Dual orthotropic damage-effect tensors with complementary structures

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Abstract

This paper deals with secant constitutive relations of orthotropic elastic damage based on the so-called damage-effect tensors, namely the fourth-order operators that define the linear transformations between nominal and effective stress and strain quantities. The damage-effect tensors are expressed by orthotropic representations in terms of symmetric second-order damage tensor variables. The paper provides a set of new dual orthotropic damage-effect tensors that possess complementary structures in the dual compliance- and stiffness-based derivations. More specifically, each orthotropic damage-effect tensor of the solution set possesses an inverse (its dual counterpart) that displays the structure of the (major) transpose of the tensor obtained by replacing the adopted second-order damage tensor variables with their inverses.

Keywords: Continuum Damage Mechanics, anisotropic damage, fourth-order damage-effect tensor, second-order damage tensor, tensor inverses.

⋆*Dedicated to Prof. Kaspar Willam on the occasion of his 60th birthday: Zum Geburtstag viel Glück!*

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1 Introduction

Stemming from the pioneering works by Kachanov (1958) and Rabotnov (1969), which introduced the original concepts of reduction of stress-carrying area and corresponding effective stress acting on the intact material between microcracks, Continuum Damage Mechanics (CDM) has reached nowadays a rather mature stage of development. Among other features, this includes the modeling of *anisotropic* elastic stiffness degradation in quasi-brittle materials such as concrete and rocks. These formulations are based on damage tensors of various orders (see the extensive reference lists provided e.g. in Zheng and Betten, 1996; Carol et al., 1994, 2001a,b; and in the relevant monographs on the subject, e.g. Lemaitre, 1992; Krajcinovic, 1995).

In a sequence of papers, the present authors have contributed to the topic with a proposal of a unified theoretical framework of elastic stiffness degradation and damage based on a loading surface (Carol et al., 1994), and with the formulation of constitutive models for anisotropic stiffness degradation in initially-isotropic materials (Carol et al., 2001a,b,c; Rizzi and Carol, 2002). The latter models are characterized by second-order symmetric damage tensor variables with evolution laws expressed in terms of a (non-holonomic) pseudo-logarithmic damage rate. The resulting secant elastic relations correspond to a restricted form of orthotropic material behavior (Valanis, 1990; Zysset and Curnier, 1995). Along the course of these investigations the request of seeking more general forms of orthotropic elastic degradation, together with the desire of preserving *duality* between alternative compliance- and stiffness-based derivations of the constitutive equations spontaneously arose.

In this respect, a major step in deriving the final secant compliance and stiffness relations of the elastic-damage model is the introduction of the so-called fourth-order *damage-effect tensors*, which, based on the underlying damage tensor variables, define the linear transformations between nominal and effective stress and strain quantities. Such typical approach of CDM is very convenient, at least from a methodological point of view, since it provides a modular structure of the constitutive equations. This is not only useful to derive and interpret the final secant relations of the pure elastic-damage model, but also makes eventually possible to include different material couplings and to enlarge the scope of the constitutive formulation to describe more realistic or different material behaviors, as e.g. elastoplastic, viscoelastic, initially anisotropic, etc.

Different damage-effect tensors have been proposed in the literature by Murakami and Ohno (1980), Cordebois and Sidoroff (1982), Betten (1983) and Lu and Chow (1990) (a summary of the different proposals on the subject is available e.g. in Lam and Zhang, 1995; Zheng and Betten, 1996; Voyiadjis and Park, 1997). The above-quoted formulations of the present authors also embed additional proposals of such damage-effect tensors. The present paper attempts a generalization of these previous propositions by providing a solution set of new orthotropic damage-effect tensors possessing *dual inverses with complementary structures* in the sense specified in the next section.

The main CDM procedures conceived to deduce the secant relations of an elastic-damage model are recalled first in Section 2. There, the definition of the damage-effect tensors is made and interpreted on the light of the properties sought in the present paper. Section 3 introduces and discusses the general orthotropic representations of fourth-order secant compliance and stiffness tensors, and damage-effect tensors, in terms of 'shear-like' and 'non-shear' coefficients and corresponding tensor addends. The problem statement is precisely addressed in Section 3.1, where the known existing solutions are listed as well. Section 4 constitutes the core of the paper and outlines the solution set of damage-effect tensors with complementary structures: Section 4.1 presents first solutions that can be directly guessed from the existing ones; Section 4.2 addresses the general problem, which is solved for the three 'shear-like' coefficients in Section 4.3; Section 4.4 collects new solution instances involving subsets of the 'non-shear' coefficients. The way these latter solutions are derived is commented in Section 5, which makes use of multiplication tables (Section 5.1) and of Sherman-Morrison's formulas (Section 5.2). A few final remarks and perspectives of the present study are outlined in the closing section.

For the ease of reading, important technical notions that are central to the derivations developed throughout the paper are collected in Appendices: Appendix A presents the details of the orthotropic representations, including the expressions of the fourth-order tensor components in matrix form and of the relevant coefficients; Appendix B gathers Rivlin's tensorial identities; Appendix C gives Sherman-Morrison's inversion formulas for one and two rank-one updates of a given non-singular fourth-order tensor; Appendix D includes an example of the multiplication tables that can be used to generate instances of the solution set. Finally, Appendix E reports the lengthy expressions of some of the factors entering *Solution (25)*, that is the more general solution instance found here, which lacks only two coefficients from the full orthotropic representation (one 'shear-like' and the other 'non-shear'). These expressions are given here not only for the sake of completeness but also because they help to show how the complete solution of the problem at hand turns out to be rather involved to be expressed in closed form.

