a}_6 = \frac{1}{a_6}; \quad \bar{a}_1 = a_9^2 \, \frac{2a_7({}^{\mathrm{w}}I_2{}^2 - 3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3) + a_9{}^{\mathrm{w}}I_1({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)}{a_6\,(3a_9+a_6)^2\,(3a_7+a_9{}^{\mathrm{w}}I_1)\,{}^{\mathrm{w}}I_3{}^2}, \quad \bar{a}_3 = -\frac{a_7}{a_6\,(3a_7+a_9{}^{\mathrm{w}}I_1)},$$

$$\bar{a}_7 = \frac{a_7\,a_9\,{}^{\mathrm{w}}I_2}{a_6\,(3a_9+a_6)\,(3a_7+a_9{}^{\mathrm{w}}I_1)\,{}^{\mathrm{w}}I_3}, \qquad \bar{a}_9 = -\frac{a_9}{a_6\,(3a_9+a_6)}. \tag{84}$$

Solution (6.4.58) is obtained as a particular case of *Solution (6.6)* by setting a_5=a_8=0, $\bar{a}_5$=$\bar{a}_8$=0, which leads to the two constraints (83). Once again, if constraint (83b) is placed on a_7 instead on a_3, Eq. (84c) for $\bar{a}_3$ transforms to the typical inversion relation $\bar{a}_3$=$-a_3/[a_6(3a_3+a_6)]$ displayed by *Solution (6.1)*. The similar dual relation between a_9 and $\bar{a}_9$ also holds in Eq. (84e).

Solution (6.4.78). Symmetric damage-effect tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ with the five-coefficients sets $(a_1, a_3, a_5, a_6, a_9)$ and $(\bar{a}_1, \bar{a}_3, \bar{a}_5, \bar{a}_6, \bar{a}_9)$ (lacking only 'shear-like' coefficients a_2, a_4 and $\bar{a}_2, \bar{a}_4$, and 'non-shear' coefficients a_7, a_8 and $\bar{a}_7, \bar{a}_8$) form the dual inverse pair:

$$\begin{aligned}\mathbb{A} &= a_6\;\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w} + a_1\;\mathbf{I}\otimes\mathbf{I} + a_3\;\mathbf{w}\otimes\mathbf{w} + a_5\;\mathbf{w}^2\otimes\mathbf{w}^2 + a_9\;(\mathbf{w}^2\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{w}^2)\,;\\ \bar{\mathbb{A}} &= \bar{a}_6\;\bar{\mathbf{w}}\,\underline{\overline{\otimes}}\,\bar{\mathbf{w}} + \bar{a}_1\;\mathbf{I}\otimes\mathbf{I} + \bar{a}_3\;\bar{\mathbf{w}}\otimes\bar{\mathbf{w}} + \bar{a}_5\;\bar{\mathbf{w}}^2\otimes\bar{\mathbf{w}}^2 + \bar{a}_9\;(\bar{\mathbf{w}}^2\otimes\mathbf{I} + \mathbf{I}\otimes\bar{\mathbf{w}}^2)\,,\end{aligned} \tag{85}$$

provided that

$$a_1 = -a_9\,\frac{{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3}{{}^{\mathrm{w}}I_2}, \quad a_5 = -a_9\,\frac{{}^{\mathrm{w}}I_2}{{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3}; \qquad \bar{a}_1 = -\bar{a}_9\,\frac{{}^{\bar{\mathrm{w}}}I_1{}^{\bar{\mathrm{w}}}I_3}{{}^{\bar{\mathrm{w}}}I_2}, \quad \bar{a}_5 = -\bar{a}_9\,\frac{{}^{\bar{\mathrm{w}}}I_2}{{}^{\bar{\mathrm{w}}}I_1{}^{\bar{\mathrm{w}}}I_3}, \tag{86}$$

