Tensor formalism in the formulation of Elastic Degradation and Damage

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Dec/2001

Technical Report No. GT-022
ETSECCPB-UPC, E-08034 Barcelona, Spain

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Abstract. An anisotropic elastic-damage model for initially isotropic materials is presented. The model is based on a pseudo-logarithmic second-order damage tensor rate. To derive the complete form of the tangent stiffness, various tensor operations and derivatives of tensor functions must be developed. Such derivations have been performed in compact form. Some useful tensor derivatives and a table of tensor algebra operations are given in the Appendices. This note should interest engineering researchers involved in the development of constitutive models through tensor formalism.

Keywords: Continuum Damage Mechanics, anisotropic damage, second-order damage tensor, tensor algebra, tensor derivatives.

1. Introduction

While developing a constitutive model apt to describe anisotropic elastic stiffness degradation (or elastic-damage, in the context of Continuum Damage Mechanics, CDM) the authors encountered unexpected difficulties in developing analytically some of the terms involved in the formulation, namely the quantities that required various tensor operations and derivatives of tensor functions to be performed. In the original formulation of the model (Carol et al., 2001a,b) the authors managed to recover the final analytical form of the constitutive equations only for a specific simple type of elastic-damage based on an associative pseudo-Rankine loading function: the required derivatives were performed by resorting to spectral decomposition. However, the desire of more general compact form expressions of the constitutive terms and of the derivatives involved, as well as the need for appropriate tools apt to expand the various tensor operations, spontaneously arose.

Technical Report, UPC Barcelona, October 2001.

This note presents such compact form derivations and focuses on the required technical aspects of tensor formalism. Its purpose is twofold: first, the general explicit expressions of the constitutive equations that are derived with the present approach enhance and complete the formulation of the proposed constitutive model; second, though the adequate sources of information concerning tensor operations and derivatives of tensor functions can be found in the literature, the rather involved assembly of these tools should interest by itself the community of elasticians, and, more generally, constitutive laws developers.

The constitutive framework for elastic degradation and damage at small strains developed by I. Carol and coworkers (Carol et al., 1994, 1998, 2001a,b,c) is briefly summarized first. This model accounts for a selected form of orthotropic damage of initially isotropic materials based on a second-order pseudo-logarithmic damage tensor rate. The derivation of all the terms involved in the fourth-order tangent stiffness tensor which defines the constitutive response in terms of rates is then performed. Compact form expressions are obtained, which are valid generally for any specific model belonging to the class of constitutive laws considered here. Notation details, some useful derivatives and a table of tensor algebra operations are collected in the Appendices.

Notation in brief: Compact or index tensor notation is used throughout. Vectors and second-order tensors are identified by boldface characters, whereas fourth-order tensors are denoted by blackboard-bold fonts. Superscript $^{\mathrm{T}}$ indicates the transpose operation, while tr is the trace operator. Symbols '$\cdot$' and ':' denote the inner products with single and double contraction. The dyadic product is indicated with '$\otimes$', whereas '$\underline{\overline{\otimes}}$' denotes the symmetrized outer product. $\mathbf{I}$ and $\mathbb{I}^s = \mathbf{I} \underline{\overline{\otimes}} \mathbf{I}$ are the second-order and symmetric fourth-order identity tensors.

2. Constitutive equations of elastic-damage

In Carol et al. (1994) a general theoretical framework for elastic degradation and damage based on a loading surface was provided; such constitutive framework has then been complemented by the introduction of a pseudo-logarithmic second-order damage tensor rate (Carol et al., 2001a,b), while an 'extended' formulation of such model based on volumetric/deviatoric decomposition has been recently completed (Carol et al., 2001c); an earlier but preliminary account of these latest developments is available in Carol et al. (1998). In this section, the basics of this constitutive framework are presented.

2.1. Elastic degradation and tangent stiffness

The constitutive behavior under consideration is fully characterized by the following secant relations (Figure 1):

$$\boldsymbol{\sigma} = \mathbb{E} : \boldsymbol{\epsilon} \; ; \qquad \boldsymbol{\epsilon} = \mathbb{C} : \boldsymbol{\sigma} \; , \tag{1}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$ are the nominal stress and (infinitesimal) strain tensors, and $\mathbb{E}$ and $\mathbb{C}$ are the current secant stiffness and compliance, respectively. Fourth-order tensors $\mathbb{E}$ and $\mathbb{C}$ are assumed to be constant on each unloading/reloading branch, fully symmetric (minor and major symmetries) and Positive Definite (PD). They may be derived from a positive definite quadratic strain (complementary) energy function, $u{=}\boldsymbol{\epsilon}{:}\mathbb{E}{:}\boldsymbol{\epsilon}/2{=}\boldsymbol{\sigma}{:}\mathbb{C}{:}\boldsymbol{\sigma}/2$. Thus tensors $\mathbb{E}$ and $\mathbb{C}$ are invertible, and are the inverse of each other: $\mathbb{E}{:}\mathbb{C}{=}\mathbb{C}{:}\mathbb{E}{=}\mathbb{I}^s$; their initial values at the beginning of the loading process (undamaged material) are denoted by $\mathbb{E}_0$ and $\mathbb{C}_0$. Unloading always leads to the origin of the stress/strain curve, so that no plastic (irreversible) strains are present. However, energy dissipation takes place in the form of stiffness degradation or compliance increase, which are related by $\dot{\mathbb{E}}{=}{-}\mathbb{E}{:}\dot{\mathbb{C}}{:}\mathbb{E}$; $\dot{\mathbb{C}}{=}{-}\mathbb{C}{:}\dot{\mathbb{E}}{:}\mathbb{C}$. Differentiation of eqn $(1)_b$ with respect to time yields the following strain rate decomposition:

$$\dot{\boldsymbol{\epsilon}} = \mathbb{C} : \dot{\boldsymbol{\sigma}} + \dot{\mathbb{C}} : \boldsymbol{\sigma} = \dot{\boldsymbol{\epsilon}}_e + \dot{\boldsymbol{\epsilon}}_d \; , \tag{2}$$

where $\dot{\boldsymbol{\epsilon}}_e{=}\mathbb{C}{:}\dot{\boldsymbol{\sigma}}$ and $\dot{\boldsymbol{\epsilon}}_d{=}\dot{\mathbb{C}}{:}\boldsymbol{\sigma}$ are the elastic and degrading strain rates as depicted in Figure 1 (Dougill, 1976).

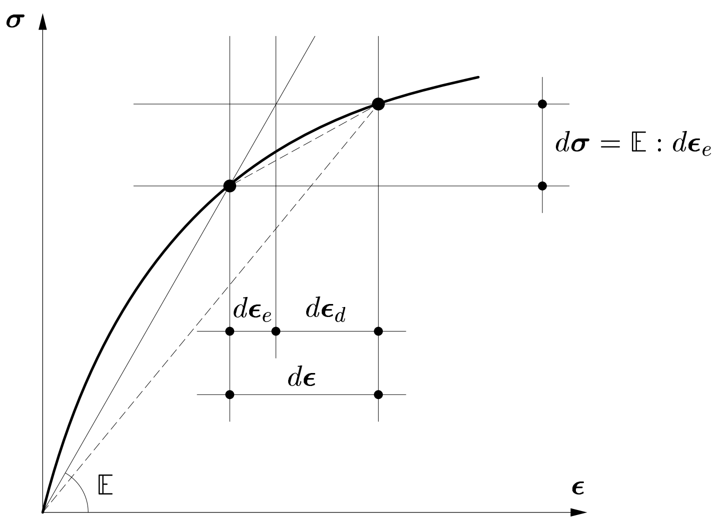

Figure 1. Stress/strain curve and current secant stiffness. Definition of the elastic and degrading strain increments.

Stiffness degradation (compliance increase) is then brought to strain level and, by introducing a loading function in stress space, $F(\boldsymbol{\sigma})$, which defines the current amplitude of the domain where no further elastic degradation takes place, $F(\boldsymbol{\sigma})<0$, eqn $(1)_a$ can be recast in the following plastic-type rate equations:

$$\begin{aligned}
&\dot{\boldsymbol{\sigma}} = \mathbb{E} : (\dot{\boldsymbol{\epsilon}} - \dot{\boldsymbol{\epsilon}}_d) \ ; \\
&\dot{\boldsymbol{\epsilon}}_d = \dot{\lambda}\, \mathbf{m} \ ; \\
&\dot{F} = \mathbf{n} : \dot{\boldsymbol{\sigma}} - H\, \dot{\lambda} \ , \quad \mathbf{n} = \left.\frac{\partial F}{\partial \boldsymbol{\sigma}}\right|_{\lambda} , \quad H = -\left.\frac{\partial F}{\partial \lambda}\right|_{\boldsymbol{\sigma}} ,
\end{aligned} \tag{3}$$

where $\dot{\lambda}$ is the inelastic multiplier, $\mathbf{m}$ the direction of the degrading strain rate, generally distinct from $\mathbf{n}$, that is the local normal to the loading surface (for associative degradation rules $\mathbf{m}$ is proportional or equal to $\mathbf{n}$), and H is the hardening/softening parameter. The possible activation of further elastic degradation is governed by the complementarity conditions $F\leq 0$, $\dot{\lambda}\geq 0$, $F\,\dot{\lambda}=0$, while the occurrence of further loading in the inelastic range is ruled by the consistency conditions $\dot{F}\leq 0$, $\dot{F}\,\dot{\lambda}=0$. In case of further elastic degradation, $\dot{\lambda}>0$, the consistency condition $\dot{F}=0$ is solved for $\dot{\lambda}$, which in turn is back-substituted into eqns $(3)_{1,2}$ to give the following rate relation:

$$\begin{aligned}
&\dot{\boldsymbol{\sigma}} = \mathbb{E}_t : \dot{\boldsymbol{\epsilon}} \ ; \\
&\mathbb{E}_t = \mathbb{E} - \frac{\mathbb{E} : \mathbf{m} \otimes \mathbf{n} : \mathbb{E}}{\bar{H}} \ , \qquad \bar{H} = H + \mathbf{n} : \mathbb{E} : \mathbf{m} \ ,
\end{aligned} \tag{4}$$

where $\mathbb{E}_t$ is the fourth-order tangent stiffness tensor of the rate constitutive law $(4)_1$ and the hardening parameter $\bar{H}$ is assumed to be strictly positive.

2.2. Damage variables and CDM concepts

Comparing eqns (2) and $(3)_2$ it appears clear that tensor $\mathbf{m}$ cannot be specified independently, but should derive from the compliance evolution law. By formulating such evolution rule in a form dual to $(3)_2$ (Maier and Hueckel, 1979; Ortiz, 1985) we get:

$$\dot{\mathbb{C}} = \dot{\lambda}\, \mathbb{M} \ ; \qquad \mathbf{m} = \mathbb{M} : \boldsymbol{\sigma} \ , \tag{5}$$

where the PD (fully symmetric) fourth-order tensor $\mathbb{M}$ defines the direction of compliance increase. In the general anisotropic case, eqn (5) requires 21 independent parameters to be specified. However, in view of reducing the number of material parameters, a generic *second-order damage tensor variable* $\boldsymbol{\mathcal{D}}$ is introduced:

$$\mathbb{C} = \mathbb{C}(\mathbb{C}_0, \boldsymbol{\mathcal{D}}) \ ; \qquad \dot{\boldsymbol{\mathcal{D}}} = \dot{\lambda}\, \boldsymbol{\mathcal{M}} \ , \tag{6}$$

where $\boldsymbol{\mathcal{M}}$ is the second-order tensor defining the damage rate direction. Eqn (6), through eqn (5), allows then to define in cascade both tensors $\mathbb{M}$ and $\mathbf{m}$:

$$\mathbf{m} = \mathbb{M} : \boldsymbol{\sigma} = \left(\frac{\partial \mathbb{C}}{\partial \boldsymbol{\mathcal{D}}} : \boldsymbol{\mathcal{M}} \right) : \boldsymbol{\sigma} \, . \tag{7}$$

As shown by Cowin (1985), the highest type of anisotropic elasticity than can be generally described through the use of a second-order fabric tensor is orthotropy. In the present case, due to the assumptions that will be made to characterize eqn $(6)_a$, only a restricted form of orthotropic damage will be described. In CDM the damage-state relation $(6)_a$ is usually derived indirectly through a series of steps (see the schematic representation sketched in Figure 2 and e.g. Carol et al., 2001a, with references quoted therein). A constitutive law for the undamaged material is first introduced, which is expressed in terms of effective stress and strain quantities, $\boldsymbol{\sigma}_{\text{eff}}$ and $\boldsymbol{\epsilon}_{\text{eff}}$:

$$\boldsymbol{\sigma}_{\text{eff}} = \mathbb{E}_0 : \boldsymbol{\epsilon}_{\text{eff}} \; ; \qquad \boldsymbol{\epsilon}_{\text{eff}} = \mathbb{C}_0 : \boldsymbol{\sigma}_{\text{eff}} \, . \tag{8}$$

Then, one of the relations between nominal and effective (stress or strain) quantities is assumed, often in linear form, by introducing a non-singular fourth-order damage-effect tensor $\mathbb{A}$, e.g. $\boldsymbol{\sigma}_{\text{eff}}=\mathbb{A}:\boldsymbol{\sigma}$ (see also Lam and Zhang, 1995; Zheng and Betten, 1996; Voyiadjis and Park, 1997), together with a second relation expressed through a so-called equivalence principle (strain equivalence if $\boldsymbol{\epsilon}=\boldsymbol{\epsilon}_{\text{eff}}$, stress equivalence if $\boldsymbol{\sigma}=\boldsymbol{\sigma}_{\text{eff}}$, or energy equivalence if $\boldsymbol{\sigma}:\boldsymbol{\epsilon}/2=\boldsymbol{\sigma}_{\text{eff}}:\boldsymbol{\epsilon}_{\text{eff}}/2$).

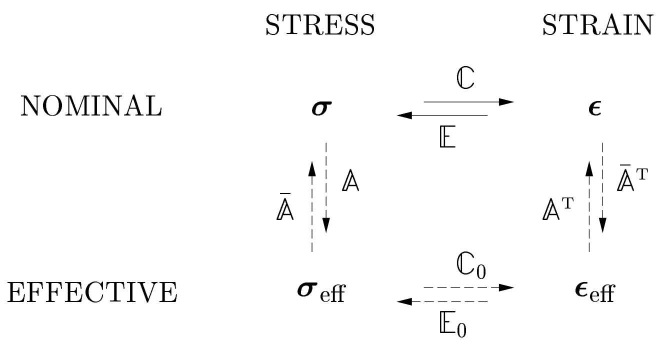

Figure 2. Relations between effective (undamaged) and nominal (damaged) secant stress/strain laws, and between nominal and effective stress/strain measures: schematic representation.

The *energy equivalence* approach (Cordebois and Sidoroff, 1982) is adopted here, since it allows to derive secant stiffness and compliance which automatically embed the required major symmetry. The elastic energy at any time of the degradation process is then expressed by any of the following forms:

$$\begin{aligned} u &= \frac{1}{2}\,\boldsymbol{\sigma} \,:\boldsymbol{\epsilon} = \frac{1}{2}\,\boldsymbol{\epsilon} \,:\boldsymbol{\sigma} = \frac{1}{2}\,\boldsymbol{\sigma} \,:\mathbb{C}\,:\boldsymbol{\sigma} = \frac{1}{2}\,\boldsymbol{\epsilon} \,:\mathbb{E}\,:\boldsymbol{\epsilon} \\ &= \frac{1}{2}\,\boldsymbol{\sigma}_{\text{eff}}:\boldsymbol{\epsilon}_{\text{eff}} = \frac{1}{2}\,\boldsymbol{\epsilon}_{\text{eff}}:\boldsymbol{\sigma}_{\text{eff}} = \frac{1}{2}\,\boldsymbol{\sigma}_{\text{eff}}:\mathbb{C}_0:\boldsymbol{\sigma}_{\text{eff}} = \frac{1}{2}\,\boldsymbol{\epsilon}_{\text{eff}}:\mathbb{E}_0:\boldsymbol{\epsilon}_{\text{eff}}\;. \end{aligned} \tag{9}$$

The following relations are then consistently assumed/obtained:

$$\boldsymbol{\sigma}_{\text{eff}} = \mathbb{A} : \boldsymbol{\sigma}\;; \qquad \boldsymbol{\epsilon} = \mathbb{A}^{\mathrm{T}} : \boldsymbol{\epsilon}_{\text{eff}}\;, \tag{10}$$

or

$$\boldsymbol{\epsilon}_{\text{eff}} = \bar{\mathbb{A}}^{\mathrm{T}} : \boldsymbol{\epsilon}\;; \qquad \boldsymbol{\sigma} = \bar{\mathbb{A}} : \boldsymbol{\sigma}_{\text{eff}}\;, \tag{11}$$

and

$$\mathbb{E} = \bar{\mathbb{A}} : \mathbb{E}_0 : \bar{\mathbb{A}}^{\mathrm{T}}\;; \qquad \mathbb{C} = \mathbb{A}^{\mathrm{T}} : \mathbb{C}_0 : \mathbb{A}\;, \tag{12}$$

where $\bar{\mathbb{A}}$ and $\mathbb{A}$ are dual quantities (meaning and discussion on dual quantities are available in Carol et al., 1994), inverse of each other, that is $\mathbb{A}{:}\bar{\mathbb{A}}{=}\bar{\mathbb{A}}{:}\mathbb{A}{=}\mathbb{I}^s$, and endowed with minor, but not necessarily major, symmetries (Carol et al., 2001a).