Notation. Compact or index tensor notation is used throughout. Intrinsic summation convention on repeated indices is *not* adopted in the paper. Vectors and second-order tensors are identified by boldface characters, whereas fourth-order tensors are denoted by blackboard-bold fonts (e.g. $\mathbb{A}, \mathbb{C}, \mathbb{E}$). Superscript $^{\mathrm{T}}$ indicates the transpose operation (on first and second couple of indices for fourth-order tensors, i.e. componentwise $(A^{\mathrm{T}})_{ijkl}=A_{klij}$), while 'tr' is the trace operator. Symbols '$\cdot$' and ':' denote the inner products with single and double contraction. The dyadic product is indicated with '$\otimes$', whereas '$\underline{\overline{\otimes}}$' denotes the symmetrized outer product defined as $(\mathbf{A}\,\underline{\overline{\otimes}}\,\mathbf{B}){:}\mathbf{C}=\mathbf{A}\cdot\mathbf{C}^{s}\cdot\mathbf{B}^{\mathrm{T}}$, for any arbitrary second-order tensors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, where $\mathbf{C}^{s}=(\mathbf{C}+\mathbf{C}^{\mathrm{T}})/2$ is the symmetric part of $\mathbf{C}$; componentwise $(\mathbf{A}\,\underline{\overline{\otimes}}\,\mathbf{B})_{ijkl}=(A_{ik}B_{jl}+A_{il}B_{jk})/2$. $\mathbf{I}$ and $\mathbb{I}^{s}=\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}$ are respectively the second-order and symmetric fourth-order identity tensors. For more detailed definitions see e.g. Rizzi and Carol (2002), Appendix A.

2 Secant relations of elastic damage

In the simplest setting of pure elastic stiffness degradation, the material behavior can be characterized by a secant linear hyper-elastic constitutive law. At any state of the damage process, the *nominal* (small) strain tensor $\boldsymbol{\epsilon}$ and stress tensor $\boldsymbol{\sigma}$ are related by

$$\boldsymbol{\epsilon} = \mathbb{C}(\mathbb{C}_0, \boldsymbol{\mathcal{D}}) : \boldsymbol{\sigma} \; ; \qquad \boldsymbol{\sigma} = \mathbb{E}(\mathbb{E}_0, \bar{\boldsymbol{\mathcal{D}}}) : \boldsymbol{\epsilon} \; , \tag{1}$$

where $\mathbb{C}$ and $\mathbb{E}$ are the current positive-definite fourth-order compliance and stiffness tensors, respectively, which are endowed with both major and minor symmetries and are the inverse of each other, i.e. $\mathbb{C}$:$\mathbb{E}$=$\mathbb{E}$:$\mathbb{C}$=$\mathbb{I}^s$. Compliance $\mathbb{C}(\mathbb{C}_0, \boldsymbol{\mathcal{D}})$ and stiffness $\mathbb{E}(\mathbb{E}_0, \bar{\boldsymbol{\mathcal{D}}})$ are functions of generally-defined damage variables $\boldsymbol{\mathcal{D}}$, or of dual damage variables $\bar{\boldsymbol{\mathcal{D}}}$, which may be scalar, vector- or tensor-valued, and, obviously, of the initial values $\mathbb{C}_0$, $\mathbb{E}_0$ of compliance and stiffness in the undamaged state. In the present paper restriction is made to *initially-isotropic materials*, namely, in terms of undamaged Young's modulus E_0 and Poisson's ratio ν_0, or Lamé's constant Λ_0 and shear modulus G_0:

$$\mathbb{C}_0 = -\frac{\nu_0}{E_0}\,\mathbf{I}\otimes\mathbf{I} + \frac{1+\nu_0}{E_0}\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}\,; \qquad \mathbb{E}_0 = \Lambda_0\,\mathbf{I}\otimes\mathbf{I} + 2\,G_0\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}\,. \tag{2}$$

Undamaged bulk modulus K_0 could also be used in eqn (2) instead of Lamé's constant Λ_0 by employing the classical relations $3K_0{=}3\Lambda_0{+}2G_0{=}E_0/(1{-}2\nu_0)$.

In CDM, the damage-state relations $\mathbb{C}{=}\mathbb{C}(\mathbb{C}_0, \boldsymbol{\mathcal{D}})$ and $\mathbb{E}{=}\mathbb{E}(\mathbb{E}_0, \bar{\boldsymbol{\mathcal{D}}})$ are usually derived indirectly through a series of conceptual steps (see the references quoted in the Introduction and e.g. the sketches drawn in Rizzi and Carol, 2002; Carol et al., 2001c). A constitutive law for the undamaged material is introduced first in terms of initial moduli and relates *effective* strain and stress quantities, $\boldsymbol{\epsilon}_{\text{eff}}$ and $\boldsymbol{\sigma}_{\text{eff}}$, namely strains and stresses acting at the level of the intact material between microcracks: $\boldsymbol{\epsilon}_{\text{eff}}{=}\mathbb{C}_0{:}\boldsymbol{\sigma}_{\text{eff}}$, $\boldsymbol{\sigma}_{\text{eff}}{=}\mathbb{E}_0{:}\boldsymbol{\epsilon}_{\text{eff}}$. Then, two further links are prescribed: one of the relations between nominal and effective (stress or strain) quantities is assumed, in linear form, by introducing a non-singular fourth-order *damage-effect tensor* expressed in terms of the damage variables, e.g. $\mathbb{A}(\boldsymbol{\mathcal{D}})$ in the stress relation $\boldsymbol{\sigma}_{\text{eff}}{=}\mathbb{A}(\boldsymbol{\mathcal{D}}){:}\boldsymbol{\sigma}$; a second relation is postulated by means of an *'equivalence principle'*: 'strain equivalence' if $\boldsymbol{\epsilon}{=}\boldsymbol{\epsilon}_{\text{eff}}$ (Lemaitre and Chaboche, 1978), 'stress equivalence' if $\boldsymbol{\sigma}{=}\boldsymbol{\sigma}_{\text{eff}}$ (Simo and Ju, 1987), or 'energy equivalence' if $\boldsymbol{\sigma}{:}\boldsymbol{\epsilon}/2{=}\boldsymbol{\sigma}_{\text{eff}}{:}\boldsymbol{\epsilon}_{\text{eff}}/2$ (Cordebois and Sidoroff, 1982). The *'energy equivalence'* approach is adopted here, since it allows to derive secant stiffness and compliance satisfying directly the hyper-elastic requirement of major symmetry. The following nominal/effective relations are then consistently assumed/obtained:

$$\boldsymbol{\sigma}_{\text{eff}} = \mathbb{A}\,(\boldsymbol{\mathcal{D}}) : \boldsymbol{\sigma}\;, \quad \boldsymbol{\epsilon} = \mathbb{A}^{\text{T}}(\boldsymbol{\mathcal{D}}) : \boldsymbol{\epsilon}_{\text{eff}}\;; \qquad \text{or} \qquad \boldsymbol{\epsilon}_{\text{eff}} = \bar{\mathbb{A}}^{\text{T}}(\bar{\boldsymbol{\mathcal{D}}}) : \boldsymbol{\epsilon}\;, \quad \boldsymbol{\sigma} = \bar{\mathbb{A}}\,(\bar{\boldsymbol{\mathcal{D}}}) : \boldsymbol{\sigma}_{\text{eff}}\;, \tag{3}$$

and the hyperelastic compliance and stiffness are expressed by the symmetric forms:

$$\mathbb{C}(\mathbb{C}_0, \boldsymbol{\mathcal{D}}) = \mathbb{A}^{\text{T}}(\boldsymbol{\mathcal{D}}) : \mathbb{C}_0 : \mathbb{A}\,(\boldsymbol{\mathcal{D}})\;; \qquad \mathbb{E}(\mathbb{C}_0, \bar{\boldsymbol{\mathcal{D}}}) = \bar{\mathbb{A}}\,(\bar{\boldsymbol{\mathcal{D}}}) : \mathbb{E}_0 : \bar{\mathbb{A}}^{\text{T}}(\bar{\boldsymbol{\mathcal{D}}})\;, \tag{4}$$

where $\mathbb{A}\,(\boldsymbol{\mathcal{D}}){=}\bar{\mathbb{A}}^{-1}(\bar{\boldsymbol{\mathcal{D}}})$ and $\bar{\mathbb{A}}\,(\bar{\boldsymbol{\mathcal{D}}}){=}\mathbb{A}^{-1}(\boldsymbol{\mathcal{D}})$ are *dual* non-singular fourth-order damage-effect tensors, inverse of each other, and endowed with minor symmetries (but not necessarily major symmetry). The damage-effect tensors $\mathbb{A}(\boldsymbol{\mathcal{D}})$ and $\bar{\mathbb{A}}(\bar{\boldsymbol{\mathcal{D}}})$ are the main concern of the derivations that follow in the paper. Notice from eqns (3), (4) that $\mathbb{A}(\boldsymbol{\mathcal{D}})$ and $\bar{\mathbb{A}}(\bar{\boldsymbol{\mathcal{D}}})$ play the same role in transforming nominal and effective stresses and strains, and initial compliance and stiffness, except for a *transpose operation.*

Concerning the dual underlying damage variables $\boldsymbol{\mathcal{D}}$ and $\bar{\boldsymbol{\mathcal{D}}}$, in the present paper positive-definite symmetric second-order tensor variables are assumed, namely the integrity tensor $\bar{\boldsymbol{\phi}}$, varying between $\mathbf{I}$ and $\mathbf{0}$ (Valanis, 1990), or its dual inverse integrity variable $\boldsymbol{\phi}{=}\bar{\boldsymbol{\phi}}^{-1}$, with complementary variation between $\mathbf{I}$ and $\boldsymbol{\infty}$. Also, for the ease of representation, the square-root tensors $\mathbf{w}=\sqrt{\boldsymbol{\phi}}$, $\bar{\mathbf{w}}=\sqrt{\bar{\boldsymbol{\phi}}}$ are as well employed in notation. Since the damage-effect tensors $\mathbb{A}(\boldsymbol{\phi})$ and $\bar{\mathbb{A}}(\bar{\boldsymbol{\phi}})$ can be postulated independently from either $\boldsymbol{\phi}$ or $\bar{\boldsymbol{\phi}}$ (or either $\mathbf{w}$ or $\bar{\mathbf{w}}$), they might be respectively given independent polynomial orthotropic representations in terms of $\boldsymbol{\phi}$ or in terms of $\bar{\boldsymbol{\phi}}$. Then, we are interested in seeking particular instances of $\mathbb{A}(\boldsymbol{\phi})$ and $\bar{\mathbb{A}}(\bar{\boldsymbol{\phi}})$ with the property that their inverses become the transposes of the tensors obtained by replacing $\boldsymbol{\phi}$ with its inverse $\bar{\boldsymbol{\phi}}$ (or viceversa), and by substituting the scalar coefficients pre-multiplying the various tensor addends of the orthotropic representation with dual (to be determined) ones. The resulting damage-effect tensors are said then to possess *(dual) inverses with complementary structures.*