with

$$\bar{a}_6 = \frac{1}{a_6}; \quad \bar{a}_1 = \frac{a_9\,{}^{\mathrm{w}}I_2{}^2}{a_6\,\bar{d}_{78}}, \quad \bar{a}_3 = -\frac{a_3}{a_6\,(3a_3+a_6)}, \quad \bar{a}_5 = \frac{a_9\,{}^{\mathrm{w}}I_1{}^2\,{}^{\mathrm{w}}I_3{}^2}{a_6\,\bar{d}_{78}},$$

$$\bar{a}_9 = -\frac{a_9\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3}{a_6\,\bar{d}_{78}}, \tag{87}$$

where

$$\bar{d}_{78} = a_6\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3 - 2\,a_9\left({}^{\mathrm{w}}I_2{}^2\,({}^{\mathrm{w}}I_1{}^2 - {}^{\mathrm{w}}I_2) - {}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3\,({}^{\mathrm{w}}I_1{}^2 + 3\,{}^{\mathrm{w}}I_2)\right). \tag{88}$$

Solution (6.4.78) is obtained as a particular case of *Solution (6.6)* by setting a_7=a_8=0, $\bar{a}_7$=$\bar{a}_8$=0, which leads to the two constraints (86). Notice the typical inversion relation between a_3 and $\bar{a}_3$ in Eq. (87c).

A solution based on three 'non-shear' coefficients is considered next:

Solution (6.3). Symmetric damage-effect tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ with the four-coefficients sets (a_1, a_5, a_6, a_9) and $(\bar{a}_1, \bar{a}_5, \bar{a}_6, \bar{a}_9)$ (lacking 'shear-like' coefficients a_2, a_4 and $\bar{a}_2, \bar{a}_4$, and 'non-shear' coefficients a_3, a_7, a_8 and $\bar{a}_3, \bar{a}_7, \bar{a}_8$) form the dual inverse pair:

$$\begin{aligned} \mathbb{A} &= a_6 \, \mathbf{w} \underline{\overline{\otimes}} \mathbf{w} + a_1 \, \mathbf{I} \otimes \mathbf{I} + a_5 \, \mathbf{w}^2 \otimes \mathbf{w}^2 + a_9 \left(\mathbf{w}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{w}^2 \right); \\ \bar{\mathbb{A}} &= \bar{a}_6 \, \bar{\mathbf{w}} \underline{\overline{\otimes}} \bar{\mathbf{w}} + \bar{a}_1 \, \mathbf{I} \otimes \mathbf{I} + \bar{a}_5 \, \bar{\mathbf{w}}^2 \otimes \bar{\mathbf{w}}^2 + \bar{a}_9 \left(\bar{\mathbf{w}}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \bar{\mathbf{w}}^2 \right), \end{aligned} \tag{89}$$

with

$$\begin{aligned} \bar{a}_6 &= \frac{1}{a_6} \, ; \quad \bar{a}_1 = \frac{(a_9{}^2 - a_1 a_5)\, ({}^{\mathrm{w}}I_2{}^2 - 2\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3) - a_5 a_6\, {}^{\mathrm{w}}I_3{}^2}{a_6\, \bar{d}} \, , \\ \bar{a}_5 &= -\frac{a_1 a_6 - (a_9{}^2 - a_1 a_5)\, ({}^{\mathrm{w}}I_1{}^2 - 2\, {}^{\mathrm{w}}I_2)}{a_6\, \bar{d}}\, {}^{\mathrm{w}}I_3{}^2 \, , \quad \bar{a}_9 = -\frac{a_6 a_9 + 3(a_9{}^2 - a_1 a_5)}{a_6\, \bar{d}}\, {}^{\mathrm{w}}I_3{}^2 \, , \end{aligned} \tag{90}$$

where

$$\begin{aligned} \bar{d} = &\Big(a_6 (a_6 + 6 a_9) + 9 (a_9{}^2 - a_1 a_5) + a_5 a_6 \, ({}^{\mathrm{w}}I_1{}^2 - 2\, {}^{\mathrm{w}}I_2) \Big)\, {}^{\mathrm{w}}I_3{}^2 \\ &+ \Big(a_1 a_6 - (a_9{}^2 - a_1 a_5)\, ({}^{\mathrm{w}}I_1{}^2 - 2\, {}^{\mathrm{w}}I_2) \Big) \, ({}^{\mathrm{w}}I_2{}^2 - 2\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3) \, . \end{aligned} \tag{91}$$

Solution (6.3) is obtained from *Solution (6.6)* just by setting $a_3{=}a_7{=}a_8{=}0$, $\bar{a}_3{=}\bar{a}_7{=}\bar{a}_8{=}0$ and works *without constraints on the coefficients*.