The damage-effect tensors $\mathbb{A}$, $\bar{\mathbb{A}}$ are expressed in terms of the damage variables $\boldsymbol{\mathcal{D}}$ adopted in the model. In our case we use PD symmetric *second-order tensor variables*: the so-called integrity tensor $\bar{\boldsymbol{\phi}}$ (Valanis, 1990) and its square root $\bar{\mathbf{w}}{=}\bar{\boldsymbol{\phi}}^{1/2}$, both varying between $\mathbf{I}$ and $\mathbf{0}$, or their inverses $\boldsymbol{\phi}{=}\bar{\boldsymbol{\phi}}^{-1}$, $\mathbf{w}{=}\bar{\mathbf{w}}^{-1}{=}\boldsymbol{\phi}^{1/2}$, varying from $\mathbf{I}$ to ∞. To obtain product-type symmetrization (Cordebois and Sidoroff, 1982) we adopt the following (fully symmetric) *damage-effect tensors*:

$$\mathbb{A} = \mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w} = \bar{\mathbb{A}}^{-1}\;; \qquad \bar{\mathbb{A}} = \bar{\mathbf{w}}\,\underline{\overline{\otimes}}\,\bar{\mathbf{w}} = \mathbb{A}^{-1}\;. \tag{13}$$

The fact that $\mathbb{A}$ and $\bar{\mathbb{A}}$ are indeed inverse of each other may be read through property $(\mathrm{B.3})_b$, Appendix B, and can be checked immediately by using the tensor contraction $(\mathrm{C.6})_2$, Appendix C, with $\mathbf{w}{\cdot}\bar{\mathbf{w}}{=}\mathbf{I}$. This is the reason why, in contrast to other approaches such as sum-type symmetrization, stress- and strain-based derivations of the damage model are fully equivalent (Carol et al., 2001a). From eqns (10), (11) and (13), the following product-type nominal/effective relations are then obtained:

$$\boldsymbol{\sigma}_{\text{eff}} = \mathbf{w}{\cdot}\boldsymbol{\sigma}{\cdot}\mathbf{w}\;,\;\; \boldsymbol{\epsilon} = \mathbf{w}{\cdot}\boldsymbol{\epsilon}_{\text{eff}}{\cdot}\mathbf{w}\;; \qquad \boldsymbol{\sigma} = \bar{\mathbf{w}}{\cdot}\boldsymbol{\sigma}_{\text{eff}}{\cdot}\bar{\mathbf{w}}\;,\;\; \boldsymbol{\epsilon}_{\text{eff}} = \bar{\mathbf{w}}{\cdot}\boldsymbol{\epsilon}{\cdot}\bar{\mathbf{w}}\;. \tag{14}$$

Before arriving at the expressions of the secant stiffness and compliance in eqn (12), the undamaged behavior has to be prescribed. In the present context we deal only with *initially isotropic materials*, namely, in terms of undamaged Lamé's constant Λ_0 and shear modulus G_0, and Young's modulus E_0 and Poisson's ratio ν_0:

$$\mathbb{E}_0 = \Lambda_0 \, \mathbf{I} \otimes \mathbf{I} + 2\, G_0 \, \mathbf{I} \,\underline{\overline{\otimes}}\, \mathbf{I} \, ; \quad \mathbb{C}_0 = -\frac{\nu_0}{E_0} \, \mathbf{I} \otimes \mathbf{I} + \frac{1+\nu_0}{E_0} \, \mathbf{I} \,\underline{\overline{\otimes}}\, \mathbf{I} \, , \quad (15)$$

where the two sets of isotropic undamaged parameters are linked by the classical relations

$$\Lambda_0 = \frac{\nu_0 \, E_0}{(1+\nu_0)\,(1-2\nu_0)} \, , \qquad G_0 = \frac{E_0}{2(1+\nu_0)} \, ;$$
$$E_0 = G_0 \, \frac{3\Lambda_0 + 2\, G_0}{\Lambda_0 + G_0} \, , \qquad \nu_0 = \frac{\Lambda_0}{2(\Lambda_0 + G_0)} \, . \quad (16)$$

Then, by developing the inner products in eqn (12) (see Appendix C), from eqns (13) and (15) we recover the following simple forms of secant stiffness and compliance which were proposed directly by Valanis (1990) and have been reconsidered more recently by Zysset and Curnier (1995); the second-order identity tensor $\mathbf{I}$ is simply replaced either by the integrity tensor $\bar{\boldsymbol{\phi}}$ in the stiffness form, or by its inverse $\boldsymbol{\phi}$ in the compliance form:

$$\mathbb{E} = \Lambda_0 \, \bar{\boldsymbol{\phi}} \otimes \bar{\boldsymbol{\phi}} + 2\, G_0 \, \bar{\boldsymbol{\phi}} \,\underline{\overline{\otimes}}\, \bar{\boldsymbol{\phi}} \, ; \quad \mathbb{C} = -\frac{\nu_0}{E_0} \, \boldsymbol{\phi} \otimes \boldsymbol{\phi} + \frac{1+\nu_0}{E_0} \, \boldsymbol{\phi} \,\underline{\overline{\otimes}}\, \boldsymbol{\phi} \, . \quad (17)$$

Positive definiteness of $\mathbb{E}$ and $\mathbb{C}$ is guaranteed by isotropic undamaged elastic constants belonging to the usual range providing $\mathbb{E}_0$ and $\mathbb{C}_0$ PD, namely $\Lambda_0 > -2/3\, G_0$, $G_0 > 0$ and $E_0 > 0$, $-1 < \nu_0 < 1/2$. The selected form of orthotropy is characterized by only 5 parameters, that is two undamaged isotropic elastic constants and the three principal values of $\boldsymbol{\phi}$ or $\bar{\boldsymbol{\phi}}$. In the principal axes of damage, the 9 orthotropic material parameters may be expressed in terms of Young's and shear moduli and Poisson's ratios, which take then the simple forms

$$E_I = \bar{\phi}_I^2 \, E_0 \, , \qquad I = 1,2,3 \, ;$$
$$G_{IJ} = \bar{\phi}_I \bar{\phi}_J \, G_0 \, , \quad I, J = 1,2 \, ; \; 2,3 \, ; \; 3,1 \, ; \quad (18)$$
$$\nu_{IJ} = \frac{\bar{\phi}_I}{\bar{\phi}_J} \, \nu_0 \, , \qquad J \neq I = 1,2,3 \, .$$

In the previously-mentioned 'extended' formulation of the model (Carol et al., 2001c) expressions (17) and (18) are further generalized by introducing an additional degree of freedom (still obtaining however only a restricted form of orthotropic damage counting on 6 parameters instead of 9).

2.3. Energy dissipation and pseudo-logarithmic damage rate

For isothermal processes, the (non-negative) rate of energy dissipation induced by the process of elastic degradation may be expressed as the difference between the stress power supplied to the material per unit volume, $\boldsymbol{\sigma}:\dot{\boldsymbol{\epsilon}}$, and the accumulation rate of (recoverable) elastic energy, $\dot{u}$. By assuming that $u=u(\boldsymbol{\sigma},\boldsymbol{\mathcal{D}})$, from eqn $(9)_1$ the dissipation rate may be rephrased in terms of the so-called thermodynamic force $-\boldsymbol{\mathcal{Y}}^{\mathcal{D}}$ conjugate to $\dot{\boldsymbol{\mathcal{D}}}$:

$$\dot{d} = \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} - \dot{u} = -\boldsymbol{\mathcal{Y}}^{\mathcal{D}} : \dot{\boldsymbol{\mathcal{D}}} \geq 0 \; ; \qquad -\boldsymbol{\mathcal{Y}}^{\mathcal{D}} = \left. \frac{\partial u}{\partial \boldsymbol{\mathcal{D}}} \right|_{\boldsymbol{\sigma}} , \tag{19}$$

where, following previous CDM literature, the symbol representing the conjugate force embeds a minus sign to reflect the fact that a positive dissipation rate at increasing damage actually corresponds to a decrease of elastic energy.

Now, if either of the four second-order damage tensors $\boldsymbol{\phi}$, $\bar{\boldsymbol{\phi}}$, or $\mathbf{w}$, $\bar{\mathbf{w}}$ introduced in the previous section is chosen as the underlying damage variable $\boldsymbol{\mathcal{D}}$, the resulting conjugate force acquires rather involved expressions, which also lack any clear physical meaning. The same happens for other alternative damage tensors usually considered, such as higher or fractional powers, logarithm, or other functions of $\boldsymbol{\phi}$, $\bar{\boldsymbol{\phi}}$, or $\mathbf{w}$, $\bar{\mathbf{w}}$. Although some of these choices have been followed in the past, the corresponding non-physical conjugate force makes it difficult to propose loading functions and evolution laws with desired specific features. A solution to this problem has been recently advanced by Carol et al. (2001a), by introducing the following *pseudo-logarithmic damage rate* $\dot{\mathbf{L}}$,

$$\dot{\mathbf{L}} = 2\, \bar{\mathbb{A}} : \dot{\boldsymbol{\phi}} = 2\, \bar{\mathbf{w}} \cdot \dot{\boldsymbol{\phi}} \cdot \bar{\mathbf{w}} \; ; \qquad \dot{\boldsymbol{\phi}} = \frac{1}{2}\, \mathbb{A} : \dot{\mathbf{L}} = \frac{1}{2}\, \mathbf{w} \cdot \dot{\mathbf{L}} \cdot \mathbf{w} \, , \tag{20}$$

and considering this as the rate of the damage variable, $\dot{\boldsymbol{\mathcal{D}}}=\dot{\mathbf{L}}$. By doing so, the thermodynamic force $-\boldsymbol{\mathcal{Y}}=-\boldsymbol{\mathcal{Y}}^{\mathbf{L}}$ conjugate to $\dot{\mathbf{L}}$ becomes just as simple as (see derivation below):

$$-\boldsymbol{\mathcal{Y}} = \frac{1}{2}\, \boldsymbol{\sigma}_{\text{eff}} \cdot \boldsymbol{\epsilon}_{\text{eff}} \, . \tag{21}$$

As a counterpart to this advantage, such choice also exhibits some minor complexities. Although having the simple expression (20), the rate $\dot{\mathbf{L}}$ does not represent in general an exact differential, i.e. the total value of $\mathbf{L}$ is not defined uniquely, and therefore such total variable $\mathbf{L}$ does not exist. This is however not a serious difficulty, since $\dot{\mathbf{L}}$ is directly related to $\dot{\boldsymbol{\phi}}$, eqn $(20)_b$, which is indeed an exact differential and can be integrated. Therefore, while damage rates and evolution laws can be conceived in terms of $\dot{\mathbf{L}}$, in real implementation the total damage values can be updated and stored in terms of $\boldsymbol{\phi}$ or $\bar{\boldsymbol{\phi}}$, which directly enter the expression of the secant stiffness or compliance, $\mathbb{E}$ or $\mathbb{C}$, eqn (17). As a matter of analogy, the relation between $\dot{\mathbf{L}}$ and $\boldsymbol{\phi}^2$ is similar to the relation between the rate-of-deformation tensor (or stretching tensor) $\mathbf{D}$ and the logarithm of the left strain tensor $\mathbf{V}$ in the theory of finite deformations (Gurtin and Spear, 1983; Hoger, 1986). In fact, for the special case with no rotation of principal damage directions, $\dot{\mathbf{L}}$ is coaxial with $\boldsymbol{\phi}$, $\bar{\boldsymbol{\phi}}$, $\mathbf{w}$ and $\bar{\mathbf{w}}$ and $\dot{\mathbf{L}}$ becomes the exact rate of the logarithmic damage tensor $ln\,\boldsymbol{\phi}^2$. Also note that, even if not for the whole tensor, when the principal axes rotate a similar relation is maintained for the first invariant of $\dot{\mathbf{L}}$, which turns out to be the exact differential of the first invariant of the logarithmic damage tensor. These and other interesting properties concerning the volumetric/deviatoric split of $\dot{\mathbf{L}}$ are developed in Carol et al. (2001a).

Going back to the thermodynamic force $-\boldsymbol{\mathcal{Y}}$, eqn (21), its physical meaning can be interpreted through the following considerations. Since we have assumed undamaged isotropic elasticity, eqns (8) and (15), the thermodynamic force is coaxial with both $\boldsymbol{\sigma}_{\text{eff}}$ and $\boldsymbol{\epsilon}_{\text{eff}}$, thus these three tensors share the same principal axes and their products commute; the principal values of $-\boldsymbol{\mathcal{Y}}$ are then simply given by $-\mathcal{Y}_I{=}\sigma_I^{\text{eff}}\,\epsilon_I^{\text{eff}}/2$, $I{=}1,2,3$. Also, the first invariant of the thermodynamic force equals the value of the current elastic energy, namely $tr(-\boldsymbol{\mathcal{Y}}){=}\boldsymbol{\sigma}_{\text{eff}}{:}\boldsymbol{\epsilon}_{\text{eff}}/2{=}u$, where the last equality holds due to the underlying assumption of energy equivalence, eqn (9). Furthermore, the volumetric and deviatoric parts of $-\boldsymbol{\mathcal{Y}}$, which are associated to the respective components of $\dot{\mathbf{L}}$ in the dissipation inequality $(19)_a$, yield a clear decomposition of effects between increments of isotropic damage (volumetric component of $\dot{\mathbf{L}}$) and anisotropic degradation rates (deviatoric component of $\dot{\mathbf{L}}$). This aspect is crucial for the convenient formulation and interpretation of loading functions and damage evolution rules in the space of thermodynamic forces (Carol et al., 2001a,b). Through eqns (8) and (15), the thermodynamic force (21) may be expressed as follows, either in terms of $\boldsymbol{\sigma}_{\text{eff}}$ or $\boldsymbol{\epsilon}_{\text{eff}}$:

$$
\begin{aligned}
-\boldsymbol{\mathcal{Y}} &= \frac{1}{2}\left(-\frac{\nu_0}{E_0}\; tr\,\boldsymbol{\sigma}_{\text{eff}}\;\boldsymbol{\sigma}_{\text{eff}} + \frac{1+\nu_0}{E_0}\;{\boldsymbol{\sigma}_{\text{eff}}}^2\right); \\
&= \frac{1}{2}\left(\;\Lambda_0\; tr\,\boldsymbol{\epsilon}_{\text{eff}}\;\;\boldsymbol{\epsilon}_{\text{eff}} + \;2\,G_0\;\;{\boldsymbol{\epsilon}_{\text{eff}}}^2\right).
\end{aligned} \tag{22}
$$

Derivation of eqn (21) is indeed quite straight forward in compact form. In fact, from the definition of the thermodynamic force, eqn $(19)_b$, for the force $-\boldsymbol{\mathcal{Y}}{=}-\boldsymbol{\mathcal{Y}}^{\mathbf{L}}$ conjugate to $\dot{\mathbf{L}}$ we can write

$$-\boldsymbol{\mathcal{Y}} = \left.\frac{\partial u}{\partial \mathbf{L}}\right|_{\boldsymbol{\sigma}} = \left.\frac{\partial u}{\partial \boldsymbol{\phi}}\right|_{\boldsymbol{\sigma}} : \frac{\partial \boldsymbol{\phi}}{\partial \mathbf{L}} \,, \tag{23}$$

where the chain rule has been used. Now, from the definition of the pseudo-log rate, eqn $(20)_b$, we have $2\,\partial\boldsymbol{\phi}/\partial\mathbf{L}{=}\mathbb{A}{=}\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}$; the energy u and its derivative with respect to $\boldsymbol{\phi}$ at constant $\boldsymbol{\sigma}$ (that is the thermodynamic force $-\boldsymbol{\mathcal{Y}}^{\phi}$ conjugate to $\dot{\boldsymbol{\phi}}$) can also be calculated by using the formulas listed in the Appendices:

$$\begin{aligned} u &= \frac{1}{2}\,\boldsymbol{\sigma} : \mathbb{C} : \boldsymbol{\sigma} = -\frac{\nu_0}{2E_0}\,(\boldsymbol{\phi} : \boldsymbol{\sigma})^2 + \frac{1+\nu_0}{2E_0}\,tr(\boldsymbol{\sigma}\cdot\boldsymbol{\phi})^2 \,; \\ -\boldsymbol{\mathcal{Y}}^{\phi} &= \left.\frac{\partial u}{\partial \boldsymbol{\phi}}\right|_{\boldsymbol{\sigma}} = -\frac{\nu_0}{E_0}\,(\boldsymbol{\phi} : \boldsymbol{\sigma})\,\boldsymbol{\sigma} + \frac{1+\nu_0}{E_0}\,\boldsymbol{\sigma}\cdot\boldsymbol{\phi}\cdot\boldsymbol{\sigma} \,. \end{aligned} \tag{24}$$

Finally, by developing the inner product in (23), and using eqn $(14)_a$ to notice that $\boldsymbol{\phi}{:}\boldsymbol{\sigma}{=}tr\,\boldsymbol{\sigma}_{\text{eff}}$ and to back-substitute $\boldsymbol{\sigma}_{\text{eff}}$ in place of $\boldsymbol{\sigma}$, expression $(22)_1$ is precisely recovered, and so is (21).