3 Fourth-order tensors in orthotropic damage

General expressions of orthotropic fourth-order tensors can be deduced by algebraic decomposition (e.g. Walpole, 1984) or through representation theorems (e.g. Boehler, 1987). Taking e.g. the latter approach, the secant linear hyperelastic compliance $\mathbb{C}$ can be derived by double differentiation with respect to $\boldsymbol{\sigma}$ of an orthotropic quadratic complementary energy function of $\boldsymbol{\sigma}$, namely an isotropic function of $\boldsymbol{\sigma}$ and of the relevant orthotropic structural tensors. In the present case the role of structural tensors is played by the second-order inverse integrity tensor $\boldsymbol{\phi}$. Then, the complementary energy function is expressed in terms of the following set of ten irreducibile invariants: $\mathrm{tr}\,\boldsymbol{\sigma}$, $\mathrm{tr}\,\boldsymbol{\sigma}^2$, $\mathrm{tr}\,\boldsymbol{\sigma}^3$, $\mathrm{tr}\,\boldsymbol{\phi}$, $\mathrm{tr}\,\boldsymbol{\phi}^2$, $\mathrm{tr}\,\boldsymbol{\phi}^3$, $\mathrm{tr}\,(\boldsymbol{\sigma}\cdot\boldsymbol{\phi})$, $\mathrm{tr}\,(\boldsymbol{\sigma}^2\cdot\boldsymbol{\phi})$, $\mathrm{tr}\,(\boldsymbol{\sigma}\cdot\boldsymbol{\phi}^2)$, $\mathrm{tr}\,(\boldsymbol{\sigma}\cdot\boldsymbol{\phi})^2$. Following the ordering adopted by Zysset and Curnier (1995) in their formulation of orthotropic fabric elasticity, the general orthotropic damaged compliance can then be expressed by the nine-coefficients polynomial expansion:

$$
\begin{aligned}
\mathbb{C} = {} & c_1\,\mathbf{I}\otimes\mathbf{I} + c_2\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I} + c_3\,\boldsymbol{\phi}\otimes\boldsymbol{\phi} + c_4\,(\boldsymbol{\phi}\,\underline{\overline{\otimes}}\,\mathbf{I} + \mathbf{I}\,\underline{\overline{\otimes}}\,\boldsymbol{\phi}) + c_5\,\boldsymbol{\phi}^2\otimes\boldsymbol{\phi}^2 + c_6\,\boldsymbol{\phi}\,\underline{\overline{\otimes}}\,\boldsymbol{\phi} \\
& + c_7\,(\boldsymbol{\phi}\otimes\mathbf{I} + \mathbf{I}\otimes\boldsymbol{\phi}) + c_8\,(\boldsymbol{\phi}^2\otimes\boldsymbol{\phi} + \boldsymbol{\phi}\otimes\boldsymbol{\phi}^2) + c_9\,(\boldsymbol{\phi}^2\otimes\mathbf{I} + \mathbf{I}\otimes\boldsymbol{\phi}^2)\,,
\end{aligned}
\tag{5}
$$

where the nine scalar coefficients c_i, i=1-9, are any arbitrary polynomial functions of the three principal invariants of $\boldsymbol{\phi}$.

Notice that only the three terms embedding symmetrized dyadic products in representation (5) (*'shear-like'* coefficients c_2, c_4, c_6) affect the shear moduli entering the lower three diagonal entries in a 6×6 matrix representation of the compliance tensor in the principal axes of damage (see Appendix A). The same terms also affect the upper three diagonal entries of the same matrix representation. The six supplemental rank-one updates given by the dyadic product addends (remaining *'non-shear'* coefficients c_1, c_3, c_5, c_7, c_8, c_9) only affect instead the upper-left 3×3 submatrix representation of $\mathbb{C}$. Properties of tensors differing by rank-one updates have been studied in different mechanical contexts, e.g. by Curnier et al. (1995), in inspecting the constitutive elastic properties of bimodular materials, by Loret et al. (2001), in exploring the relations between drained and undrained moduli in anisotropic poroelasticity, and by Rizzi et al. (1996), in conjunction with the study of strain localization phenomena in multi-dissipative solids.

Furthermore, note that, as a result of representation theorems, only powers of $\boldsymbol{\phi}$ up to the second degree enter eqn (5). Indeed, higher powers can be reduced by the application of the Cayley-Hamilton theorem applied to $\boldsymbol{\phi}$. Also, notice that, as compared to the rank-one terms including standard dyadic products '$\otimes$', the three symmetrized dyadic product addends that would contain $\boldsymbol{\phi}^2$, i.e. $(\boldsymbol{\phi}^2\underline{\overline{\otimes}}\mathbf{I}+\mathbf{I}\underline{\overline{\otimes}}\boldsymbol{\phi}^2)$, $(\boldsymbol{\phi}^2\underline{\overline{\otimes}}\boldsymbol{\phi}+\boldsymbol{\phi}\underline{\overline{\otimes}}\boldsymbol{\phi}^2)$ and $\boldsymbol{\phi}^2\underline{\overline{\otimes}}\boldsymbol{\phi}^2$ are missing in representation (5). In fact, the latter terms are not independent from the tensor terms already included in eqn (5), since they can be reduced, respectively, through the use of Rivlin's tensorial identities (B.1), (B.2), (B.3), Appendix B. Actually, through Rivlin's tensorial identities, the three missing terms span together all the tensor addends of representation (5).