Two solutions embedding the two 'non-shear' coefficients indicated in the last two label digits are now reported:

Solution (6.2.59). Another solution instance arises as a particular case of *Solution (6.3)* above by setting $a_1{=}0$, $\bar{a}_1{=}0$, namely by taking symmetric damage-effect tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ with only the three-coefficients sets (a_5, a_6, a_9) and $(\bar{a}_5, \bar{a}_6, \bar{a}_9)$ (lacking 'shear-like' coefficients a_2, a_4 and $\bar{a}_2, \bar{a}_4$, and 'non-shear' coefficients a_1, a_3, a_7, a_8 and $\bar{a}_1, \bar{a}_3, \bar{a}_7, \bar{a}_8$):

$$\begin{aligned} \mathbb{A} &= a_6 \, \mathbf{w} \underline{\overline{\otimes}} \mathbf{w} + a_5 \, \mathbf{w}^2 \otimes \mathbf{w}^2 + a_9 \left(\mathbf{w}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{w}^2 \right); \\ \bar{\mathbb{A}} &= \bar{a}_6 \, \bar{\mathbf{w}} \underline{\overline{\otimes}} \bar{\mathbf{w}} + \bar{a}_5 \, \bar{\mathbf{w}}^2 \otimes \bar{\mathbf{w}}^2 + \bar{a}_9 \left(\bar{\mathbf{w}}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \bar{\mathbf{w}}^2 \right), \end{aligned} \tag{92}$$

provided that

$$a_5 = \frac{a_9^2\, ({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)}{a_6\, {}^{\mathrm{w}}I_3{}^2} \, ; \qquad \bar{a}_5 = \frac{\bar{a}_9^2\, ({}^{\bar{\mathrm{w}}}I_2{}^2 - 2{}^{\bar{\mathrm{w}}}I_1{}^{\bar{\mathrm{w}}}I_3)}{\bar{a}_6\, {}^{\bar{\mathrm{w}}}I_3{}^2} \, , \tag{93}$$

with

$$\bar{a}_6 = \frac{1}{a_6} \, ; \quad \bar{a}_5 = \frac{a_9^2\, ({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2)}{a_6\, (3 a_9 + a_6)^2} \, , \quad \bar{a}_9 = -\frac{a_9}{a_6\, (3 a_9 + a_6)} \, . \tag{94}$$

Notice that the conditions $a_1{=}0$, $\bar{a}_1{=}0$ imposed into *Solution (6.3)* lead to the constraint (93).

Solution (6.2.19). A further solution instance which is an alternative particular case of *Solution (6.3)* is obtained by setting a_5=0, $\bar{a}_5$=0, namely by taking symmetric damage-effect tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ with only the three-coefficients sets (a_1, a_6, a_9) and $(\bar{a}_1, \bar{a}_6, \bar{a}_9)$ (lacking 'shear-like' coefficients a_2, a_4 and $\bar{a}_2, \bar{a}_4$, and 'non-shear' coefficients a_3, a_5, a_7, a_8 and $\bar{a}_3, \bar{a}_5, \bar{a}_7, \bar{a}_8$):

$$\begin{aligned} \mathbb{A} &= a_6\, \mathbf{w} \,\underline{\overline{\otimes}}\, \mathbf{w} + a_1\, \mathbf{I} \otimes \mathbf{I} + a_9\, (\mathbf{w}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{w}^2)\,; \\ \bar{\mathbb{A}} &= \bar{a}_6\, \bar{\mathbf{w}} \,\underline{\overline{\otimes}}\, \bar{\mathbf{w}} + \bar{a}_1\, \mathbf{I} \otimes \mathbf{I} + \bar{a}_9\, (\bar{\mathbf{w}}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \bar{\mathbf{w}}^2)\,, \end{aligned} \tag{95}$$