3. Explicit expression of the tangent stiffness

Now that the elastic-damage model has been briefly presented, we can move to the compact form derivation of all the quantities involved in the expression of the tangent stiffness $\mathbb{E}_t$, eqn (4).

First of all let us postulate that the current elastic domain is defined in the space of the thermodynamic forces $-\boldsymbol{\mathcal{Y}}$ by the following rather general isotropic hardening/softening form:

$$F = f(-\boldsymbol{\mathcal{Y}}) - r(\boldsymbol{\mathcal{D}}) \leq 0 \,, \tag{25}$$

where the function $f(-\boldsymbol{\mathcal{Y}})$ specifies the shape of the damage domain and $r(\boldsymbol{\mathcal{D}})$ is the hardening/softening function, normally expressed in terms of the underlying damage variable $\boldsymbol{\mathcal{D}}$, the rate of which depends eventually on $\dot{\lambda}$ through eqn $(6)_b$. The model would then be characterized by specific choices of the functions above, according to the material behavior to be described. Concerning $f(-\boldsymbol{\mathcal{Y}})$, as a first approach, such function may be expressed in terms of the invariants of $-\boldsymbol{\mathcal{Y}}$, as customary in isotropic elastic-plasticity, and the surface $F(-\boldsymbol{\mathcal{Y}}){=}0$ may be represented in the space of principal thermodynamic forces (see Figure 3 later shown). Note that, due to the structure of $-\boldsymbol{\mathcal{Y}}$, eqn (21), F may be expressed in terms of effective stress and effective strain or, through eqn (22), in terms of either or them exclusively, which greatly simplifies the model and its physical interpretation.

3.1. Terms $\mathbf{n}$, $\mathbf{n}:\mathbb{E}$, and $\mathbf{m}$, $\mathbb{E}:\mathbf{m}$

The gradient of the loading function in stress space, that is the second-order tensor $\mathbf{n}$, eqn $(3)_{3b}$, will now be related to the gradient of the loading function in the conjugate force space, $\boldsymbol{\mathcal{N}}$, through the following chain rule relation:

$$\mathbf{n} = \left.\frac{\partial F}{\partial \boldsymbol{\sigma}}\right|_{\lambda} = \left.\frac{\partial f}{\partial (-\boldsymbol{\mathcal{Y}})}\right|_{\lambda} : \left.\frac{\partial (-\boldsymbol{\mathcal{Y}})}{\partial \boldsymbol{\sigma}}\right|_{\lambda} = \boldsymbol{\mathcal{N}} : \frac{\partial (-\boldsymbol{\mathcal{Y}})}{\partial \boldsymbol{\sigma}_{\text{eff}}} : \left.\frac{\partial \boldsymbol{\sigma}_{\text{eff}}}{\partial \boldsymbol{\sigma}}\right|_{\lambda} , \tag{26}$$

where, due to eqns $(10)_a$ and $(13)_a$, the derivative $\partial\boldsymbol{\sigma}_{\text{eff}}/\partial\boldsymbol{\sigma}$ at constant λ (that is $\mathbf{w}$) is given by $\mathbb{A}=\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w}$. From eqn $(22)_1$, by using the derivatives reported in Appendices A and B, eqns $(\text{B}.8)_1$, (A.23) and $(\text{B}.4)_2$, the derivative $\partial(-\boldsymbol{\mathcal{Y}})/\partial\boldsymbol{\sigma}_{\text{eff}}$ is also obtained as

$$\begin{aligned} \frac{\partial (-\boldsymbol{\mathcal{Y}})}{\partial \boldsymbol{\sigma}_{\text{eff}}} = & -\frac{\nu_0}{2\,E_0} \left(\boldsymbol{\sigma}_{\text{eff}} \otimes \mathbf{I} + tr\,\boldsymbol{\sigma}_{\text{eff}}\, \mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I} \right) \\ & + \frac{1+\nu_0}{2\,E_0} \left(\boldsymbol{\sigma}_{\text{eff}}\,\underline{\overline{\otimes}}\,\mathbf{I} + \mathbf{I}\,\underline{\overline{\otimes}}\,\boldsymbol{\sigma}_{\text{eff}} \right), \end{aligned} \tag{27}$$

so that, by double contraction of the r.h.s of eqn (27) with $\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w}$, from eqn (26) we get

$$\begin{aligned} \left.\frac{\partial (-\boldsymbol{\mathcal{Y}})}{\partial \boldsymbol{\sigma}}\right|_{\lambda} = & -\frac{\nu_0}{2\,E_0} \left(\boldsymbol{\sigma}_{\text{eff}} \otimes \mathbf{w}^2 + tr\,\boldsymbol{\sigma}_{\text{eff}}\, \mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w} \right) \\ & + \frac{1+\nu_0}{2\,E_0} \left((\boldsymbol{\sigma}_{\text{eff}} \cdot \mathbf{w})\,\underline{\overline{\otimes}}\,\mathbf{w} + \mathbf{w}\,\underline{\overline{\otimes}}\,(\boldsymbol{\sigma}_{\text{eff}} \cdot \mathbf{w}) \right), \end{aligned} \tag{28}$$

and finally, by contracting this last expression with $\boldsymbol{\mathcal{N}}$, we arrive at

$$\begin{aligned} \mathbf{n} = & -\frac{\nu_0}{2\,E_0} \left((\boldsymbol{\mathcal{N}} : \boldsymbol{\sigma}_{\text{eff}})\, \mathbf{w}^2 + tr\,\boldsymbol{\sigma}_{\text{eff}}\;\, \mathbf{w} \cdot \boldsymbol{\mathcal{N}} \cdot \mathbf{w} \right) \\ & + \frac{1+\nu_0}{2\,E_0} \left(\mathbf{w} \cdot \boldsymbol{\mathcal{N}} \cdot \boldsymbol{\sigma}_{\text{eff}} \cdot \mathbf{w} + \mathbf{w} \cdot \boldsymbol{\sigma}_{\text{eff}} \cdot \boldsymbol{\mathcal{N}} \cdot \mathbf{w} \right). \end{aligned} \tag{29}$$

Now, from eqns (29) and $(17)_a$, the term $\mathbf{n}:\mathbb{E}$ appearing at the numerator of the expression of the tangent stiffness, eqn $(4)_{2a}$, can be evaluated as

$$\mathbf{n} : \mathbb{E} = \Lambda_0\;\, tr(\bar{\mathbf{w}} \cdot \mathbf{n} \cdot \bar{\mathbf{w}})\;\, \bar{\mathbf{w}}^2 + 2\,G_0\;\, \bar{\mathbf{w}}^2 \cdot \mathbf{n} \cdot \bar{\mathbf{w}}^2 , \tag{30}$$

where Λ_0 and G_0 may be expressed in terms of E_0 and ν_0 through eqn $(16)_1$, and

$$\begin{aligned} tr(\bar{\mathbf{w}} \cdot \mathbf{n} \cdot \bar{\mathbf{w}}) = & \;\frac{2-\nu_0}{2\,E_0}\, \boldsymbol{\mathcal{N}} : \boldsymbol{\sigma}_{\text{eff}} - \frac{\nu_0}{2\,E_0}\; tr\,\boldsymbol{\mathcal{N}}\;\; tr\,\boldsymbol{\sigma}_{\text{eff}} , \\ \bar{\mathbf{w}}^2 \cdot \mathbf{n} \cdot \bar{\mathbf{w}}^2 = & -\frac{\nu_0}{2\,E_0} \left((\boldsymbol{\mathcal{N}} : \boldsymbol{\sigma}_{\text{eff}})\, \bar{\mathbf{w}}^2 + tr\,\boldsymbol{\sigma}_{\text{eff}}\;\, \bar{\mathbf{w}} \cdot \boldsymbol{\mathcal{N}} \cdot \bar{\mathbf{w}} \right) \\ & + \frac{1+\nu_0}{2\,E_0} \left(\bar{\mathbf{w}} \cdot \boldsymbol{\mathcal{N}} \cdot \boldsymbol{\sigma}_{\text{eff}} \cdot \bar{\mathbf{w}} + \bar{\mathbf{w}} \cdot \boldsymbol{\sigma}_{\text{eff}} \cdot \boldsymbol{\mathcal{N}} \cdot \bar{\mathbf{w}} \right). \end{aligned} \tag{31}$$

The degrading strain direction $\mathbf{m}$ can be derived from eqn (7) and takes finally a form dual to (29), where $\boldsymbol{\mathcal{N}}$ is simply replaced by $\boldsymbol{\mathcal{M}}$, namely the direction of the pseudo-log rate $\dot{\mathbf{L}}=\dot{\lambda}\,\boldsymbol{\mathcal{M}}$; similarly, the term $\mathbb{E}{:}\mathbf{m}$ at the numerator of $(4)_{2a}$ is obtained just in the same way as $\mathbf{n}{:}\mathbb{E}$. For the sake of completeness we report below all the relevant expressions:

$$\begin{aligned} \mathbf{m} = &-\frac{\nu_0}{2\,E_0}\ \Big((\boldsymbol{\mathcal{M}} : \boldsymbol{\sigma}_{\text{eff}})\ \mathbf{w}^2 + tr\,\boldsymbol{\sigma}_{\text{eff}}\ \ \mathbf{w}\cdot\boldsymbol{\mathcal{M}}\cdot\mathbf{w} \Big) \\ &+ \frac{1+\nu_0}{2\,E_0}\ \Big(\mathbf{w}\cdot\boldsymbol{\mathcal{M}}\cdot\boldsymbol{\sigma}_{\text{eff}}\cdot\mathbf{w} + \mathbf{w}\cdot\boldsymbol{\sigma}_{\text{eff}}\cdot\boldsymbol{\mathcal{M}}\cdot\mathbf{w} \Big), \end{aligned} \tag{32}$$

and

$$\begin{aligned} \mathbf{m} : \mathbb{E} = &\quad \Lambda_0\ \ tr(\bar{\mathbf{w}}\cdot\mathbf{m}\cdot\bar{\mathbf{w}})\ \ \bar{\mathbf{w}}^2 + 2\,G_0\ \ \bar{\mathbf{w}}^2\cdot\mathbf{m}\cdot\bar{\mathbf{w}}^2\ ; \\ tr(\bar{\mathbf{w}}\cdot\mathbf{m}\cdot\bar{\mathbf{w}}) = &\quad \frac{2-\nu_0}{2\,E_0}\ \ \boldsymbol{\mathcal{M}} : \boldsymbol{\sigma}_{\text{eff}} - \frac{\nu_0}{2\,E_0}\ \ tr\,\boldsymbol{\mathcal{M}}\ \ tr\,\boldsymbol{\sigma}_{\text{eff}}\ , \\ \bar{\mathbf{w}}^2\cdot\mathbf{m}\cdot\bar{\mathbf{w}}^2 = &-\frac{\nu_0}{2\,E_0}\ \ \Big((\boldsymbol{\mathcal{M}} : \boldsymbol{\sigma}_{\text{eff}})\ \bar{\mathbf{w}}^2 + tr\,\boldsymbol{\sigma}_{\text{eff}}\ \ \bar{\mathbf{w}}\cdot\boldsymbol{\mathcal{M}}\cdot\bar{\mathbf{w}} \Big) \\ &+ \frac{1+\nu_0}{2\,E_0} \Big(\bar{\mathbf{w}}\cdot\boldsymbol{\mathcal{M}}\cdot\boldsymbol{\sigma}_{\text{eff}}\cdot\bar{\mathbf{w}} + \bar{\mathbf{w}}\cdot\boldsymbol{\sigma}_{\text{eff}}\cdot\boldsymbol{\mathcal{M}}\cdot\bar{\mathbf{w}} \Big). \end{aligned} \tag{33}$$

3.2. Hardening parameters H and $\bar{H}$

We are now in the position to form the term $\mathbf{n}{:}\mathbb{E}{:}\mathbf{m}$, which makes the difference between the hardening parameters $\bar{H}$ and H, eqn $(4)_{2b}$. From eqns (30) and (32) we get:

$$\mathbf{n} : \mathbb{E} : \mathbf{m} = \Lambda_0\ tr(\bar{\mathbf{w}}\cdot\mathbf{n}\cdot\bar{\mathbf{w}})\ tr(\bar{\mathbf{w}}\cdot\mathbf{m}\cdot\bar{\mathbf{w}}) + 2\,G_0\ tr(\bar{\mathbf{w}}^2\cdot\mathbf{n}\cdot\bar{\mathbf{w}}^2\cdot\mathbf{m})\ , \tag{34}$$

where the terms $tr(\bar{\mathbf{w}}{\cdot}\mathbf{n}{\cdot}\bar{\mathbf{w}})$ and $tr(\bar{\mathbf{w}}{\cdot}\mathbf{m}{\cdot}\bar{\mathbf{w}})$ are given in eqns $(31)_1$ and $(33)_2$, and

$$\begin{aligned} &tr(\bar{\mathbf{w}}^2\cdot\mathbf{n}\cdot\bar{\mathbf{w}}^2\cdot\mathbf{m}) = \\ &= \left(\frac{1+\nu_0}{E_0}\right)^2\ \frac{1}{2}\ \Big(tr(\boldsymbol{\mathcal{N}}\cdot\boldsymbol{\sigma}^2_{\text{eff}}\cdot\boldsymbol{\mathcal{M}}) + tr(\boldsymbol{\mathcal{N}}\cdot\boldsymbol{\sigma}_{\text{eff}}\cdot\boldsymbol{\mathcal{M}}\cdot\boldsymbol{\sigma}_{\text{eff}}) \Big) \\ &- \frac{\nu_0\,(4+\nu_0)}{(2\,E_0)^2}\ (\boldsymbol{\mathcal{N}} : \boldsymbol{\sigma}_{\text{eff}})\,(\boldsymbol{\mathcal{M}} : \boldsymbol{\sigma}_{\text{eff}}) \\ &+ \frac{\nu_0}{(2\,E_0)^2}\ tr\,\boldsymbol{\sigma}_{\text{eff}}\ \Big[-4\,(1+\nu_0)\ tr(\boldsymbol{\mathcal{N}}\cdot\boldsymbol{\sigma}_{\text{eff}}\cdot\boldsymbol{\mathcal{M}}) \\ &\quad + \nu_0\ \Big((\boldsymbol{\mathcal{N}} : \boldsymbol{\sigma}_{\text{eff}})\,tr\,\boldsymbol{\mathcal{M}} + (\boldsymbol{\mathcal{M}} : \boldsymbol{\sigma}_{\text{eff}})\,tr\,\boldsymbol{\mathcal{N}} + tr\,\boldsymbol{\sigma}_{\text{eff}}\,tr(\boldsymbol{\mathcal{N}}\cdot\boldsymbol{\mathcal{M}}) \Big) \Big]. \end{aligned} \tag{35}$$

After some rearrangement of terms in (34) we get the following rather involved expression of $\mathbf{n}{:}\mathbb{E}{:}\mathbf{m}$ as a function of $\boldsymbol{\sigma}_{\text{eff}}$:

$$\begin{aligned}
\mathbf{n} : \mathbb{E} : \mathbf{m} = \\
= & \frac{1+\nu_0}{2\,E_0} \Big(tr(\boldsymbol{\mathcal{N}} \cdot \boldsymbol{\sigma}_{\text{eff}}^2 \cdot \boldsymbol{\mathcal{M}}) + tr(\boldsymbol{\mathcal{N}} \cdot \boldsymbol{\sigma}_{\text{eff}} \cdot \boldsymbol{\mathcal{M}} \cdot \boldsymbol{\sigma}_{\text{eff}}) \Big) \\
& + \frac{\nu_0}{4E_0(1-2\nu_0)} \Bigg[3\nu_0 \, (\boldsymbol{\mathcal{N}} : \boldsymbol{\sigma}_{\text{eff}})\,(\boldsymbol{\mathcal{M}} : \boldsymbol{\sigma}_{\text{eff}}) \\
& \quad - tr\,\boldsymbol{\sigma}_{\text{eff}} \Bigg(\; 4\,(1-2\nu_0)\; tr(\boldsymbol{\mathcal{N}} \cdot \boldsymbol{\sigma}_{\text{eff}} \cdot \boldsymbol{\mathcal{M}}) \\
& \qquad + \nu_0 \Big((\boldsymbol{\mathcal{N}} : \boldsymbol{\sigma}_{\text{eff}})\, tr\,\boldsymbol{\mathcal{M}} + (\boldsymbol{\mathcal{M}} : \boldsymbol{\sigma}_{\text{eff}})\, tr\,\boldsymbol{\mathcal{N}} \Big) \Bigg) \\
& \quad + \frac{\nu_0}{1+\nu_0}\, tr^2\,\boldsymbol{\sigma}_{\text{eff}} \Big((1-2\nu_0)\; tr(\boldsymbol{\mathcal{N}} \cdot \boldsymbol{\mathcal{M}}) + \nu_0\; tr\,\boldsymbol{\mathcal{N}}\; tr\,\boldsymbol{\mathcal{M}} \Big) \Bigg].
\end{aligned} \tag{36}$$

However, this should not be discouraging. In fact, to form $\bar{H}$, eqn $(4)_{2b}$, the term $\mathbf{n}{:}\mathbb{E}{:}\mathbf{m}$ must be added to the part of H arising from the function $f(-\boldsymbol{\mathcal{Y}})$ appearing in eqn (25). So, let us first calculate the hardening parameter H, as defined in eqn $(3)_{3c}$:

$$H = -\left.\frac{\partial F}{\partial \lambda}\right|_{\boldsymbol{\sigma}} = \frac{\partial r}{\partial \lambda} - \left.\frac{\partial f}{\partial \lambda}\right|_{\boldsymbol{\sigma}} = H_r + H_f \,, \tag{37}$$

where the first part of H may be obtained as $H_r{=}(\partial r/\partial \mathbf{L}){:}\boldsymbol{\mathcal{M}}$, while its second part may be expanded through the chain rule as follows

$$H_f = -\left.\frac{\partial f}{\partial \lambda}\right|_{\boldsymbol{\sigma}} = -\frac{\partial f}{\partial(-\boldsymbol{\mathcal{Y}})} : \left.\frac{\partial(-\boldsymbol{\mathcal{Y}})}{\partial \mathbf{L}}\right|_{\boldsymbol{\sigma}} : \frac{\partial \mathbf{L}}{\partial \lambda} = -\boldsymbol{\mathcal{N}} : \left.\frac{\partial(-\boldsymbol{\mathcal{Y}})}{\partial \mathbf{L}}\right|_{\boldsymbol{\sigma}} : \boldsymbol{\mathcal{M}} \,. \tag{38}$$

So, we still have to evaluate the derivative $\partial(-\boldsymbol{\mathcal{Y}})/\partial\mathbf{L}$ at constant $\boldsymbol{\sigma}$.