Indeed, a crucial factor in a formal comparison of the possible different orthotropic representations is the list of irreducible invariants adopted in the source functional and the resulting tensorial terms. As pointed out by Zysset and Curnier (1995) and Lam and Zhang (1995), the adoption followed here of the invariant $\mathrm{tr}\,(\boldsymbol{\sigma}\cdot\boldsymbol{\phi})^2$ in place of the more commonly used $\mathrm{tr}\,(\boldsymbol{\sigma}^2\cdot\boldsymbol{\phi}^2)$ (see e.g. Cowin, 1985; Zheng and Betten, 1996), leads to the term $\boldsymbol{\phi}\,\underline{\overline{\otimes}}\,\boldsymbol{\phi}$ in place of $(\boldsymbol{\phi}^2\underline{\overline{\otimes}}\mathbf{I}+\mathbf{I}\underline{\overline{\otimes}}\boldsymbol{\phi}^2)$. The two options are equivalent, since the two invariants are related by an identity involving the remaining invariants of the representation (Boehler, 1987; Lam and Zhang, 1995, eqn (3.2)) and the resulting tensorial terms are related by Rivlin's tensorial identity (B.1). However, the term $\boldsymbol{\phi}\,\underline{\overline{\otimes}}\,\boldsymbol{\phi}$ seems far more convenient than the term $(\boldsymbol{\phi}^2\underline{\overline{\otimes}}\mathbf{I}+\mathbf{I}\underline{\overline{\otimes}}\boldsymbol{\phi}^2)$ for the analytical developments in the dual compliance- and stiffness-based derivations, since it possesses a very convenient inverse, i.e. simply $(\boldsymbol{\phi}\,\underline{\overline{\otimes}}\,\boldsymbol{\phi})^{-1}=(\boldsymbol{\phi}^{-1}\underline{\overline{\otimes}}\,\boldsymbol{\phi}^{-1})=\bar{\boldsymbol{\phi}}\,\underline{\overline{\otimes}}\,\bar{\boldsymbol{\phi}}$, while the inverse of $(\boldsymbol{\phi}^2\underline{\overline{\otimes}}\mathbf{I}+\mathbf{I}\underline{\overline{\otimes}}\boldsymbol{\phi}^2)$ may be expressed by a more involved relation that can be adapted from that of $(\boldsymbol{\phi}\,\underline{\overline{\otimes}}\mathbf{I}+\mathbf{I}\underline{\overline{\otimes}}\boldsymbol{\phi})^{-1}$ (Hoger and Carlson, 1984; Rosati, 2000; Rizzi and Carol, 2002, Appendix A). On the light of these remarks, orthotropic stiffness representations of fabric elasticity by Cowin (1985) and Zysset and Curnier (1995) can be considered to be fully equivalent. Also, the formulation of elastic damage proposed by Valanis (1990) and the equivalent specific instance of fabric elasticity presented by Zysset and Curnier (1995) turn out to be particular cases of the Cowin (1985) representation.

An orthotropic fourth-order representation similar to that of eqn (5) also applies to the general symmetric damage-effect tensor apt to represent orthotropic damage in initially-isotropic materials, namely an isotropic tensor-valued function of $\boldsymbol{\phi}$ (Lam and Zhang, 1995). However, to allow broader generality (which is very relevant to the present context for the solution set of the problem under consideration), the following *non-symmetric* orthotropic polynomial form of damage-effect tensor in terms of 12 scalar coefficients is adopted as in Zheng and Betten (1996) (here within the above-commented preferential choice of the tensorial term $\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w}$ in place of $\mathbf{w}^2\,\underline{\overline{\otimes}}\,\mathbf{I}+\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{w}^2$):

$$\begin{aligned}\mathbb{A} = {} & a_1\,\mathbf{I}\otimes\mathbf{I} + a_2\,\mathbf{I}\underline{\overline{\otimes}}\mathbf{I} + a_3\,\mathbf{w}\otimes\mathbf{w} + a_4\,(\mathbf{w}\underline{\overline{\otimes}}\mathbf{I} + \mathbf{I}\underline{\overline{\otimes}}\mathbf{w}) + a_5\,\mathbf{w}^2\otimes\mathbf{w}^2 + a_6\,\mathbf{w}\underline{\overline{\otimes}}\mathbf{w} \\ & + a_{71}\,\mathbf{w}\otimes\mathbf{I} + a_{72}\,\mathbf{I}\otimes\mathbf{w} + a_{81}\,\mathbf{w}^2\otimes\mathbf{w} + a_{82}\,\mathbf{w}\otimes\mathbf{w}^2 + a_{91}\,\mathbf{w}^2\otimes\mathbf{I} + a_{92}\,\mathbf{I}\otimes\mathbf{w}^2\,, \end{aligned} \tag{6}$$