provided that

$$a_1 = \frac{a_9^2\,({}^{\mathrm{w}}I_1{}^2 - 2\,{}^{\mathrm{w}}I_2)}{a_6}\;; \qquad \bar{a}_1 = \frac{\bar{a}_9^2\,({}^{\bar{\mathrm{w}}}I_1{}^2 - 2\,{}^{\bar{\mathrm{w}}}I_2)}{\bar{a}_6}\,, \tag{96}$$

with

$$\bar{a}_6 = \frac{1}{a_6}\;; \quad \bar{a}_1 = \frac{a_9^2\,({}^{\mathrm{w}}I_2{}^2 - 2\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3)}{a_6\,(3a_9 + a_6)^2\;{}^{\mathrm{w}}I_3{}^2}\,, \qquad \bar{a}_9 = -\frac{a_9}{a_6\,(3a_9 + a_6)}\,. \tag{97}$$

Notice that the conditions a_5=0, $\bar{a}_5$=0 imposed into *Solution (6.3)* lead to the constraint (96). The presence of coefficients $a_9, \bar{a}_9$ is common to both *Solutions (6.2.59)* and *(6.2.19)*. Coefficients $a_9, \bar{a}_9$ correspond to each other with the same relation (94c) or (97c), which holds as well for *Solutions (6.4.17)* and *(6.4.58)*, Eqs (81e) and (84e), and, similarly, for non-symmetric *Solution (6.1ns)*, Eq. (23b). Such relation resembles closely the typical inversion relations between dual coefficients a_1 and $\bar{a}_1$ in the isotropic structure (7) of *Solution (2.1)*, see Eq. (8b), and between dual coefficients a_3 and $\bar{a}_3$ in *Solutions (6.1)*, *(6.4.78)*, see Eqs (21b), (87c), and in *Solutions (6.4.17), (6.4.58)*, see comments following Eqs (81), (84).

The remaining solution instances of the family that embed respectively only one and none of the 'non-shear' coefficients are already given in *Solutions (6.1), (6.0)*, Section 3. The various solution instances of the family based on 'shear-like' generators $a_6\, \mathbf{w} \,\underline{\overline{\otimes}}\, \mathbf{w}$, $\bar{a}_6\, \bar{\mathbf{w}} \,\underline{\overline{\otimes}}\, \bar{\mathbf{w}}$ are summarized in Table 1 below.