This is actually the tricky part of the derivation: the interest of going through the developments presented here has arisen indeed from the evaluation of this term. Let us expand it once more by repeated application of the chain rule:

$$\left.\frac{\partial(-\boldsymbol{\mathcal{Y}})}{\partial \mathbf{L}}\right|_{\boldsymbol{\sigma}} = \left.\frac{\partial(-\boldsymbol{\mathcal{Y}})}{\partial \mathbf{w}}\right|_{\boldsymbol{\sigma}} : \frac{\partial \mathbf{w}}{\partial \mathbf{L}} \,, \tag{39}$$

where, in turns

$$\left.\frac{\partial(-\boldsymbol{\mathcal{Y}})}{\partial \mathbf{w}}\right|_{\boldsymbol{\sigma}} = \frac{\partial(-\boldsymbol{\mathcal{Y}})}{\partial \boldsymbol{\sigma}_{\text{eff}}} : \left.\frac{\partial \boldsymbol{\sigma}_{\text{eff}}}{\partial \mathbf{w}}\right|_{\boldsymbol{\sigma}} ; \qquad \frac{\partial \mathbf{w}}{\partial \mathbf{L}} = \frac{\partial \mathbf{w}}{\partial \boldsymbol{\phi}} : \frac{\partial \boldsymbol{\phi}}{\partial \mathbf{L}} \,. \tag{40}$$

Now, of the derivatives above, $\partial(-\boldsymbol{\mathcal{Y}})/\partial\boldsymbol{\sigma}_{\text{eff}}$ is given by eqn (27) and, as already noticed, $\partial\boldsymbol{\phi}/\partial\mathbf{L}{=}\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w}/2$. Furthermore, the derivative of the square root function, namely $\partial \mathbf{w}/\partial \boldsymbol{\phi}$, which was expanded through spectral representation in Carol et al. (2001a) according to Ogden (1984), p. 162-163, may actually take the compact form that can be extracted from Hoger and Carlson (1984), see Appendix B, eqn (B.11). So, only one term is still missing, which may be evaluated from the effective/nominal relation $(14)_a$:

$$\left.\frac{\partial \boldsymbol{\sigma}_{\text{eff}}}{\partial \mathbf{w}}\right|_{\boldsymbol{\sigma}} = (\boldsymbol{\sigma}_{\text{eff}} \cdot \bar{\mathbf{w}}) \,\underline{\overline{\otimes}}\, \mathbf{I} + \mathbf{I} \,\underline{\overline{\otimes}}\, (\boldsymbol{\sigma}_{\text{eff}} \cdot \bar{\mathbf{w}}) \,, \tag{41}$$

where relation $\boldsymbol{\sigma}_{\text{eff}} \cdot \bar{\mathbf{w}} = \mathbf{w} \cdot \boldsymbol{\sigma}$ from $(14)_a$ has been used. Then, by performing the inner products in (40) we get:

$$\begin{aligned}\left.\frac{\partial(-\boldsymbol{\mathcal{Y}})}{\partial \mathbf{w}}\right|_{\boldsymbol{\sigma}} = -\frac{\nu_0}{2\,E_0} \Big(& \boldsymbol{\sigma}_{\text{eff}} \otimes (\boldsymbol{\sigma}_{\text{eff}} \cdot \bar{\mathbf{w}} + \bar{\mathbf{w}} \cdot \boldsymbol{\sigma}_{\text{eff}}) \\ & + tr\, \boldsymbol{\sigma}_{\text{eff}} \Big((\boldsymbol{\sigma}_{\text{eff}} \cdot \bar{\mathbf{w}}) \,\underline{\overline{\otimes}}\, \mathbf{I} + \mathbf{I} \,\underline{\overline{\otimes}}\, (\boldsymbol{\sigma}_{\text{eff}} \cdot \bar{\mathbf{w}}) \Big) \Big) \\ + \frac{1+\nu_0}{2\,E_0} \Big(& (\boldsymbol{\sigma}_{\text{eff}} \cdot \bar{\mathbf{w}}) \,\underline{\overline{\otimes}}\, \boldsymbol{\sigma}_{\text{eff}} + \boldsymbol{\sigma}_{\text{eff}} \,\underline{\overline{\otimes}}\, (\boldsymbol{\sigma}_{\text{eff}} \cdot \bar{\mathbf{w}}) \\ & + (\boldsymbol{\sigma}^2_{\text{eff}} \cdot \bar{\mathbf{w}}) \,\underline{\overline{\otimes}}\, \mathbf{I} + \mathbf{I} \,\underline{\overline{\otimes}}\, (\boldsymbol{\sigma}^2_{\text{eff}} \cdot \bar{\mathbf{w}}) \Big) \,,\end{aligned} \tag{42}$$

and

$$\begin{aligned}\frac{\partial \mathbf{w}}{\partial \mathbf{L}} = \frac{1}{4\,({}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2 - {}^{\mathrm{w}}I_3)} \Big(& -{}^{\mathrm{w}}I_1\; (\mathbf{w}^2 \,\underline{\overline{\otimes}}\, \mathbf{w} + \mathbf{w} \,\underline{\overline{\otimes}}\, \mathbf{w}^2) \\ & -{}^{\mathrm{w}}I_3\; (\mathbf{w} \,\underline{\overline{\otimes}}\, \mathbf{I} + \mathbf{I} \,\underline{\overline{\otimes}}\, \mathbf{w}) + \mathbf{w}^2 \,\underline{\overline{\otimes}}\, \mathbf{w}^2 \\ & + ({}^{\mathrm{w}}I_1{}^2 + {}^{\mathrm{w}}I_2)\; \mathbf{w} \,\underline{\overline{\otimes}}\, \mathbf{w} + {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3\; \mathbf{I} \,\underline{\overline{\otimes}}\, \mathbf{I} \Big) \,,\end{aligned} \tag{43}$$

where, as defined in Appendix B, eqn (B.7), ${}^{\mathrm{w}}I_1, {}^{\mathrm{w}}I_2, {}^{\mathrm{w}}I_3$ are the principal invariants of $\mathbf{w}$, and the Cayley-Hamilton theorem (B.6) applied to $\mathbf{w}$ has been used to eliminate $\mathbf{w}^3 = {}^{\mathrm{w}}I_1\, \mathbf{w}^2 - {}^{\mathrm{w}}I_2\, \mathbf{w} + {}^{\mathrm{w}}I_3\, \mathbf{I}$. Finally, by performing a further inner product, from eqns (39), (42) and (43) we obtain:

$$\begin{aligned}\left.\frac{\partial(-\boldsymbol{\mathcal{Y}})}{\partial \mathbf{L}}\right|_{\boldsymbol{\sigma}} = & -\frac{\nu_0}{4\,E_0}\, \boldsymbol{\sigma}_{\text{eff}} \otimes \boldsymbol{\sigma}_{\text{eff}} - \frac{\nu_0}{8\,E_0}\, tr\, \boldsymbol{\sigma}_{\text{eff}}\; (\boldsymbol{\sigma}_{\text{eff}} \,\underline{\overline{\otimes}}\, \mathbf{I} + \mathbf{I} \,\underline{\overline{\otimes}}\, \boldsymbol{\sigma}_{\text{eff}}) \\ & + \frac{1+\nu_0}{8\,E_0}\; (2\,\boldsymbol{\sigma}_{\text{eff}} \,\underline{\overline{\otimes}}\, \boldsymbol{\sigma}_{\text{eff}} + \boldsymbol{\sigma}^2_{\text{eff}} \,\underline{\overline{\otimes}}\, \mathbf{I} + \mathbf{I} \,\underline{\overline{\otimes}}\, \boldsymbol{\sigma}^2_{\text{eff}}) \\ & + \left.\frac{\partial(-\boldsymbol{\mathcal{Y}})}{\partial \mathbf{L}}\right|^{ex}_{\boldsymbol{\sigma}} \,,\end{aligned} \tag{44}$$

where the extra terms, that is

$$
\begin{aligned}
8\,({}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2 - {}^{\mathrm{w}}I_3)\;\; & \frac{\partial(-\boldsymbol{\mathcal{Y}})}{\partial \mathbf{L}}\bigg|_{\boldsymbol{\sigma}}^{ex} = \\
= -\frac{\nu_0}{E_0}\, tr\, \boldsymbol{\sigma}_{\text{eff}} \Big[\quad & (\boldsymbol{\sigma}_{\text{eff}}\cdot\mathbf{w})\,\overline{\underline{\otimes}}\,\mathbf{w}^2 + \mathbf{w}^2\,\overline{\underline{\otimes}}\,(\boldsymbol{\sigma}_{\text{eff}}\cdot\mathbf{w}) \\
& -(\boldsymbol{\sigma}_{\text{eff}}\cdot\mathbf{w}^2)\,\overline{\underline{\otimes}}\,\mathbf{w} - \mathbf{w}\,\overline{\underline{\otimes}}\,(\boldsymbol{\sigma}_{\text{eff}}\cdot\mathbf{w}^2) \\
+{}^{\mathrm{w}}I_1 \Big(\quad & (\boldsymbol{\sigma}_{\text{eff}}\cdot\mathbf{w}^2)\,\overline{\underline{\otimes}}\,\mathbf{I} + \mathbf{I}\,\overline{\underline{\otimes}}\,(\boldsymbol{\sigma}_{\text{eff}}\cdot\mathbf{w}^2) \\
& -\boldsymbol{\sigma}_{\text{eff}}\,\overline{\underline{\otimes}}\,\mathbf{w}^2 - \mathbf{w}^2\,\overline{\underline{\otimes}}\,\boldsymbol{\sigma}_{\text{eff}}\Big) \\
+{}^{\mathrm{w}}I_1{}^2 \Big(\quad & \boldsymbol{\sigma}_{\text{eff}}\,\overline{\underline{\otimes}}\,\mathbf{w} + \mathbf{w}\,\overline{\underline{\otimes}}\,\boldsymbol{\sigma}_{\text{eff}} \\
& -(\boldsymbol{\sigma}_{\text{eff}}\cdot\mathbf{w})\,\overline{\underline{\otimes}}\,\mathbf{I} - \mathbf{I}\,\overline{\underline{\otimes}}\,(\boldsymbol{\sigma}_{\text{eff}}\cdot\mathbf{w})\Big)\Big] \\
+\frac{1+\nu_0}{E_0} \Big[\quad & (\boldsymbol{\sigma}^2_{\text{eff}}\cdot\mathbf{w})\,\overline{\underline{\otimes}}\,\mathbf{w}^2 + \mathbf{w}^2\,\overline{\underline{\otimes}}\,(\boldsymbol{\sigma}^2_{\text{eff}}\cdot\mathbf{w}) \\
& -(\boldsymbol{\sigma}^2_{\text{eff}}\cdot\mathbf{w}^2)\,\overline{\underline{\otimes}}\,\mathbf{w} - \mathbf{w}\,\overline{\underline{\otimes}}\,(\boldsymbol{\sigma}^2_{\text{eff}}\cdot\mathbf{w}^2) \\
+{}^{\mathrm{w}}I_1 \Big(\quad & (\boldsymbol{\sigma}^2_{\text{eff}}\cdot\mathbf{w}^2)\,\overline{\underline{\otimes}}\,\mathbf{I} + \mathbf{I}\,\overline{\underline{\otimes}}\,(\boldsymbol{\sigma}^2_{\text{eff}}\cdot\mathbf{w}^2) \\
& -\boldsymbol{\sigma}^2_{\text{eff}}\,\overline{\underline{\otimes}}\,\mathbf{w}^2 - \mathbf{w}^2\,\overline{\underline{\otimes}}\,\boldsymbol{\sigma}^2_{\text{eff}}\Big) \\
+{}^{\mathrm{w}}I_1{}^2 \Big(\quad & \boldsymbol{\sigma}^2_{\text{eff}}\,\overline{\underline{\otimes}}\,\mathbf{w} + \mathbf{w}\,\overline{\underline{\otimes}}\,\boldsymbol{\sigma}^2_{\text{eff}} \\
& -(\boldsymbol{\sigma}^2_{\text{eff}}\cdot\mathbf{w})\,\overline{\underline{\otimes}}\,\mathbf{I} - \mathbf{I}\,\overline{\underline{\otimes}}\,(\boldsymbol{\sigma}^2_{\text{eff}}\cdot\mathbf{w})\Big)\Big]
\end{aligned}
\tag{45}
$$

are grouped aside for the reason that will be clear now. To get the compact form expressions in eqns (44) and (45) the tensor operations listed in Boxes 1, 2 of Appendix C have been used, and, most important, the Cayley-Hamilton theorem (B.6) applied to $\mathbf{w}$ in the form ${}^{\mathrm{w}}I_3\,\bar{\mathbf{w}}=\mathbf{w}^2-{}^{\mathrm{w}}I_1\,\mathbf{w}+{}^{\mathrm{w}}I_2\,\mathbf{I}$ has been applied once again to eliminate $\bar{\mathbf{w}}$. As it can be seen in eqn (44) the expression of $\partial(-\boldsymbol{\mathcal{Y}})/\partial\mathbf{L}$ at constant $\boldsymbol{\sigma}$ is surprisingly simple, besides of course the extra terms that do not cancel out identically.[1] However, this derivative enters H_f, eqn (38), through contraction with $\boldsymbol{\mathcal{N}}$ and $\boldsymbol{\mathcal{M}}$.