where, to link representations (5) and (6) through eqn (4a), the square-root inverse integrity tensor $\mathbf{w}=\sqrt{\boldsymbol{\phi}}$ has been adopted in $\mathbb{A}$, and the 12 coefficients a_i, i=1-6; a_{i1}, a_{i2}, i=7-9, are also any arbitrary polynomial functions of the three principal invariants of $\mathbf{w}$. In the symmetric case the 12 coefficients reduce to 9, simply by posing $a_{i1}=a_{i2}=a_i$, i=7-9. Walpole's (1984) matrix representation of tensor $\mathbb{A}$ in the principle axes of damage is given in Appendix A, eqns (A.7) and (A.4)-(A.6). Consistency between eqns (6) and (5) through eqn (4a) is apparent for initially-isotropic materials, since $\mathbb{C}_0$, eqn (2a), does not embed separate structural tensors: pre-multiplication with $\mathbb{A}^{\mathrm{T}}$ and post-multiplication with $\mathbb{A}$ of terms $\mathbf{I}\otimes\mathbf{I}$ and $\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}$ in (2a) span the same tensor products space. The relations between alternative representations of $\mathbb{C}$ and $\mathbb{A}$ in terms of either $\boldsymbol{\phi}$ or $\mathbf{w}$ can be obtained through the isotropic functions $\boldsymbol{\phi}{=}\mathbf{w}^2$ and $\mathbf{w}{=}\sqrt{\boldsymbol{\phi}}$, where (Ting, 1985, eqn (2.7)):

$$\mathbf{w} = \sqrt{\boldsymbol{\phi}} = \frac{1}{{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2 - {}^{\mathrm{w}}I_3} \Big(-\boldsymbol{\phi}^2 + ({}^{\mathrm{w}}I_1{}^2 - {}^{\mathrm{w}}I_2)\,\boldsymbol{\phi} + {}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3\,\mathbf{I} \Big) \,, \tag{7}$$

and ${}^{\mathrm{w}}I_1, {}^{\mathrm{w}}I_2, {}^{\mathrm{w}}I_3$ are the classical three principal invariants of $\mathbf{w}$ as defined in Appendix B.

A natural constraint on representation (6) arises from the intrinsic meaning of the damage-effect tensor in eqns (3a,b): in the absence of damage ($\boldsymbol{\phi}{=}\mathbf{w}{=}\mathbf{I}$), $\mathbb{A}(\mathbf{I})$ must coincide with the symmetric identity $\mathbb{I}^s{=}\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}$, namely when all the coefficients are evaluated in $\mathbf{I}$:

$$[a_2 + 2a_4 + a_6](\mathbf{I}) = 1 \; ; \quad [a_1 + a_3 + a_5 + a_{71} + a_{72} + a_{81} + a_{82} + a_{91} + a_{92}](\mathbf{I}) = 0 \; . \tag{8}$$

Representations similar to (5) and (6) hold as well for the damaged stiffness $\mathbb{E}$ and the damage-effect tensor $\bar{\mathbb{A}}$ in terms of the integrity variable $\bar{\boldsymbol{\phi}}$ and its square root $\bar{\mathbf{w}}$, and of dual nine coefficients e_i, i=1-9, and 12 coefficients $\bar{a}_i$, i=1-6; $\bar{a}_{i1}$, $\bar{a}_{i2}$, i=7-9. The natural constraint (8) at zero damage must also hold for the coefficients with bars pertinent to the representation of $\bar{\mathbb{A}}$. Alternative orthotropic representations and links to the present ones are provided in Appendix A.

3.1 Problem statement and known solutions

Considering now representations (5), (6) for $\mathbb{C}$ and $\mathbb{A}$, and dual ones for $\mathbb{E}$ and $\bar{\mathbb{A}}$, we come precisely to the point under consideration here of seeking particular instances of such general representations (possibly with a limited number of convenient tensor terms) that correspond to each other through an inversion operation spanning the same set of tensor terms, within a transpose operation of non-symmetric addends attached to a_{71}, a_{72}; a_{81}, a_{82}; a_{91}, a_{92} in $\mathbb{A}$, and dual ones with bars in $\bar{\mathbb{A}}$. In other words, while for the symmetric terms the sought correspondence should be direct, e.g. a_i should mirror to $\bar{a}_i$, i=1-6, for the non-symmetric ones, the correspondence in dual complementary structures should go through the links between a_{i1} and $\bar{a}_{i2}$, and conversely between a_{i2} and $\bar{a}_{i1}$, i=7-9.

Before undergoing this search, let us remark that there are at least *three known particular instances* of orthotropic structures (5), (6) (and dual ones) that fall within the sought solution set. The *first two cases* may be clearly seen as the following apparent symmetric cases of the general representations:

Solution (0.1). The isotropic case in which only the two-coefficients sets (c_1, c_2) and (e_1, e_2) are considered in the representations of $\mathbb{C}$ and $\mathbb{E}$, which corresponds to maintain only the two-coefficients sets (a_1, a_2) and $(\bar{a}_1, \bar{a}_2)$ in the expansions of $\mathbb{A}$ and $\bar{\mathbb{A}}$:

$$\mathbb{A} = a_2\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I} + a_1\,\mathbf{I}\otimes\mathbf{I} \; ; \quad \bar{\mathbb{A}} = \bar{a}_2\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I} + \bar{a}_1\,\mathbf{I}\otimes\mathbf{I} \,, \tag{9}$$

with

$$\bar{a}_2 = \frac{1}{a_2}\,, \qquad \bar{a}_1 = -\frac{a_1}{a_2\,(3a_1 + a_2)}\,. \tag{10}$$

This leads to a general form of isotropic damage based on two independent scalar damage variables, if the two coefficient functions a_1, a_2 are independent, or to a restricted form of isotropic damage based on a single scalar damage variable, if the two coefficient functions a_1, a_2 are linked to each other (e.g. Ju, 1990);