Solution	Damage-effect tensor	Coefficients set	Constraints
(6.6)	$\mathbb{A}=a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}+a_1\,\mathbf{I}\otimes\mathbf{I}+a_3\,\mathbf{w}\otimes\mathbf{w}+a_5\,\mathbf{w}^2\otimes\mathbf{w}^2+a_7\,(\mathbf{w}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w})+a_8\,(\mathbf{w}^2\otimes\mathbf{w}+\mathbf{w}\otimes\mathbf{w}^2)+a_9\,(\mathbf{w}^2\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w}^2)$	7: $(a_1,a_3,a_5,a_6,a_7,a_8,a_9)$	none
(6.5.1)	$\mathbb{A}=a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}+a_3\,\mathbf{w}\otimes\mathbf{w}+a_5\,\mathbf{w}^2\otimes\mathbf{w}^2+a_7\,(\mathbf{w}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w})+a_8\,(\mathbf{w}^2\otimes\mathbf{w}+\mathbf{w}\otimes\mathbf{w}^2)+a_9\,(\mathbf{w}^2\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w}^2)$	6: $(a_3,a_5,a_6,a_7,a_8,a_9)$	1 on a_3
(6.5.3)	$\mathbb{A}=a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}+a_1\,\mathbf{I}\otimes\mathbf{I}+a_5\,\mathbf{w}^2\otimes\mathbf{w}^2+a_7\,(\mathbf{w}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w})+a_8\,(\mathbf{w}^2\otimes\mathbf{w}+\mathbf{w}\otimes\mathbf{w}^2)+a_9\,(\mathbf{w}^2\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w}^2)$	6: $(a_1,a_5,a_6,a_7,a_8,a_9)$	1 on a_1
(6.5.5)	$\mathbb{A}=a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}+a_1\,\mathbf{I}\otimes\mathbf{I}+a_3\,\mathbf{w}\otimes\mathbf{w}+a_7\,(\mathbf{w}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w})+a_8\,(\mathbf{w}^2\otimes\mathbf{w}+\mathbf{w}\otimes\mathbf{w}^2)+a_9\,(\mathbf{w}^2\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w}^2)$	6: $(a_1,a_3,a_6,a_7,a_8,a_9)$	1 on a_3
(6.5.7)	$\mathbb{A}=a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}+a_1\,\mathbf{I}\otimes\mathbf{I}+a_3\,\mathbf{w}\otimes\mathbf{w}+a_5\,\mathbf{w}^2\otimes\mathbf{w}^2+a_8\,(\mathbf{w}^2\otimes\mathbf{w}+\mathbf{w}\otimes\mathbf{w}^2)+a_9\,(\mathbf{w}^2\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w}^2)$	6: $(a_1,a_3,a_5,a_6,a_8,a_9)$	1 on a_5
(6.5.8)	$\mathbb{A}=a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}+a_1\,\mathbf{I}\otimes\mathbf{I}+a_3\,\mathbf{w}\otimes\mathbf{w}+a_5\,\mathbf{w}^2\otimes\mathbf{w}^2+a_7\,(\mathbf{w}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w})+a_9\,(\mathbf{w}^2\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w}^2)$	6: $(a_1,a_3,a_5,a_6,a_7,a_9)$	1 on a_1
(6.5.9)	$\mathbb{A}=a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}+a_1\,\mathbf{I}\otimes\mathbf{I}+a_3\,\mathbf{w}\otimes\mathbf{w}+a_5\,\mathbf{w}^2\otimes\mathbf{w}^2+a_7\,(\mathbf{w}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w})+a_8\,(\mathbf{w}^2\otimes\mathbf{w}+\mathbf{w}\otimes\mathbf{w}^2)$	6: $(a_1,a_3,a_5,a_6,a_7,a_8)$	1 on a_3
(6.4.17)	$\mathbb{A}=a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}+a_3\,\mathbf{w}\otimes\mathbf{w}+a_5\,\mathbf{w}^2\otimes\mathbf{w}^2+a_8\,(\mathbf{w}^2\otimes\mathbf{w}+\mathbf{w}\otimes\mathbf{w}^2)+a_9\,(\mathbf{w}^2\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w}^2)$	5: (a_3,a_5,a_6,a_8,a_9)	2 on a_3,a_5
(6.4.58)	$\mathbb{A}=a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}+a_1\,\mathbf{I}\otimes\mathbf{I}+a_3\,\mathbf{w}\otimes\mathbf{w}+a_7\,(\mathbf{w}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w})+a_9\,(\mathbf{w}^2\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w}^2)$	5: (a_1,a_3,a_6,a_7,a_9)	2 on a_1,a_3
(6.4.78)	$\mathbb{A}=a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}+a_1\,\mathbf{I}\otimes\mathbf{I}+a_3\,\mathbf{w}\otimes\mathbf{w}+a_5\,\mathbf{w}^2\otimes\mathbf{w}^2+a_9\,(\mathbf{w}^2\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w}^2)$	5: (a_1,a_3,a_5,a_6,a_9)	2 on a_1,a_5
(6.3)	$\mathbb{A}=a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}+a_1\,\mathbf{I}\otimes\mathbf{I}+a_5\,\mathbf{w}^2\otimes\mathbf{w}^2+a_9\,(\mathbf{w}^2\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w}^2)$	4: (a_1,a_5,a_6,a_9)	none
(6.2.59)	$\mathbb{A}=a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}+a_5\,\mathbf{w}^2\otimes\mathbf{w}^2+a_9\,(\mathbf{w}^2\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w}^2)$	3: (a_5,a_6,a_9)	1 on a_5
(6.2.19)	$\mathbb{A}=a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}+a_1\,\mathbf{I}\otimes\mathbf{I}+a_9\,(\mathbf{w}^2\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w}^2)$	3: (a_1,a_6,a_9)	1 on a_1
(6.1)	$\mathbb{A}=a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}+a_3\,\mathbf{w}\otimes\mathbf{w}$	2: (a_3,a_6)	none
(6.0)	$\mathbb{A}=a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}$	1: (a_6)	none