Now, by *assuming coaxiality* between $\boldsymbol{\mathcal{N}}$ and $-\boldsymbol{\mathcal{Y}}$ (and so between $\boldsymbol{\mathcal{N}}$ and $\boldsymbol{\sigma}_{\text{eff}}$), which is granted for an $f(-\boldsymbol{\mathcal{Y}})$ expressed in terms of the

[1] Notice that these extra terms are present also if minor symmetries are not directly imposed in the derivatives (see Appendix A.2): an expression similar to eqn (45), with dyadic products $\overline{\underline{\otimes}}$ replaced by $\overline{\otimes}$ (Appendix A.1), would still be obtained through the operations listed in Boxes 3, 4 of Appendix C.

invariants of $-\boldsymbol{\mathcal{Y}}$, it can be checked that by double contracting the derivative with $\boldsymbol{\mathcal{N}}$ from left, the contributions of the extra terms to H_f disappear since $\boldsymbol{\mathcal{N}}$ and $\boldsymbol{\sigma}_{\text{eff}}$ commute:

$$\boldsymbol{\mathcal{N}} : \left. \frac{\partial(-\boldsymbol{\mathcal{Y}})}{\partial \mathbf{L}} \right|_{\boldsymbol{\sigma}}^{ex} = \mathbf{0} \,. \tag{46}$$

In other words, to get the contribution H_f to the hardening parameter H, only the simple form given in the first two lines of eqn (44) can be taken into account. Then, the following compact form of H_f, eqns (37) and (38), is obtained:

$$\begin{aligned} H_f = & -\boldsymbol{\mathcal{N}} : \left. \frac{\partial(-\boldsymbol{\mathcal{Y}})}{\partial \mathbf{L}} \right|_{\boldsymbol{\sigma}} : \boldsymbol{\mathcal{M}} \\ = & \quad \frac{\nu_0}{4\,E_0} \Big((\boldsymbol{\mathcal{N}} : \boldsymbol{\sigma}_{\text{eff}})\, (\boldsymbol{\mathcal{M}} : \boldsymbol{\sigma}_{\text{eff}}) + tr\,\boldsymbol{\sigma}_{\text{eff}}\;\; tr(\boldsymbol{\mathcal{N}} \cdot \boldsymbol{\sigma}_{\text{eff}} \cdot \boldsymbol{\mathcal{M}}) \Big) \\ & - \frac{1+\nu_0}{2\,E_0}\;\; tr(\boldsymbol{\mathcal{N}} \cdot \boldsymbol{\sigma}_{\text{eff}}^2 \cdot \boldsymbol{\mathcal{M}}) \,, \end{aligned} \tag{47}$$

which, in terms of the principal cartesian components of $\boldsymbol{\sigma}_{\text{eff}}$, $\boldsymbol{\mathcal{N}}$, and of $\boldsymbol{\mathcal{M}}$, if now *$\boldsymbol{\mathcal{M}}$ is also supposed to be coaxial with $\boldsymbol{\mathcal{N}}$* and $\boldsymbol{\sigma}_{\text{eff}}$, may be expressed in another quite appealing form:

$$\begin{aligned} H_f = & -\frac{1}{2\,E_0} \sum_{I=1}^{3} \mathcal{N}_I \;\; \mathcal{M}_I \;\; (\sigma_I^{\text{eff}})^2 \\ & + \frac{\nu_0}{4\,E_0} \sum_{J>I=1}^{3} (\mathcal{N}_I + \mathcal{N}_J)\; (\mathcal{M}_I + \mathcal{M}_J)\; \sigma_I^{\text{eff}}\; \sigma_J^{\text{eff}} \,. \end{aligned} \tag{48}$$

Finally, though as said the expression (36) of $\mathbf{n}{:}\mathbb{E}{:}\mathbf{m}$ was rather lengthy, the faithful reader may be pleased to see that by adding $\mathbf{n}{:}\mathbb{E}{:}\mathbf{m}$ to H_f, the following compact forms of the hardening parameter $\bar{H}$ at the denominator of the tangent stiffness, eqn $(4)_2$, are obtained, dual to (47) and (48), if expressed in terms of $\boldsymbol{\epsilon}_{\text{eff}}$ and Λ_0, G_0 :[2]

$$\begin{aligned} \bar{H} = H_r + & \frac{\Lambda_0}{4} \Big((\boldsymbol{\mathcal{N}} : \boldsymbol{\epsilon}_{\text{eff}})\, (\boldsymbol{\mathcal{M}} : \boldsymbol{\epsilon}_{\text{eff}}) + tr\,\boldsymbol{\epsilon}_{\text{eff}}\;\; tr(\boldsymbol{\mathcal{N}} \cdot \boldsymbol{\epsilon}_{\text{eff}} \cdot \boldsymbol{\mathcal{M}}) \Big) \\ & + G_0\;\; tr(\boldsymbol{\mathcal{N}} \cdot \boldsymbol{\epsilon}_{\text{eff}}^2 \cdot \boldsymbol{\mathcal{M}}) \,; \end{aligned} \tag{49}$$

$$\begin{aligned} \bar{H} = H_r + & \frac{\Lambda_0 + 2\,G_0}{2} \sum_{I=1}^{3} \mathcal{N}_I \;\; \mathcal{M}_I \;\; (\epsilon_I^{\text{eff}})^2 \\ & + \frac{\Lambda_0}{4} \sum_{J>I=1}^{3} (\mathcal{N}_I + \mathcal{N}_J)\; (\mathcal{M}_I + \mathcal{M}_J)\; \epsilon_I^{\text{eff}}\; \epsilon_J^{\text{eff}} \,. \end{aligned} \tag{50}$$

---

[2] The reason of duality may be explored through a parallel strain-based derivation (Carol et al., 1994) of the present model.

In sum, the tangent stiffness is obtained by substituting in eqn $(4)_{2a}$ the expressions (30) and (31), and (33) in the numerator, and (49) in the denominator. The remarkably simple forms obtained in eqns (47), (48) and (49), (50) should be seen as the main rewards of the following lengthy, but, through the present compact notation approach, rather straight forward calculations: they show the potentiality of the proposed formalism and constitute the main result of the present approach as applied to the constitutive model under consideration.

3.3. Associative pseudo-Rankine model

From the general expressions just derived, different models can be characterized by defining explicitly their characteristic quantities, namely $r(\lambda)$, $\boldsymbol{\mathcal{N}}$ and $\boldsymbol{\mathcal{M}}$. If an *associative pseudo-Rankine criterion* is adopted, the simple expressions derived by Carol et al. (2001a,b) through a spectral representation of the derivatives are recovered. In fact a pseudo-Rankine criterion may be basically interpreted as a cut-off condition in the space of principal conjugate forces (Figure 3).

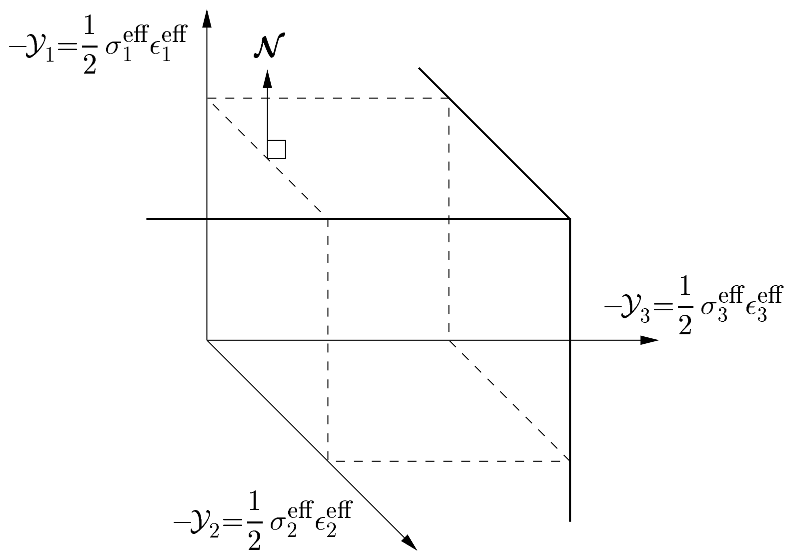

Figure 3. Representation of the loading surface in the principal thermodynamic force space: pseudo-Rankine criterion.

Let us suppose to sit on the fold of the loading surface with normal $\boldsymbol{\mathcal{N}}$ pointing e.g. in the first principal direction $\mathbf{t}_1$ of the effective quantities: in that case $\mathcal{N}_1 = 1$, $\mathcal{N}_2 = \mathcal{N}_3 = 0$. Then, assuming that the model is associative, that is $\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{N}}$, by indicating with $\mathbf{w}_1 = \mathbf{w} \cdot \mathbf{t}_1$ and $\bar{\mathbf{w}}_1 = \bar{\mathbf{w}} \cdot \mathbf{t}_1$ the vector projections of $\mathbf{w}$ and $\bar{\mathbf{w}}$ in the principal direction $\mathbf{t}_1$, we recover the following rather simple expressions of the terms entering the tangent stiffness $\mathbb{E}_t$, eqn $(4)_{2a}$:

$$
\begin{aligned}
\mathbf{m} = \mathbf{n} &= \frac{1}{2}\left(-\frac{\nu_0}{E_0}\,\sigma_1^{\text{eff}}\,\mathbf{w}^2 + \left(\frac{1+\nu_0}{E_0}\,\sigma_1^{\text{eff}} + \epsilon_1^{\text{eff}}\right)\mathbf{w}_1 \otimes \mathbf{w}_1\right);\\
\mathbb{E} : \mathbf{m} = \mathbf{n} : \mathbb{E} &= \frac{1}{2}\left(\Lambda_0\,\epsilon_1^{\text{eff}}\,\bar{\mathbf{w}}^2 + \left(2\,G_0\,\epsilon_1^{\text{eff}} + \sigma_1^{\text{eff}}\right)\bar{\mathbf{w}}_1 \otimes \bar{\mathbf{w}}_1\right);\\
H &= H_r - \frac{1}{4}\left(\sigma_1^{\text{eff}}\,\epsilon_1^{\text{eff}} + \frac{1}{E_0}\,(\sigma_1^{\text{eff}})^2\right);\\
\bar{H} &= H_r + \frac{1}{4}\left(\sigma_1^{\text{eff}}\,\epsilon_1^{\text{eff}} + (\Lambda_0 + 2\,G_0)\,(\epsilon_1^{\text{eff}})^2\right).
\end{aligned}
\tag{51}
$$

4. Conclusions

The compact tensor formalism elaborated here allowed us to obtain general and rather simple expressions of all the ingredients entering the tangent stiffness of a class of elastic-damage models. Once specific forms of the constitutive model are conceived, this should make easier its final setting and its implementation in computer codes. We believe that the derivations performed here do not constitute a mere exercise, rather, the formal tools developed should also help in the formulation of other constitutive models. In fact they allow quasi-automatic compact-form manipulations which are useful to foresee the choices of the material formulation that lead to simple and meaningful expressions of the quantities involved.

It should be clear to the reader that a rather simplified engineering notation has been used throughout this note, in place of a more rigorous, but perhaps less immediate, mathematical notation. However, an effort to reconcile the two different terminologies is attempted in the Appendices.

Acknowledgements

Author E.R. would like to acknowledge financial support from the Italian Ministry (MURST). He wishes to thank Prof. M. Vianello, Politecnico di Milano, for help in addressing him to the appropriate literature on the derivatives of tensor functions; the discussions with Prof. D. De Tommasi, colleague at DIS, Politecnico di Bari, are also gratefully acknowledged. Author I.C. wishes to acknowledge partial support from DGICYT (Madrid, Spain) under research grants PB95-0771 and PB96–0500, and travel support through NATO collaborative research grant CRG.961177 and grant GRQ93-3012 from Generalitat de Catalunya (Barcelona, Spain). A NATO fellowship for a 3-month stay of I.C. at the University of Colorado at Boulder, which made possible to have the time and appropriate environment to review some detailed aspects of the proposed derivations, is also gratefully acknowledged.

References

D. E. Carlson and A. Hoger. On the derivatives of the principal invariants of a second-order tensor. *J. of Elasticity*, 16: 221–224 (1986a).

D. E. Carlson and A. Hoger. The derivative of a tensor-valued function of a tensor. *Quarterly of Applied Mathematics*, XLIV(3): 409–423 (1986b).

I. Carol, E. Rizzi and K. Willam. A unified theory of elastic degradation and damage based on a loading surface. *Int. J. Solids Structures*, 31(20): 2835–2865 (1994).

I. Carol, E. Rizzi and K. Willam. On the formulation of isotropic and anisotropic damage. In H. Mang, N. Bićanić and R. de Borst, eds., *Computational Modelling of Concrete Structures*, p. 183-192. Proc. of EURO-C 98, Balkema, Rotterdam.

I. Carol, E. Rizzi and K. Willam. On the formulation of anisotropic elastic degradation. Part I: Theory based on a pseudo-logarithmic damage tensor rate. Part II: Generalized pseudo-Rankine model for tensile damage. *Int. J. Solids Structures*, 38(4), 491-518 (2001a), 519-546 (2001b).

I. Carol, E. Rizzi and K. Willam. An 'extended' volumetric/deviatoric formulation of anisotropic damage based on a pseudo-log rate. *Submitted* (2001c).

J. P. Cordebois and F. Sidoroff. "Endommagement anisotrope in élasticité et plasticité. *J. de Mécanique Théorique et Appliquée*, No. Special 1982: 45–60 (1982).

S. C. Cowin. The relationship between the elasticity tensor and the fabric tensor. *Mechanics of Materials*, 4: 137–147 (1985).

G. Del Piero. Some properties of the set of fourth-order tensors, with application to elasticity. *J. of Elasticity*, 9(3): 245–261 (1979).

J. W. Dougill. On stable progressively fracturing solids. *J. of Applied Mathematics and Physics*, 27: 423–437 (1976).

M. E. Gurtin. *An Introduction to Continuum Mechanics*. Academic Press (1981).

M. E. Gurtin and K. Spear. On the relationship between the logarithmic strain rate and the stretching tensor. *Int. J. Solids Structures*, 19(5): 437–444 (1983).

A. Hoger and D. E. Carlson. On the derivative of the square root of a tensor and Guo's rate theorems. *J. of Elasticity*, 14: 329–336 (1984).

A. Hoger. The material time derivative of the logarithmic strain. *Int. J. Solids Structures*, 22(9): 1019–1032 (1986).

K. Y. Lam and J. M. Zhang. On damage effect tensors of anisotropic solids. *Zeitschrift für angewandte Mathematik und Mechanik ZAMM*, 75(1): 51–59 (1995).

J. E. Marsden and T. J. R. Hughes. *Mathematical Theory of Elasticity*. Prentice-Hall (1983).

G. Maier and T. Hueckel. Nonassociated and coupled flow rules of elastoplasticity for rock-like materials. *Int. J. Rock Mech. and Mining Sciences*, 16: 77–92 (1979).

R. W. Ogden. *Non-Linear Elastic Deformation*. Wiley (1984).

M. Ortiz. A constitutive theory for the inelastic behavior of concrete. *Mechanics of Materials*, 4: 67–93 (1985).

M. Šilhavý. *The Mechanics and Thermodynamics of Continuous Media*. Springer-Verlag (1997).

K. C. Valanis. A theory of damage in brittle materials. *Engineering Fracture Mechanics*, 36: 403–416 (1990).

G. Z. Voyiadjis and T. Park. Anisotropic damage effect tensors for the symmetrization of the effective stress tensor. *J. of Applied Mechanics*, 64: 106–110 (1997).

Q.-S. Zheng and J. Betten. On damage effective stress and equivalence hypothesis. *Int. J. of Damage Mechanics*, 5: 219–240 (1996).

P. K. Zysset and A. Curnier. An alternative model for anisotropic elasticity based on fabric tensors. *Mechanics of Materials*, 21: 243–250 (1995).

Appendix A. Notation and definitions

A.1. Tensor algebra

Rephrasing the notation adopted by Del Piero (1979), let $\mathcal{V}$ be a three-dimensional vector space over the reals, *Lin* the set of all linear mappings of $\mathcal{V}$ into itself (second-order tensors), *LinLin* the set of all linear mappings of *Lin* into itself (fourth-order tensors). Let also *Sym* be the subset of *Lin* constituted by symmetric second-order tensors, namely

$$\mathbf{A}^{\mathrm{T}} = \mathbf{A} \ , \quad \forall \ \mathbf{A} \in Lin \ , \tag{A.1}$$

where $^{\mathrm{T}}$ denotes the transpose operation, that is, for any $\mathbf{A} \in Lin$

$$\mathbf{v} \cdot \mathbf{A}^{\mathrm{T}} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{A} \cdot \mathbf{v} \ , \quad \forall \ \mathbf{u}, \mathbf{v} \in \mathcal{V} \ . \tag{A.2}$$

Here, symbol '$\cdot$' denotes either the application of the linear transformation of $\mathcal{V}$ in $\mathcal{V}$, or the inner product in $\mathcal{V}$, or, more generally, the inner product with a single contraction. Component-wise, in a three-dimensional orthonormal basis

$$\mathbf{u} \cdot \mathbf{A} \cdot \mathbf{v} = u_i \, A_{ij} \, v_j \ , \quad i, j = 1, 2, 3, \tag{A.3}$$

where summation notation convention on repeated indices is implicitly assumed. Then, $\mathbf{A} \in Sym$ if $A_{ij} = A_{ji}$, $i, j = 1, 2, 3$. $\mathbf{A}$ is Positive Definite (PD) if $\mathbf{u} \cdot \mathbf{A} \cdot \mathbf{u} > 0$ for any $\mathbf{u} \neq \mathbf{0} \in \mathcal{V}$.

Symbol ':' denotes on the other hand either the application of the linear transformation of *Lin* in *Lin*, namely for $\mathbf{A}, \mathbf{B} \in Lin$, $\mathbb{C} \in LinLin$,

$$\mathbf{B} = \mathbb{C} : \mathbf{A} \ , \qquad B_{ij} = C_{ijkl} \, A_{kl} \ , \tag{A.4}$$

or the inner product in *Lin*,

$$\mathbf{A} : \mathbf{B} = tr(\mathbf{A} \cdot \mathbf{B}^{\mathrm{T}}) = tr(\mathbf{A}^{\mathrm{T}} \cdot \mathbf{B}) = A_{ij} \, B_{ij} \ , \quad \forall \ \mathbf{A}, \mathbf{B} \in Lin \ , \tag{A.5}$$

or, more generally, the inner product with double contraction, e.g. for $\mathbb{A}, \mathbb{B}, \mathbb{C} \in LinLin$

$$\mathbb{C} = \mathbb{A} : \mathbb{B} \ , \qquad C_{ijkl} = A_{ijrs} \, B_{rskl} \ . \tag{A.6}$$

In eqn (A.5), *tr* denotes the trace operator:

$$tr \, \mathbf{A} = \mathbf{I} : \mathbf{A} \ , \quad \forall \ \mathbf{A} \in Lin \ , \tag{A.7}$$

where $\mathbf{I}$ is the identity in *Lin* (second-order identity tensor), that is $\mathbf{I} \cdot \mathbf{u} = \mathbf{u}$, $\forall \ \mathbf{u} \in \mathcal{V}$, with cartesian components δ_{ij}, where δ_{ij} is the Kronecker delta ($\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$).