Solution (0.2). The Valanis-type compliance and stiffness (Valanis, 1990; Zysset and Curnier, 1995) that can be respectively obtained by replacing the identity $\mathbf{I}$ in the original isotropic compliance and stiffness (2) with the inverse integrity and integrity tensors $\boldsymbol{\phi}$ and $\bar{\boldsymbol{\phi}}$. This corresponds to keep only terms from two-coefficients sets (c_3, c_6) and (e_3, e_6) in the compliance and stiffness representations of $\mathbb{C}$ and $\mathbb{E}$ in terms of $\boldsymbol{\phi}, \bar{\boldsymbol{\phi}}$, namely $\mathbb{C}{=}c_3\,\boldsymbol{\phi}{\otimes}\boldsymbol{\phi}{+}c_6\,\boldsymbol{\phi}\,\underline{\overline{\otimes}}\,\boldsymbol{\phi}$, $\mathbb{E}{=}e_3\,\bar{\boldsymbol{\phi}}{\otimes}\bar{\boldsymbol{\phi}}{+}e_6\,\bar{\boldsymbol{\phi}}\,\underline{\overline{\otimes}}\,\bar{\boldsymbol{\phi}}$ (and only terms from five-coefficients sets $(c_3, c_4, c_5, c_6, c_7)$ and $(e_3, e_4, e_5, e_6, e_7)$ in the $\mathbf{w}, \bar{\mathbf{w}}$ expansions), and is recovered by guarding only single coefficients a_6 and $\bar{a}_6$ in the damage-effect tensors representations of $\mathbb{A}$ and $\bar{\mathbb{A}}$:

$$\mathbb{A} = a_6\ \mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w}\ ; \quad \bar{\mathbb{A}} = \bar{a}_6\ \bar{\mathbf{w}}\,\underline{\overline{\otimes}}\,\bar{\mathbf{w}}\ , \tag{11}$$

with

$$\bar{a}_6 = \frac{1}{a_6}\,. \tag{12}$$

Taking $a_6{=}\bar{a}_6{=}1$, i.e. 'basic' damage-effect tensors $\mathbb{A}_{\mathrm{bas}}{=}\sqrt{\boldsymbol{\phi}}\,\underline{\overline{\otimes}}\,\sqrt{\boldsymbol{\phi}}$ and $\bar{\mathbb{A}}_{\mathrm{bas}}{=}\sqrt{\bar{\boldsymbol{\phi}}}\,\underline{\overline{\otimes}}\,\sqrt{\bar{\boldsymbol{\phi}}}$ (Carol et al., 2001a,b), Valanis-type secant relations are indeed obtained from eqns (2), (4). This case corresponds to a restricted form of orthotropic material symmetry (Bigoni and Loret, 1999, Appendix A; see also Appendix A at the end of this paper).

Solution (0.3). A *third instance* of the sought solution set can be derived as an 'extended' version of the 'basic' Valanis-damage case (Carol et al., 2001c) by considering a non-symmetric damage-effect tensor $\mathbb{A}$, eqn (6), with only two-coefficients set (a_6, a_{92}) and a dual damage-effect tensor $\bar{\mathbb{A}}$ with only complementary two-coefficients set $(\bar{a}_6, \bar{a}_{91})$:

$$\mathbb{A} = a_6\ \mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w} + a_{92}\ \mathbf{I}\otimes\mathbf{w}^2\ ; \quad \bar{\mathbb{A}} = \bar{a}_6\ \bar{\mathbf{w}}\,\underline{\overline{\otimes}}\,\bar{\mathbf{w}} + \bar{a}_{91}\ \bar{\mathbf{w}}^2\otimes\mathbf{I}\ , \tag{13}$$

with

$$\bar{a}_6 = \frac{1}{a_6}\,, \qquad \bar{a}_{91} = -\frac{a_{92}}{a_6\,(3a_{92} + a_6)}\,. \tag{14}$$

From eqns (13), (14) it is immediate to check that $\mathbb{A}{:}\bar{\mathbb{A}}{=}\bar{\mathbb{A}}{:}\mathbb{A}{=}\mathbb{I}^s$. *Damage-effect tensors (13) give a clear example of the type of solutions that are searched here.* The arising secant compliance and stiffness still belong to the above-mentioned Valanis-type structure embedding only two-coefficients sets (c_3, c_6) and (e_3, e_6) in the tensor representations in terms of $\boldsymbol{\phi}, \bar{\boldsymbol{\phi}}$.

Moreover, a formulation of elastic damage based on volumetric/deviatoric decomposition can be obtained by taking $a_6{=}\phi^\eta$, $a_{92}{=}(\phi^{-\eta}{-}\phi^\eta)/3$, and $\bar{a}_6{=}\bar{\phi}^\eta$, $\bar{a}_{91}{=}(\bar{\phi}^{-\eta}{-}\bar{\phi}^\eta)/3$, namely (Carol et al., 2001c):

$$\mathbb{A} = \phi^\eta\ \sqrt{\boldsymbol{\phi}}\,\underline{\overline{\otimes}}\,\sqrt{\boldsymbol{\phi}} + \frac{\phi^{-\eta} - \phi^\eta}{3}\ \mathbf{I}\otimes\boldsymbol{\phi}\ ; \quad \bar{\mathbb{A}} = \bar{\phi}^\eta\ \sqrt{\bar{\boldsymbol{\phi}}}\,\underline{\overline{\otimes}}\,\sqrt{\bar{\boldsymbol{\phi}}} + \frac{\bar{\phi}^{-\eta} - \bar{\phi}^\eta}{3}\ \bar{\boldsymbol{\phi}}\otimes\mathbf{I}\ , \tag{15}$$