Table 1: Solution instances of dual symmetric damage-effect tensors based on 'shear-like' generator $a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}$, *Solution (6.0)*, missing two 'shear-like' coefficients (a_2=a_4=0, $a_6\neq 0$). *Solution (6.6)* embeds all the six 'non-shear' coefficients a_1,a_3,a_5,a_7,a_8,a_9 and misses only the two 'shear-like' coefficients a_2,a_4 from the general orthotropic representation. It contains all the following solutions as particular cases, which are obtained by suppressing sets of the 'non-shear' coefficients. Six-, five- and three-coefficients solutions work through constraints on the coefficients. Four-coefficients *Solution (6.3)* and two-coefficients *Solution (6.1)* work without constraints on the coefficients and are obtained from *Solution (6.6)* just by setting a_3=a_7=a_8=0 and a_1=a_5=a_7=a_8=a_9=0, respectively.

5 Conclusions

A complete family of symmetric orthotropic fourth-order damage-effect tensors with dual structures has been derived in full invariant form. These instances complement those already presented in [18]. The solution family is based on 'shear-like' generators $a_6\, \mathbf{w} \overline{\underline{\otimes}} \mathbf{w}$, $\bar{a}_6\, \bar{\mathbf{w}} \overline{\underline{\otimes}} \bar{\mathbf{w}}$ and includes *fifteen* solution instances (*seven* of which were not presented previously: *Solutions (6.5.1), (6.5.3), (6.5.7), (6.5.8), (6.5.9), (6.4.17), (6.2.59)*), starting with more general *Solution (6.6)*, which includes all the six 'non-shear' coefficients. Most of the obtained solutions are particular cases of others and work *through constraints on the coefficients*. However, *Solutions (6.6), (6.3), (6.1), (6.0)* succeed in reaching the dual structure *without constraints on the coefficients*. The new general solution based on 'shear-like' generators $a_2\, \mathbf{I} \overline{\underline{\otimes}} \mathbf{I}$, $\bar{a}_2\, \mathbf{I} \overline{\underline{\otimes}} \mathbf{I}$ and including all the 'non-shear' terms (*Solution (2.6)*) is given as well in the paper in full invariant form. Particular instances of such case are available in [18].

The solution instances advanced here represent new candidate propositions of damage-effect tensors allowing for dual compliance- and stiffness-based derivations of the constitutive relations in orthotropic damage. These damage-effect tensors should lead to damaged compliance and stiffness embedding less restricted forms of orthotropic damage than that of Valanis-type, with a complexity that increases with the number of additional coefficients that are kept in the representation. The ultimate convenience of any of the damage-effect tensors advanced here in the final development and implementation of a constitutive model of orthotropic elastic damage remains to be validated on physical grounds and explored both analytically and numerically.

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References

[1] Betten, J., Damage tensors in continuum mechanics, *J. de Mécanique Théorique et Appliquée*, 2(1), 13–32, 1983.

[2] Betten, J., Chapters 11-14. Applications of Tensor Functions in Solid Mechanics, Ed. J.P. Boehler, CISM Advanced School, Springer-Verlag, Wien, pp. 203–299, 1987.

[3] Betten, J., Recent Advances in Applications of Tensor Functions in Continuum Mechanics, in *Advances in Applied Mechanics*, Eds. E. van der Giessen and Th.-Y. Wu, Vol. 37, Academic Press, New York, pp. 277–364, 2001.

[4] Boehler, J.P., Chapters 1-4. Applications of Tensor Functions in Solid Mechanics, Ed. J.P. Boehler, CISM Advanced School, Springer-Verlag, Wien, pp. 3–65, 1987.