The transpose operation also applies to fourth-order tensors, namely for $\mathbb{C} \in LinLin$:

$$\mathbf{B} : \mathbb{C}^{\mathrm{T}} : \mathbf{A} = \mathbf{A} : \mathbb{C} : \mathbf{B} \,, \quad \forall \;\; \mathbf{A}, \mathbf{B} \in Lin \,. \tag{A.8}$$

When $\mathbb{C}^{\mathrm{T}}{=}\mathbb{C}$, the fourth-order tensor $\mathbb{C}$ is said to possess the major symmetry, so that in terms of cartesian components $C_{ijkl}{=}C_{klij}$. $\mathbb{C}$ is PD provided that $\mathbf{A}{:}\mathbb{C}{:}\mathbf{A}{>}0$ for any $\mathbf{A} \neq \mathbf{0} \in Lin$.

The dyadic product of two vectors or two second-order tensors is denoted by symbol $\otimes$ and defined as follows:

$$\begin{aligned} (\mathbf{u} \otimes \mathbf{v}) \cdot \mathbf{z} &= (\mathbf{v} \cdot \mathbf{z}) \; \mathbf{u} \,, \quad && \forall \;\; \mathbf{u}, \mathbf{v}, \mathbf{z} \in \mathcal{V} \,, \\ (\mathbf{A} \otimes \mathbf{B}) : \mathbf{C} &= (\mathbf{B} : \mathbf{C}) \; \mathbf{A} \,, \quad && \forall \;\; \mathbf{A}, \mathbf{B}, \mathbf{C} \in Lin \,, \end{aligned} \tag{A.9}$$

so that, in cartesian components:

$$(\mathbf{u} \otimes \mathbf{v})_{ij} = u_i \; v_j \,, \qquad (\mathbf{A} \otimes \mathbf{B})_{ijkl} = A_{ij} \; B_{kl} \,. \tag{A.10}$$

Two further outer products may also be introduced, namely $\overline{\otimes}$ and $\underline{\otimes}$, defined as

$$\left. \begin{aligned} (\mathbf{A} \,\overline{\otimes}\, \mathbf{B}) : \mathbf{C} &= \mathbf{A} \cdot \mathbf{C} \;\; \cdot \mathbf{B}^{\mathrm{T}}, \\ (\mathbf{A} \,\underline{\otimes}\, \mathbf{B}) : \mathbf{C} &= \mathbf{A} \cdot \mathbf{C}^{\mathrm{T}} \cdot \mathbf{B}^{\mathrm{T}}, \end{aligned} \right. \quad \forall \;\; \mathbf{A}, \mathbf{B}, \mathbf{C} \in Lin \,, \tag{A.11}$$

so that, in cartesian components

$$(\mathbf{A} \,\overline{\otimes}\, \mathbf{B})_{ijkl} = A_{ik} \; B_{jl} \,, \qquad (\mathbf{A} \,\underline{\otimes}\, \mathbf{B})_{ijkl} = A_{il} \; B_{jk} \,. \tag{A.12}$$

Symbol $\overline{\otimes}$ corresponds to the dyadic product $\boxtimes$ introduced by Del Piero (1979). The symmetrized dyadic product, $\overline{\underline{\otimes}}$, is also defined as

$$(\mathbf{A} \,\overline{\underline{\otimes}}\, \mathbf{B}) : \mathbf{C} = \mathbf{A} \cdot \mathbf{C}^s \cdot \mathbf{B}^{\mathrm{T}} \,, \quad \forall \;\; \mathbf{A}, \mathbf{B}, \mathbf{C} \in Lin \,, \tag{A.13}$$

where $\mathbf{C}^s \in Sym$ is the symmetric part of $\mathbf{C}$, that is

$$\mathbf{C}^s = \frac{1}{2} \left(\mathbf{C} + \mathbf{C}^{\mathrm{T}} \right) . \tag{A.14}$$

In other words,

$$\mathbf{A} \,\overline{\underline{\otimes}}\, \mathbf{B} = \frac{1}{2} \left(\mathbf{A} \,\overline{\otimes}\, \mathbf{B} + \mathbf{A} \,\underline{\otimes}\, \mathbf{B} \right) , \tag{A.15}$$

or, in cartesian components

$$(\mathbf{A} \,\overline{\underline{\otimes}}\, \mathbf{B})_{ijkl} = \frac{1}{2} \left(A_{ik} \; B_{jl} + A_{il} \; B_{jk} \right) . \tag{A.16}$$

Then, given the second-order identity tensor $\mathbf{I}$, the fourth-order identity tensor $\mathbb{I}$ is defined such that $\mathbb{I}{:}\mathbf{A}{=}\mathbf{A}$, $\forall \; \mathbf{A} \in Lin$, namely

$$\mathbb{I} = \mathbf{I} \,\overline{\otimes}\, \mathbf{I} \,, \tag{A.17}$$

and displays only major symmetry. The same happens for the transposition mapping $\mathbb{T}$ (Del Piero, 1979), which is defined such that $\mathbb{T}{:}\mathbf{A}{=}\mathbf{A}^{\mathrm{T}}$, $\forall\, \mathbf{A} \in Lin$, that is

$$\mathbb{T} = \mathbf{I} \underline{\otimes} \mathbf{I}\,. \tag{A.18}$$

A fourth-order tensor $\mathbb{A} \in LinLin$ is said to possess the first and/or the second minor symmetry respectively if

$$\mathbb{T} : \mathbb{A} = \mathbb{A} \quad \text{and/or} \quad \mathbb{A} : \mathbb{T} = \mathbb{A}\,, \tag{A.19}$$

so that, in cartesian components, $A_{ijkl}{=}A_{jikl}$ and/or $A_{ijkl}{=}A_{ijlk}$, respectively. Then, the fully symmetric fourth-order identity tensor $\mathbb{I}^s$ (denoted with $\mathbb{S}$ by Del Piero, 1979, since it defines the linear mapping of $\mathbf{A} \in Lin$ into its symmetric part, $\mathbb{I}^s{:}\mathbf{A}{=}\mathbf{A}^s \in Sym$) is defined as

$$\mathbb{I}^s = \frac{1}{2}\,(\mathbb{I} + \mathbb{T}) = \mathbf{I} \overline{\underline{\otimes}} \mathbf{I}\,, \quad I^s_{ijkl} = \frac{1}{2}\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{A.20}$$

and possesses not only the major but also both the minor symmetries. $\mathbb{I}^s$ maps any symmetric second-order tensor $\mathbf{A} \in Sym$ into itself.

A.2. Tensor derivatives

Definition of the derivatives of tensor functions may be found e.g. in Gurtin (1981), Chapter II, and Šilhavý (1997), Chapter 1. Rephrasing once more Del Piero (1979), the function $\mathbf{f}(\mathbf{A})$ is said to be differentiable at $\mathbf{A} \in Lin$ if there exists a tensor $\partial\mathbf{f}(\mathbf{A})/\partial\mathbf{A}$ such that

$$\mathbf{f}(\mathbf{A} + \mathbf{H}) = \mathbf{f}(\mathbf{A}) + \frac{\partial\mathbf{f}(\mathbf{A})}{\partial\mathbf{A}} : \mathbf{H} + \mathbf{o}(\mathbf{H}) \tag{A.21}$$

for any $\mathbf{H}$ in the neighborhood of $\mathbf{0} \in Lin$ with $\lim_{|\mathbf{H}|\to 0} \mathbf{o}(\mathbf{H})/|\mathbf{H}|{=}\mathbf{0}$, where $|\mathbf{H}|{=}\sqrt{\mathbf{H}{:}\mathbf{H}}$. The value of $\partial\mathbf{f}(\mathbf{A})/\partial\mathbf{A}$ on $\mathbf{H}$ is $(\partial\mathbf{f}(\mathbf{A})/\partial\mathbf{A}){:}\mathbf{H}$. However, $\partial\mathbf{f}(\mathbf{A})/\partial\mathbf{A}$ is said to be the derivative of $\mathbf{f}(\mathbf{A})$ at $\mathbf{A} \in Lin$, so that any reference to the argument $\mathbf{H}$ can be omitted (Del Piero, 1979, eqn (5.4), p. 251).

Then, considering e.g. $\mathbf{f}(\mathbf{A}){=}\mathbf{A}$, for any $\mathbf{A} \in Lin$, the value of $\partial\mathbf{A}/\partial\mathbf{A}$ on $\mathbf{H} \in Lin$ is $\mathbf{H}$, namely

$$\frac{\partial\mathbf{A}}{\partial\mathbf{A}} = \mathbb{I}\,, \quad \mathbf{A} \in Lin\,, \tag{A.22}$$

since this derivative defines the identity transformation of Lin in Lin. However, if $\mathbf{A} \in Sym$, the derivative should be expressed by the identity mapping of Sym in Sym, that is $\mathbb{I}^s$:[3]

$$\frac{\partial\mathbf{A}}{\partial\mathbf{A}} = \mathbb{I}^s, \quad \mathbf{A} \in Sym\,. \tag{A.23}$$

[3] This dstinction is not really crucial, since $\mathbb{I}{:}\mathbf{A}{=}\mathbb{I}^s{:}\mathbf{A}{=}\mathbf{A}$ if $\mathbf{A} \in Sym$, if not for the fact that $\mathbb{I}^s$ automatically embeds the minor symmetries.

Appendix B. Useful tensor derivatives

Derivatives of general tensor functions may be expanded through spectral representation following Ogden (1984), p. 162-163, Carlson and Hoger (1986b), Hoger (1986). In such cases, however, one may have to deal with the problem of repeated eigenvalues. Instead, in some simple cases as the ones reported below, the derivatives may be found in compact form by direct differentiation. In particular, this is the notable case of the derivative of the square root function, of primary concern in the paper, where a compact polynomial expression with invariant coefficients is directly available from Hoger and Carlson (1984).

B.1. Derivatives of the inverse and square

Let $\mathbf{A}$ be an invertible second-order tensor $\in Lin$. Then, differentiation of the identity $\mathbf{A}\cdot\mathbf{A}^{-1}=\mathbf{I}$ leads to (Del Piero, 1979; Gurtin, 1981, p. 28; Marsden and Hughes, 1983, p. 222; Šilhavý, 1997, p. 11):

$$\frac{\partial \mathbf{A}^{-1}}{\partial \mathbf{A}} = -\mathbf{A}^{-1}\,\overline{\otimes}\,\mathbf{A}^{-\mathrm{T}}\,, \quad \mathbf{A}\in Lin\,, \tag{B.1}$$

whereas, if $\mathbf{A}$ is symmetric, one should write

$$\frac{\partial \mathbf{A}^{-1}}{\partial \mathbf{A}} = -\mathbf{A}^{-1}\,\underline{\overline{\otimes}}\,\mathbf{A}^{-1}\,, \quad \mathbf{A}\in Sym\,, \tag{B.2}$$

so that, from an algebraic point of view, for any invertible $\mathbf{A}\in Lin$:

$$(\mathbf{A}\,\overline{\otimes}\,\mathbf{A}^{\mathrm{T}})^{-1} = \mathbf{A}^{-1}\,\overline{\otimes}\,\mathbf{A}^{-\mathrm{T}}\,, \qquad (\mathbf{A}\,\underline{\overline{\otimes}}\,\mathbf{A})^{-1} = \mathbf{A}^{-1}\,\underline{\overline{\otimes}}\,\mathbf{A}^{-1}\,. \tag{B.3}$$

Considering now the derivative of $\mathbf{f}(\mathbf{A})=\mathbf{A}^2$ one also gets (Del Piero, 1979; Gurtin, 1981, p. 22, Šilhavý, 1997, p. 11):

$$\begin{aligned} \frac{\partial \mathbf{A}^2}{\partial \mathbf{A}} &= \mathbf{A}\,\overline{\otimes}\,\mathbf{I} + \mathbf{I}\,\overline{\otimes}\,\mathbf{A}^{\mathrm{T}}, \quad && \mathbf{A}\in Lin\,, \\ \frac{\partial \mathbf{A}^2}{\partial \mathbf{A}} &= \mathbf{A}\,\underline{\overline{\otimes}}\,\mathbf{I} + \mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{A}\,, \quad && \mathbf{A}\in Sym\,. \end{aligned} \tag{B.4}$$

More generally, for any positive integer n (Šilhavý, 1997, p. 11):

$$\begin{aligned} \frac{\partial \mathbf{A}^n}{\partial \mathbf{A}} &= \sum_{k=0}^{n-1} \mathbf{A}^k\,\overline{\otimes}\,(\mathbf{A}^{\mathrm{T}})^{n-k-1}, \quad && \mathbf{A}\in Lin\,, \\ \frac{\partial \mathbf{A}^n}{\partial \mathbf{A}} &= \sum_{k=0}^{n-1} \mathbf{A}^k\,\underline{\overline{\otimes}}\,\mathbf{A}^{n-k-1}, \quad && \mathbf{A}\in Sym\,. \end{aligned} \tag{B.5}$$

B.2. Derivatives of the principal invariants

Now, let $\mathbf{A}$ be a second-order tensor $\in Lin$. The Cayley-Hamilton theorem applied to $\mathbf{A}$ poses

$$\mathbf{A}^3 - {}^{\mathrm{A}}I_1\ \mathbf{A}^2 + {}^{\mathrm{A}}I_2\ \mathbf{A} - {}^{\mathrm{A}}I_3\ \mathbf{I} = \mathbf{0}\ , \tag{B.6}$$

where ${}^{\mathrm{A}}I_1$, ${}^{\mathrm{A}}I_2$, ${}^{\mathrm{A}}I_3$ are the principal invariants of $\mathbf{A}$, namely (see e.g. Marsden and Hughes, 1983, p. 220-221):

$$\begin{aligned} {}^{\mathrm{A}}I_1 &= tr\,\mathbf{A}\ , \\ {}^{\mathrm{A}}I_2 &= \frac{1}{2}\,(tr^2\,\mathbf{A} - tr\,\mathbf{A}^2)\ , \\ {}^{\mathrm{A}}I_3 &= det\,\mathbf{A} = \frac{1}{3}\,tr\,\mathbf{A}^3 + \frac{1}{6}\,tr^3\,\mathbf{A} - \frac{1}{2}\,tr\,\mathbf{A}\ \ tr\,\mathbf{A}^2\ . \end{aligned} \tag{B.7}$$

The derivatives of the principal invariants of $\mathbf{A}$ are expressed by the following second-order tensors (Carlson and Hoger, 1986a; Šilhavý, 1997, p. 12):

$$\begin{aligned} \frac{\partial\,{}^{\mathrm{A}}I_1}{\partial \mathbf{A}} &= \mathbf{I}\ , \\ \frac{\partial\,{}^{\mathrm{A}}I_2}{\partial \mathbf{A}} &= {}^{\mathrm{A}}I_1\ \mathbf{I} - \mathbf{A}^{\mathrm{T}}\ , \\ \frac{\partial\,{}^{\mathrm{A}}I_3}{\partial \mathbf{A}} &= (\mathbf{A}^2 - {}^{\mathrm{A}}I_1\ \mathbf{A} + {}^{\mathrm{A}}I_2\ \mathbf{I})^{\mathrm{T}}\ . \end{aligned} \tag{B.8}$$

Also, if the second-order tensor $\mathbf{A}$ is invertible, the following expressions may be recalled (see Marsden and Hughes, 1983, p. 221; Carlson and Hoger, 1986a; Šilhavý, 1997, p. 12):

$${}^{\mathrm{A}}I_2 = det\,\mathbf{A}\ \ tr\,\mathbf{A}^{-1}\ , \tag{B.9}$$

and

$$\frac{\partial\,{}^{\mathrm{A}}I_3}{\partial \mathbf{A}} = det\,\mathbf{A}\ \ \mathbf{A}^{-\mathrm{T}}\ . \tag{B.10}$$

B.3. Derivative of the square root

Finally, consider the square root function $\mathbf{B}=\mathbf{f}(\mathbf{A})=\mathbf{A}^{1/2}$, where $\mathbf{A}$ and $\mathbf{B}$ are positive definite second-order tensors $\in Sym$. Considering the derivative of the inverse function $\mathbf{A}(\mathbf{B})$ it has been seen from eqn $(\text{B}.4)_2$ that $\partial\mathbf{B}^2/\partial\mathbf{B}=\mathbf{B}\,\underline{\overline{\otimes}}\,\mathbf{I}+\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{B}$. Then, from an algebraic point of view, the inverse of this fourth-order tensor should express the derivative of the square root function $\mathbf{B}(\mathbf{A})$, which exists (Gurtin, 1981, p. 23), and has been obtained by Hoger and Carlson (1984) by solving the tensor equation arising from the direct differentiation of the identity $\mathbf{B}\cdot\mathbf{B}=\mathbf{A}$:

$$
\begin{aligned}
\frac{\partial\mathbf{B}}{\partial\mathbf{A}} &= (\mathbf{B}\,\underline{\overline{\otimes}}\,\mathbf{I}+\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{B})^{-1}\\
&= \frac{1}{2\,{}^{\mathrm{B}}\!I_3\,({}^{\mathrm{B}}\!I_1\,{}^{\mathrm{B}}\!I_2-{}^{\mathrm{B}}\!I_3)}\Big[\ {}^{\mathrm{B}}\!I_1\,\mathbf{B}^2\,\underline{\overline{\otimes}}\,\mathbf{B}^2\\
&\qquad -{}^{\mathrm{B}}\!I_1^{\;2}\,(\mathbf{B}^2\,\underline{\overline{\otimes}}\,\mathbf{B}+\mathbf{B}\,\underline{\overline{\otimes}}\,\mathbf{B}^2)\\
&\qquad +({}^{\mathrm{B}}\!I_1\,{}^{\mathrm{B}}\!I_2-{}^{\mathrm{B}}\!I_3)\,(\mathbf{B}^2\,\underline{\overline{\otimes}}\,\mathbf{I}+\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{B}^2)\\
&\qquad +({}^{\mathrm{B}}\!I_1^{\;3}+{}^{\mathrm{B}}\!I_3)\,\mathbf{B}\,\underline{\overline{\otimes}}\,\mathbf{B}\\
&\qquad -{}^{\mathrm{B}}\!I_1^{\;2}\,{}^{\mathrm{B}}\!I_2\,(\mathbf{B}\,\underline{\overline{\otimes}}\,\mathbf{I}+\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{B})\\
&\qquad +\Big({}^{\mathrm{B}}\!I_1^{\;2}\,{}^{\mathrm{B}}\!I_3+{}^{\mathrm{B}}\!I_2\,({}^{\mathrm{B}}\!I_1\,{}^{\mathrm{B}}\!I_2-{}^{\mathrm{B}}\!I_3)\Big)\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}\ \Big],
\end{aligned}
\tag{B.11}
$$

Notice that ${}^{\mathrm{B}}\!I_3\,({}^{\mathrm{B}}\!I_1\,{}^{\mathrm{B}}\!I_2-{}^{\mathrm{B}}\!I_3)>0$ since second-order tensor $\mathbf{B}\in Sym$ is positive definite:

$$
\begin{aligned}
{}^{\mathrm{B}}\!I_1\,{}^{\mathrm{B}}\!I_2-{}^{\mathrm{B}}\!I_3 &= \frac{1}{3}\,\left(tr^3\,\mathbf{B}-tr\,\mathbf{B}^3\right)\\
&= det\,\mathbf{B}\,\left(tr\,\mathbf{B}\ \ tr\,\mathbf{B}^{-1}-1\right)\\
&= B_1^2(B_2+B_3)+B_2^2(B_1+B_3)+B_3^2(B_1+B_2)+2\,B_1B_2B_3\,,\\
&= (B_1+B_2)\,(B_2+B_3)\,(B_3+B_1)\,,
\end{aligned}
\tag{B.12}
$$

where B_1, B_2, B_3 are the (real and positive) principal values of $\mathbf{B}$ that can be obtained from the characteristic equation associated to the Cayley-Hamilton theorem (B.6) applied to $\mathbf{B}$.

Appendix C. Table of tensor algebra operations

In the following four boxes we list two series of tensor algebra operations; the first involves the symmetrized dyadic product $\underline{\overline{\otimes}}$, the second concerns the dyadic product $\overline{\otimes}$. Most of the algebraic operations have been used for the derivations presented in text; the other ones are listed for the sake of completeness. Tensors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are all second-order tensors $\in Lin$, while $\mathbf{I}$ is the identity in Lin. The symbols used are defined in Appendix A.1.

Box 1. Tensor operations with outer product $\overline{\underline{\otimes}}$.

$$
\begin{aligned}
\mathbf{C}\cdot(\mathbf{A}\,\overline{\underline{\otimes}}\,\mathbf{B}) &= (\mathbf{C}\cdot\mathbf{A})\,\overline{\underline{\otimes}}\,\mathbf{B}\\
(\mathbf{A}\,\overline{\underline{\otimes}}\,\mathbf{B})\cdot\mathbf{D} &= \frac{1}{2}\Big(\mathbf{A}\,\overline{\otimes}\,(\mathbf{B}\cdot\mathbf{D}) + (\mathbf{A}\cdot\mathbf{D})\,\underline{\otimes}\,\mathbf{B}\Big)\\
\mathbf{C}\cdot(\mathbf{A}\,\overline{\underline{\otimes}}\,\mathbf{B})\cdot\mathbf{D} &= \frac{1}{2}\Big((\mathbf{C}\cdot\mathbf{A})\,\overline{\otimes}\,(\mathbf{B}\cdot\mathbf{D}) + (\mathbf{C}\cdot\mathbf{A}\cdot\mathbf{D})\,\underline{\otimes}\,\mathbf{B}\Big)
\end{aligned}
\tag{C.1}
$$

$$
\begin{aligned}
\mathbf{C}\cdot(\mathbf{A}\,\overline{\underline{\otimes}}\,\mathbf{B}+\mathbf{B}\,\overline{\underline{\otimes}}\,\mathbf{A}) &= (\mathbf{C}\cdot\mathbf{A})\,\overline{\underline{\otimes}}\,\mathbf{B} + (\mathbf{C}\cdot\mathbf{B})\,\overline{\underline{\otimes}}\,\mathbf{A}\\
(\mathbf{A}\,\overline{\underline{\otimes}}\,\mathbf{B}+\mathbf{B}\,\overline{\underline{\otimes}}\,\mathbf{A})\cdot\mathbf{D} &= \frac{1}{2}\Big(\mathbf{A}\,\overline{\otimes}\,(\mathbf{B}\cdot\mathbf{D}) + (\mathbf{B}\cdot\mathbf{D})\,\underline{\otimes}\,\mathbf{A}\\
&\quad + \mathbf{B}\,\overline{\otimes}\,(\mathbf{A}\cdot\mathbf{D}) + (\mathbf{A}\cdot\mathbf{D})\,\underline{\otimes}\,\mathbf{B}\Big)\\
\mathbf{C}\cdot(\mathbf{A}\,\overline{\underline{\otimes}}\,\mathbf{B}+\mathbf{B}\,\overline{\underline{\otimes}}\,\mathbf{A})\cdot\mathbf{D} &= \frac{1}{2}\Big((\mathbf{C}\cdot\mathbf{A})\,\overline{\otimes}\,(\mathbf{B}\cdot\mathbf{D}) + (\mathbf{C}\cdot\mathbf{B}\cdot\mathbf{D})\,\underline{\otimes}\,\mathbf{A}\\
&\quad + (\mathbf{C}\cdot\mathbf{B})\,\overline{\otimes}\,(\mathbf{A}\cdot\mathbf{D}) + (\mathbf{C}\cdot\mathbf{A}\cdot\mathbf{D})\,\underline{\otimes}\,\mathbf{B}\Big)
\end{aligned}
\tag{C.2}
$$

$$
\begin{aligned}
\mathbf{C}:(\mathbf{A}\,\overline{\underline{\otimes}}\,\mathbf{B}) &= (\mathbf{A}^{\mathrm{T}}\cdot\mathbf{C}\cdot\mathbf{B})^{s}\\
(\mathbf{A}\,\overline{\underline{\otimes}}\,\mathbf{B}):\mathbf{D} &= \mathbf{A}\cdot\mathbf{D}^{s}\cdot\mathbf{B}^{\mathrm{T}}\\
\mathbf{C}:(\mathbf{A}\,\overline{\underline{\otimes}}\,\mathbf{B}):\mathbf{D} &= tr(\mathbf{C}^{\mathrm{T}}\cdot\mathbf{A}\cdot\mathbf{D}^{s}\cdot\mathbf{B}^{\mathrm{T}})
\end{aligned}
\tag{C.3}
$$

$$
\begin{aligned}
\mathbf{C}:(\mathbf{A}\,\overline{\underline{\otimes}}\,\mathbf{B}+\mathbf{B}\,\overline{\underline{\otimes}}\,\mathbf{A}) &= 2\,(\mathbf{A}^{\mathrm{T}}\cdot\mathbf{C}^{s}\cdot\mathbf{B})^{s}\\
(\mathbf{A}\,\overline{\underline{\otimes}}\,\mathbf{B}+\mathbf{B}\,\overline{\underline{\otimes}}\,\mathbf{A}):\mathbf{D} &= 2\,(\mathbf{A}\cdot\mathbf{D}^{s}\cdot\mathbf{B}^{\mathrm{T}})^{s}\\
\mathbf{C}:(\mathbf{A}\,\overline{\underline{\otimes}}\,\mathbf{B}+\mathbf{B}\,\overline{\underline{\otimes}}\,\mathbf{A}):\mathbf{D} &= 2\,tr(\mathbf{C}^{s}\cdot\mathbf{A}\cdot\mathbf{D}^{s}\cdot\mathbf{B}^{\mathrm{T}})
\end{aligned}
\tag{C.4}
$$

$$
\begin{aligned}
&(\mathbf{A}\,\overline{\underline{\otimes}}\,\mathbf{B}) : (\mathbf{C}\,\overline{\underline{\otimes}}\,\mathbf{D}) =\\
&\qquad = \frac{1}{2}\Big((\mathbf{A}\cdot\mathbf{C})\,\overline{\underline{\otimes}}\,(\mathbf{B}\cdot\mathbf{D}) + (\mathbf{A}\cdot\mathbf{D})\,\overline{\underline{\otimes}}\,(\mathbf{B}\cdot\mathbf{C})\Big)\\
&(\mathbf{A}\,\overline{\underline{\otimes}}\,\mathbf{B}+\mathbf{B}\,\overline{\underline{\otimes}}\,\mathbf{A}) : (\mathbf{C}\,\overline{\underline{\otimes}}\,\mathbf{D}) =\\
&\qquad = \frac{1}{2}\Big((\mathbf{A}\cdot\mathbf{C})\,\overline{\underline{\otimes}}\,(\mathbf{B}\cdot\mathbf{D}) + (\mathbf{B}\cdot\mathbf{D})\,\overline{\underline{\otimes}}\,(\mathbf{A}\cdot\mathbf{C})\\
&\qquad\quad + (\mathbf{B}\cdot\mathbf{C})\,\overline{\underline{\otimes}}\,(\mathbf{A}\cdot\mathbf{D}) + (\mathbf{A}\cdot\mathbf{D})\,\overline{\underline{\otimes}}\,(\mathbf{B}\cdot\mathbf{C})\Big)\\
&(\mathbf{A}\,\overline{\underline{\otimes}}\,\mathbf{B}+\mathbf{B}\,\overline{\underline{\otimes}}\,\mathbf{A}) : (\mathbf{C}\,\overline{\underline{\otimes}}\,\mathbf{D}+\mathbf{D}\,\overline{\underline{\otimes}}\,\mathbf{C}) =\\
&\qquad = (\mathbf{A}\cdot\mathbf{C})\,\overline{\underline{\otimes}}\,(\mathbf{B}\cdot\mathbf{D}) + (\mathbf{B}\cdot\mathbf{D})\,\overline{\underline{\otimes}}\,(\mathbf{A}\cdot\mathbf{C})\\
&\qquad + (\mathbf{B}\cdot\mathbf{C})\,\overline{\underline{\otimes}}\,(\mathbf{A}\cdot\mathbf{D}) + (\mathbf{A}\cdot\mathbf{D})\,\overline{\underline{\otimes}}\,(\mathbf{B}\cdot\mathbf{C})
\end{aligned}
\tag{C.5}
$$

$$
\begin{aligned}
(\mathbf{A}\,\overline{\underline{\otimes}}\,\mathbf{B}):(\mathbf{C}\,\overline{\underline{\otimes}}\,\mathbf{C}) &= (\mathbf{A}\cdot\mathbf{C})\,\overline{\underline{\otimes}}\,(\mathbf{B}\cdot\mathbf{C})\\
(\mathbf{A}\,\overline{\underline{\otimes}}\,\mathbf{A}):(\mathbf{C}\,\overline{\underline{\otimes}}\,\mathbf{C}) &= (\mathbf{A}\cdot\mathbf{C})\,\overline{\underline{\otimes}}\,(\mathbf{A}\cdot\mathbf{C})\\
(\mathbf{A}\,\overline{\underline{\otimes}}\,\mathbf{B}+\mathbf{B}\,\overline{\underline{\otimes}}\,\mathbf{A}):(\mathbf{C}\,\overline{\underline{\otimes}}\,\mathbf{C}) &= (\mathbf{A}\cdot\mathbf{C})\,\overline{\underline{\otimes}}\,(\mathbf{B}\cdot\mathbf{C}) + (\mathbf{B}\cdot\mathbf{C})\,\overline{\underline{\otimes}}\,(\mathbf{A}\cdot\mathbf{C})
\end{aligned}
\tag{C.6}
$$

Box 2. Tensor operations with outer product $\overline{\underline{\otimes}}$ involving the identity $\mathbf{I}$.

$$
\begin{aligned}
\mathbf{C}\cdot(\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I}) &= \mathbf{C}\,\overline{\underline{\otimes}}\,\mathbf{I} \\
(\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I})\cdot\mathbf{D} &= \frac{1}{2}\Big(\ \mathbf{I}\,\overline{\otimes}\,\mathbf{D}+\mathbf{D}\,\underline{\otimes}\,\mathbf{I}\Big) \\
\mathbf{C}\cdot(\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I})\cdot\mathbf{D} &= \frac{1}{2}\Big(\mathbf{C}\,\overline{\otimes}\,\mathbf{D}+(\mathbf{C}\cdot\mathbf{D})\,\underline{\otimes}\,\mathbf{I}\Big)
\end{aligned}
\tag{C.7}
$$

$$
\begin{aligned}
\mathbf{C}\cdot(\mathbf{A}\,\overline{\underline{\otimes}}\,\mathbf{I}+\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{A}) &= (\mathbf{C}\cdot\mathbf{A})\,\overline{\underline{\otimes}}\,\mathbf{I}+\mathbf{C}\,\overline{\underline{\otimes}}\,\mathbf{A} \\
(\mathbf{A}\,\overline{\underline{\otimes}}\,\mathbf{I}+\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{A})\cdot\mathbf{D} &= \frac{1}{2}\Big(\ \mathbf{A}\,\overline{\otimes}\,\mathbf{D}+\mathbf{D}\,\underline{\otimes}\,\mathbf{A} \\
&\quad +\ \ \mathbf{I}\,\overline{\otimes}\,(\mathbf{A}\cdot\mathbf{D})+(\mathbf{A}\cdot\mathbf{D})\,\underline{\otimes}\,\mathbf{I}\Big) \\
\mathbf{C}\cdot(\mathbf{A}\,\overline{\underline{\otimes}}\,\mathbf{I}+\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{A})\cdot\mathbf{D} &= \frac{1}{2}\Big((\mathbf{C}\cdot\mathbf{A})\,\overline{\otimes}\,\mathbf{D}+(\mathbf{C}\cdot\mathbf{D})\,\underline{\otimes}\,\mathbf{A} \\
&\quad +\ \ \mathbf{C}\,\overline{\otimes}\,(\mathbf{A}\cdot\mathbf{D})+(\mathbf{C}\cdot\mathbf{A}\cdot\mathbf{D})\,\underline{\otimes}\,\mathbf{I}\Big)
\end{aligned}
\tag{C.8}
$$

$$
\begin{aligned}
\mathbf{C}:(\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I}) &= \mathbf{C}^s \\
(\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I}):\mathbf{D} &= \mathbf{D}^s \\
\mathbf{C}:(\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I}):\mathbf{D} &= \mathbf{C}:\mathbf{D}^s = \mathbf{C}^s:\mathbf{D}
\end{aligned}
\tag{C.9}
$$

$$
\begin{aligned}
\mathbf{C}:(\mathbf{A}\,\overline{\underline{\otimes}}\,\mathbf{I}+\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{A}) &= 2\,(\mathbf{A}^{\mathrm{T}}\cdot\mathbf{C}^s)^s = \mathbf{A}^{\mathrm{T}}\cdot\mathbf{C}^s+\mathbf{C}^s\cdot\mathbf{A} \\
(\mathbf{A}\,\overline{\underline{\otimes}}\,\mathbf{I}+\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{A}):\mathbf{D} &= 2\,(\mathbf{A}\ \ \cdot\mathbf{D}^s)^s = \mathbf{A}\ \ \cdot\mathbf{D}^s+\mathbf{D}^s\cdot\mathbf{A}^{\mathrm{T}} \\
\mathbf{C}:(\mathbf{A}\,\overline{\underline{\otimes}}\,\mathbf{I}+\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{A}):\mathbf{D} &= 2\,tr(\mathbf{C}^s\cdot\mathbf{A}\cdot\mathbf{D}^s)
\end{aligned}
\tag{C.10}
$$