[5] Carol, I., Rizzi, E. and Willam, K., A unified theory of elastic degradation and damage based on a loading surface, *Int. J. of Solids and Structures*, 31(20), 2835–2865, 1994.

[6] Carol, I., Rizzi, E. and Willam, K., On the formulation of anisotropic elastic degradation. Part I: Theory based on a pseudo-logarithmic damage tensor rate. Part II: Generalized pseudo-Rankine model for tensile damage, *Int. J. of Solids and Structures*, 38(4), 491–518, 519–546, 2001a,b.

[7] Carol, I., Rizzi, E. and Willam, K., An 'extended' volumetric/deviatoric formulation of anisotropic damage based on a pseudo-log rate, *European J. of Mechanics A/Solids*, in press, 2002.

[8] Cordebois, J.P. and Sidoroff, F., Endommagement anisotrope en élasticité et plasticité, *J. de Mécanique Théorique et Appliquée*, N. Special 1982, 45–60, 1982.

[9] Ju, J.W., Isotropic and anisotropic damage variables in continuum mechanics, *J. of Engineering Mechanics*, ASCE, 116(12), 2764–2770, 1990.

[10] Kachanov, L.M., Time rupture process under creep conditions (in Russian), *Izv. Akad. Nauk USSR Otd. Tekh. Nauk*, 8, 26–31, 1958.

[11] Krajcinovic, D., *Damage Mechanics*, North-Holland, Elsevier Science, Amsterdam, 1996.

[12] Lam, K.Y. and Zhang, J.M., On damage effect tensors of anisotropic solids, *Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM)*, 75(1), 51–59, 1995.

[13] Lemaitre, J., *A Course on Damage Mechanics*, Springer-Verlag, Berlin, 1992.

[14] Lu, T.J. and Chow, C.L., On constitutive equations of inelastic solids with anisotropic damage, *Theoretical and Applied Fracture Mechanics*, 14, 187–218, 1990.

[15] Murakami, S. and Ohno, N., A continuum theory of creep and creep damage, in *Creep in Structures*, Eds. A.R.S. Ponter and D.R. Hayhurst, Third IUTAM Symposium, Sept. 8-12, Leichester, UK, Springer-Verlag, New York, pp. 422–443, 1980.

[16] Rabotnov, Y.N., *Creep Problems in Structural Members*, North-Holland, Amsterdam, 1969.

[17] Rizzi, E. and Carol, I., A formulation of anisotropic elastic damage using compact tensor formalism, *J. of Elasticity*, 64(2–3), 85–109, 2001.

[18] Rizzi, E. and Carol, I., Dual orthotropic damage-effect tensors with complementary structures, *Int. J. of Engineering Science*, accepted, 2002.

[19] Skrzypek, J. and Ganczarski, A., *Modeling of Material Damage and Failure of Structures*, Springer–Verlag, Berlin, 1999.

[20] Ting, T.C.T., Determination of $\mathbf{C}^{1/2}$, $\mathbf{C}^{-1/2}$ and more general isotropic functions of $\mathbf{C}$, *J. of Elasticity*, 15, 319–323, 1985.

[21] Valanis, K.C., A theory of damage in brittle materials, *Engineering Fracture Mechanics*, 36, 403–416, 1990.

[22] Voyiadjis, G.Z. and Park, T., Anisotropic damage effect tensors for the symmetrization of the effective stress tensor, *J. of Applied Mechanics*, ASME, 64, 106–110, 1997.

[23] Zheng, Q.-S. and Betten, J., On damage effective stress and equivalence hypothesis, *Int. J. of Damage Mechanics*, 5, 219–240, 1996.

[24] Zysset, P.K. and Curnier, A., An alternative model for anisotropic elasticity based on fabric tensors, *Mechanics of Materials*, 21, 243–250, 1995.

[25] Walpole, L.J., Fourth-rank tensors of the thirty-two crystal classes: multiplication tables, *Proceedings of the Royal Society of London*, 391, 149–179, 1984.