$$
\begin{aligned}
&(\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I})\ :\ (\mathbf{C}\,\overline{\underline{\otimes}}\,\mathbf{D}) = \frac{1}{2}\Big(\mathbf{C}\,\overline{\underline{\otimes}}\,\mathbf{D}+\mathbf{D}\,\overline{\underline{\otimes}}\,\mathbf{C}\Big) \\
&(\mathbf{A}\,\overline{\underline{\otimes}}\,\mathbf{I}+\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{A})\ :\ (\mathbf{C}\,\overline{\underline{\otimes}}\,\mathbf{D}) = \\
&\quad = \frac{1}{2}\Big((\mathbf{A}\cdot\mathbf{C})\overline{\underline{\otimes}}\,\mathbf{D}+\mathbf{D}\,\overline{\underline{\otimes}}\,(\mathbf{A}\cdot\mathbf{C})+\mathbf{C}\,\overline{\underline{\otimes}}\,(\mathbf{A}\cdot\mathbf{D})+(\mathbf{A}\cdot\mathbf{D})\,\overline{\underline{\otimes}}\,\mathbf{C}\Big) \\
&(\mathbf{A}\,\overline{\underline{\otimes}}\,\mathbf{I}+\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{A})\ :\ (\mathbf{C}\,\overline{\underline{\otimes}}\,\mathbf{D}+\mathbf{D}\,\overline{\underline{\otimes}}\,\mathbf{C}) = \\
&\quad = \quad (\mathbf{A}\cdot\mathbf{C})\overline{\underline{\otimes}}\,\mathbf{D}+\mathbf{D}\,\overline{\underline{\otimes}}\,(\mathbf{A}\cdot\mathbf{C})+\mathbf{C}\,\overline{\underline{\otimes}}\,(\mathbf{A}\cdot\mathbf{D})+(\mathbf{A}\cdot\mathbf{D})\,\overline{\underline{\otimes}}\,\mathbf{C}
\end{aligned}
\tag{C.11}
$$

$$
\begin{aligned}
&(\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I}):(\mathbf{C}\,\overline{\underline{\otimes}}\,\mathbf{C}) = \mathbf{C}\,\overline{\underline{\otimes}}\,\mathbf{C} \\
&(\mathbf{A}\,\overline{\underline{\otimes}}\,\mathbf{I}+\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{A}):(\mathbf{C}\,\overline{\underline{\otimes}}\,\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\,\overline{\underline{\otimes}}\,\mathbf{C}+\mathbf{C}\,\overline{\underline{\otimes}}\,(\mathbf{A}\cdot\mathbf{C}) \\
&(\mathbf{A}\,\overline{\underline{\otimes}}\,\mathbf{I}+\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{A}):(\mathbf{C}\,\overline{\underline{\otimes}}\,\mathbf{I}+\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{C}) = \\
&\qquad\qquad = (\mathbf{A}\cdot\mathbf{C})\,\overline{\underline{\otimes}}\,\mathbf{I}+\mathbf{I}\,\overline{\underline{\otimes}}\,(\mathbf{A}\cdot\mathbf{C})+\mathbf{C}\,\overline{\underline{\otimes}}\,\mathbf{A}+\mathbf{A}\,\overline{\underline{\otimes}}\,\mathbf{C}
\end{aligned}
\tag{C.12}
$$

Box 3. Tensor operations with outer product $\overline{\otimes}$.

$$\begin{aligned} \mathbf{C}\cdot(\mathbf{A}\,\overline{\otimes}\,\mathbf{B}) &= (\mathbf{C}\cdot\mathbf{A})\,\overline{\otimes}\,\mathbf{B} \\ (\mathbf{A}\,\overline{\otimes}\,\mathbf{B})\cdot\mathbf{D} &= \mathbf{A}\,\overline{\otimes}\,(\mathbf{B}\cdot\mathbf{D}) \\ \mathbf{C}\cdot(\mathbf{A}\,\overline{\otimes}\,\mathbf{B})\cdot\mathbf{D} &= (\mathbf{C}\cdot\mathbf{A})\,\overline{\otimes}\,(\mathbf{B}\cdot\mathbf{D}) \end{aligned} \tag{C.13}$$

$$\begin{aligned} \mathbf{C}\cdot(\mathbf{A}\,\overline{\otimes}\,\mathbf{B}+\mathbf{B}\,\overline{\otimes}\,\mathbf{A}) &= (\mathbf{C}\cdot\mathbf{A})\,\overline{\otimes}\,\mathbf{B}+(\mathbf{C}\cdot\mathbf{B})\,\overline{\otimes}\,\mathbf{A} \\ (\mathbf{A}\,\overline{\otimes}\,\mathbf{B}+\mathbf{B}\,\overline{\otimes}\,\mathbf{A})\cdot\mathbf{D} &= \mathbf{A}\,\overline{\otimes}\,(\mathbf{B}\cdot\mathbf{D})+\mathbf{B}\,\overline{\otimes}\,(\mathbf{A}\cdot\mathbf{D}) \\ \mathbf{C}\cdot(\mathbf{A}\,\overline{\otimes}\,\mathbf{B}+\mathbf{B}\,\overline{\otimes}\,\mathbf{A})\cdot\mathbf{D} &= (\mathbf{C}\cdot\mathbf{A})\,\overline{\otimes}\,(\mathbf{B}\cdot\mathbf{D})+(\mathbf{C}\cdot\mathbf{B})\,\overline{\otimes}\,(\mathbf{A}\cdot\mathbf{D}) \end{aligned} \tag{C.14}$$

$$\begin{aligned} \mathbf{C}:(\mathbf{A}\,\overline{\otimes}\,\mathbf{B}) &= \mathbf{A}^{\mathrm{T}}\cdot\mathbf{C}\cdot\mathbf{B} \\ (\mathbf{A}\,\overline{\otimes}\,\mathbf{B}):\mathbf{D} &= \mathbf{A}\cdot\mathbf{D}\cdot\mathbf{B}^{\mathrm{T}} \\ \mathbf{C}:(\mathbf{A}\,\overline{\otimes}\,\mathbf{B}):\mathbf{D} &= tr(\mathbf{C}^{\mathrm{T}}\cdot\mathbf{A}\cdot\mathbf{D}\cdot\mathbf{B}^{\mathrm{T}}) \end{aligned} \tag{C.15}$$

$$\begin{aligned} \mathbf{C}:(\mathbf{A}\,\overline{\otimes}\,\mathbf{B}+\mathbf{B}\,\overline{\otimes}\,\mathbf{A}) &= \mathbf{A}^{\mathrm{T}}\cdot\mathbf{C}\cdot\mathbf{B}+\mathbf{B}^{\mathrm{T}}\cdot\mathbf{C}\cdot\mathbf{A} \\ (\mathbf{A}\,\overline{\otimes}\,\mathbf{B}+\mathbf{B}\,\overline{\otimes}\,\mathbf{A}):\mathbf{D} &= \mathbf{A}\cdot\mathbf{D}\cdot\mathbf{B}^{\mathrm{T}}+\mathbf{B}\cdot\mathbf{D}\cdot\mathbf{A}^{\mathrm{T}} \\ \mathbf{C}:(\mathbf{A}\,\overline{\otimes}\,\mathbf{B}+\mathbf{B}\,\overline{\otimes}\,\mathbf{A}):\mathbf{D} &= tr(\mathbf{C}^{\mathrm{T}}\cdot\mathbf{A}\cdot\mathbf{D}\cdot\mathbf{B}^{\mathrm{T}}+\mathbf{C}^{\mathrm{T}}\cdot\mathbf{B}\cdot\mathbf{D}\cdot\mathbf{A}^{\mathrm{T}}) \end{aligned} \tag{C.16}$$

$$\begin{aligned} (\mathbf{A}\,\overline{\otimes}\,\mathbf{B}):(\mathbf{C}\,\overline{\otimes}\,\mathbf{D}) &= (\mathbf{A}\cdot\mathbf{C})\,\overline{\otimes}\,(\mathbf{B}\cdot\mathbf{D}) \\ (\mathbf{A}\,\overline{\otimes}\,\mathbf{B}+\mathbf{B}\,\overline{\otimes}\,\mathbf{A}):(\mathbf{C}\,\overline{\otimes}\,\mathbf{D}) &= (\mathbf{A}\cdot\mathbf{C})\,\overline{\otimes}\,(\mathbf{B}\cdot\mathbf{D})+(\mathbf{B}\cdot\mathbf{C})\,\overline{\otimes}\,(\mathbf{A}\cdot\mathbf{D}) \\ (\mathbf{A}\,\overline{\otimes}\,\mathbf{B}+\mathbf{B}\,\overline{\otimes}\,\mathbf{A}):(\mathbf{C}\,\overline{\otimes}\,\mathbf{D}+\mathbf{D}\,\overline{\otimes}\,\mathbf{C}) &= \\ &= (\mathbf{A}\cdot\mathbf{C})\,\overline{\otimes}\,(\mathbf{B}\cdot\mathbf{D})+(\mathbf{B}\cdot\mathbf{D})\,\overline{\otimes}\,(\mathbf{A}\cdot\mathbf{C}) \\ &+ (\mathbf{B}\cdot\mathbf{C})\,\overline{\otimes}\,(\mathbf{A}\cdot\mathbf{D})+(\mathbf{A}\cdot\mathbf{D})\,\overline{\otimes}\,(\mathbf{B}\cdot\mathbf{C}) \end{aligned} \tag{C.17}$$

$$\begin{aligned} (\mathbf{A}\,\overline{\otimes}\,\mathbf{B}):(\mathbf{C}\,\overline{\otimes}\,\mathbf{C}) &= (\mathbf{A}\cdot\mathbf{C})\,\overline{\otimes}\,(\mathbf{B}\cdot\mathbf{C}) \\ (\mathbf{A}\,\overline{\otimes}\,\mathbf{A}):(\mathbf{C}\,\overline{\otimes}\,\mathbf{C}) &= (\mathbf{A}\cdot\mathbf{C})\,\overline{\otimes}\,(\mathbf{A}\cdot\mathbf{C}) \\ (\mathbf{A}\,\overline{\otimes}\,\mathbf{B}+\mathbf{B}\,\overline{\otimes}\,\mathbf{A}):(\mathbf{C}\,\overline{\otimes}\,\mathbf{C}) &= (\mathbf{A}\cdot\mathbf{C})\,\overline{\otimes}\,(\mathbf{B}\cdot\mathbf{C})+(\mathbf{B}\cdot\mathbf{C})\,\overline{\otimes}\,(\mathbf{A}\cdot\mathbf{C}) \end{aligned} \tag{C.18}$$

Box 4. Tensor operations with outer product $\overline{\otimes}$ involving the identity $\mathbf{I}$.

$$
\begin{aligned}
\mathbf{C}\cdot(\mathbf{I}\,\overline{\otimes}\,\mathbf{I}) &= \mathbf{C}\,\overline{\otimes}\,\mathbf{I} \\
(\mathbf{I}\,\overline{\otimes}\,\mathbf{I})\cdot\mathbf{D} &= \mathbf{I}\,\overline{\otimes}\,\mathbf{D} \\
\mathbf{C}\cdot(\mathbf{I}\,\overline{\otimes}\,\mathbf{I})\cdot\mathbf{D} &= \mathbf{C}\,\overline{\otimes}\,\mathbf{D}
\end{aligned}
\tag{C.19}
$$

$$
\begin{aligned}
\mathbf{C}\cdot(\mathbf{A}\,\overline{\otimes}\,\mathbf{I}+\mathbf{I}\,\overline{\otimes}\,\mathbf{A}) &= (\mathbf{C}\cdot\mathbf{A})\,\overline{\otimes}\,\mathbf{I}+\mathbf{C}\,\overline{\otimes}\,\mathbf{A} \\
(\mathbf{A}\,\overline{\otimes}\,\mathbf{I}+\mathbf{I}\,\overline{\otimes}\,\mathbf{A})\cdot\mathbf{D} &= \mathbf{A}\,\overline{\otimes}\,\mathbf{D}+\mathbf{I}\,\overline{\otimes}\,(\mathbf{A}\cdot\mathbf{D}) \\
\mathbf{C}\cdot(\mathbf{A}\,\overline{\otimes}\,\mathbf{I}+\mathbf{I}\,\overline{\otimes}\,\mathbf{A})\cdot\mathbf{D} &= (\mathbf{C}\cdot\mathbf{A})\,\overline{\otimes}\,\mathbf{D}+\mathbf{C}\,\overline{\otimes}\,(\mathbf{A}\cdot\mathbf{D})
\end{aligned}
\tag{C.20}
$$

$$
\begin{aligned}
\mathbf{C}:(\mathbf{I}\,\overline{\otimes}\,\mathbf{I}) &= \mathbf{C} \\
(\mathbf{I}\,\overline{\otimes}\,\mathbf{I}):\mathbf{D} &= \mathbf{D} \\
\mathbf{C}:(\mathbf{I}\,\overline{\otimes}\,\mathbf{I}):\mathbf{D} &= \mathbf{C}:\mathbf{D}
\end{aligned}
\tag{C.21}
$$

$$
\begin{aligned}
\mathbf{C}:(\mathbf{A}\,\overline{\otimes}\,\mathbf{I}+\mathbf{I}\,\overline{\otimes}\,\mathbf{A}) &= \quad \mathbf{A}^{\mathrm{T}}\cdot\mathbf{C}+\mathbf{C}\ \cdot\mathbf{A} \\
(\mathbf{A}\,\overline{\otimes}\,\mathbf{I}+\mathbf{I}\,\overline{\otimes}\,\mathbf{A}):\mathbf{D} &= \quad \mathbf{A}\ \cdot\mathbf{D}+\mathbf{D}\ \cdot\mathbf{A}^{\mathrm{T}} \\
\mathbf{C}:(\mathbf{A}\,\overline{\otimes}\,\mathbf{I}+\mathbf{I}\,\overline{\otimes}\,\mathbf{A}):\mathbf{D} &= tr(\mathbf{C}^{\mathrm{T}}\cdot\mathbf{A}\ \cdot\mathbf{D}+\mathbf{C}^{\mathrm{T}}\cdot\mathbf{D}\cdot\mathbf{A}^{\mathrm{T}})
\end{aligned}
\tag{C.22}
$$

$$
\begin{aligned}
&(\mathbf{I}\,\overline{\otimes}\,\mathbf{I})\,:\,(\mathbf{C}\,\overline{\otimes}\,\mathbf{D}) = \mathbf{C}\,\overline{\otimes}\,\mathbf{D} \\
&(\mathbf{A}\,\overline{\otimes}\,\mathbf{I}+\mathbf{I}\,\overline{\otimes}\,\mathbf{A})\,:\,(\mathbf{C}\,\overline{\otimes}\,\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})\,\overline{\otimes}\,\mathbf{D}+\mathbf{C}\,\overline{\otimes}\,(\mathbf{A}\cdot\mathbf{D}) \\
&(\mathbf{A}\,\overline{\otimes}\,\mathbf{I}+\mathbf{I}\,\overline{\otimes}\,\mathbf{A})\,:\,(\mathbf{C}\,\overline{\otimes}\,\mathbf{D}+\mathbf{D}\,\overline{\otimes}\,\mathbf{C}) = \\
&\quad = (\mathbf{A}\cdot\mathbf{C})\,\overline{\otimes}\,\mathbf{D}+\mathbf{D}\,\overline{\otimes}\,(\mathbf{A}\cdot\mathbf{C})+\mathbf{C}\,\overline{\otimes}\,(\mathbf{A}\cdot\mathbf{D})+(\mathbf{A}\cdot\mathbf{D})\,\overline{\otimes}\,\mathbf{C}
\end{aligned}
\tag{C.23}
$$

$$
\begin{aligned}
&(\mathbf{I}\,\overline{\otimes}\,\mathbf{I}):(\mathbf{C}\,\overline{\otimes}\,\mathbf{C}) = \mathbf{C}\,\overline{\otimes}\,\mathbf{C} \\
&(\mathbf{A}\,\overline{\otimes}\,\mathbf{I}+\mathbf{I}\,\overline{\otimes}\,\mathbf{A}):(\mathbf{C}\,\overline{\otimes}\,\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\,\overline{\otimes}\,\mathbf{C}+\mathbf{C}\,\overline{\otimes}\,(\mathbf{A}\cdot\mathbf{C}) \\
&(\mathbf{A}\,\overline{\otimes}\,\mathbf{I}+\mathbf{I}\,\overline{\otimes}\,\mathbf{A}):(\mathbf{C}\,\overline{\otimes}\,\mathbf{I}+\mathbf{I}\,\overline{\otimes}\,\mathbf{C}) = \\
&\quad =(\mathbf{A}\cdot\mathbf{C})\,\overline{\otimes}\,\mathbf{I}+\mathbf{I}\,\overline{\otimes}\,(\mathbf{A}\cdot\mathbf{C})+\mathbf{C}\,\overline{\otimes}\,\mathbf{A}+\mathbf{A}\,\overline{\otimes}\,\mathbf{C}
\end{aligned}
\tag{C.24}
$$