where ϕ and $\bar{\phi}$ are the 1/3 powers of the determinants of $\boldsymbol{\phi}$ and $\bar{\boldsymbol{\phi}}=\boldsymbol{\phi}^{-1}$, $\phi=(\det\boldsymbol{\phi})^{1/3}$, $\bar{\phi}=(\det\bar{\boldsymbol{\phi}})^{1/3}$, and η is a power constant with values between -1 and 1. The damage-effect tensors (15) can also be conveniently decomposed in product form in isotropic and anisotropic parts, e.g. $\mathbb{A}=\hat{\mathbb{A}}_{\rm iso}:\mathbb{A}_{\rm bas}$, where $\mathbb{A}_{\rm bas}=\sqrt{\boldsymbol{\phi}}\,\underline{\overline{\otimes}}\,\sqrt{\boldsymbol{\phi}}$ is the 'basic' damage-effect tensor leading to the Valanis-type secant compliance, and $\hat{\mathbb{A}}_{\rm iso}$ is the isotropic tensor $\hat{\mathbb{A}}_{\rm iso}=\phi^{-\eta}\,\mathbb{P}_V+\phi^{\eta}\,\mathbb{P}_D$, where $\mathbb{P}_V=\mathbf{I}\otimes\mathbf{I}/3$ and $\mathbb{P}_D=\mathbb{I}^s-\mathbf{I}\otimes\mathbf{I}/3$ are the classical fourth-order idempotent volumetric and deviatoric projection operators, respectively. This leads to the following compliance and stiffness expressions that still embed the Valanis-type structure,

$$\mathbb{C}=-\frac{\hat{\nu}}{\hat{E}}\,\boldsymbol{\phi}\otimes\boldsymbol{\phi}+\frac{1+\hat{\nu}}{\hat{E}}\,\boldsymbol{\phi}\,\underline{\overline{\otimes}}\,\boldsymbol{\phi}\;;\qquad \mathbb{E}=\hat{\Lambda}\,\bar{\boldsymbol{\phi}}\otimes\bar{\boldsymbol{\phi}}+2\,\hat{G}\,\bar{\boldsymbol{\phi}}\,\underline{\overline{\otimes}}\,\bar{\boldsymbol{\phi}}\;, \tag{16}$$

but that include undamaged elastic parameters replaced by degraded ones with hats determined as follows: $\hat{K}=\phi^{2\eta}\,K_0=\phi^2 K$, $\hat{G}=\phi^{-2\eta}\,G_0=\phi^2 G$, and similarly for parameters $3\hat{\Lambda}=3\hat{K}-2\hat{G}$, $\hat{E}=9\hat{K}\hat{G}/(3\hat{K}+\hat{G})$, $2\hat{\nu}=(3\hat{K}-2\hat{G})/(3\hat{K}+\hat{G})$. Notice that bulk and shear moduli $\hat{K}$, $\hat{G}$ differ from the real current secant moduli $K=\phi^{-2(1-\eta)}K_0$ and $G=\phi^{-2(1+\eta)}G_0$. The assumption above corresponds to assign different weights to bulk and shear damage components according to the values taken for constant η ($\eta=1$: pure deviatoric damage, $-1<\eta<1$: mixed deviatoric/volumetric damage; $\eta=-1$: pure volumetric damage; see Carol et al., 2001c for the details). Eqns (15), (16) include as particular cases both the instances previously mentioned: *Solution (0.1)* when the damage tensors become spherical, $\boldsymbol{\phi}=\phi\,\mathbf{I}$ and $\bar{\boldsymbol{\phi}}=\bar{\phi}\,\mathbf{I}$, since moduli $\hat{K}$ and $\hat{G}$ come to coincide with the current bulk and shear moduli K and G, and similarly for the other elastic material parameters, so that an isotropic damage model with linear links between bulk and shear moduli degradations is recovered; *Solution (0.2)* for $\eta=0$, since $\hat{K}=K_0$, $\hat{G}=G_0$, and similarly for the other parameters, so that the 'basic' orthotropic Valanis-damage secant relations embedding undamaged elastic parameters are retrieved.

4 Solution set of damage-effect tensors with complementary structures

4.1 Additional foreseen solutions

Before undergoing a more general treatment, let us remark first that *ten additional instances* (*nine symmetric* and *one non-symmetric*) of the sought solution set of dual damage-effect tensors with complementary structures can be already advanced at the present stage by guess and a posteriori check, starting right-away from the known solutions reported above:

Solution (1). By taking the symmetric Valanis-type structure of compliance and stiffness directly for the damage-effect tensors, namely by keeping only the complementary two-coefficients sets (a_6, a_3) and $(\bar{a}_6, \bar{a}_3)$ in $\mathbb{A}$ and $\bar{\mathbb{A}}$:

$$\mathbb{A}=a_6\;\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w}+a_3\;\mathbf{w}\otimes\mathbf{w}\;;\quad \bar{\mathbb{A}}=\bar{a}_6\;\bar{\mathbf{w}}\,\underline{\overline{\otimes}}\,\bar{\mathbf{w}}+\bar{a}_3\;\bar{\mathbf{w}}\otimes\bar{\mathbf{w}}\;, \tag{17}$$

with

$$\bar{a}_6=\frac{1}{a_6}\;,\qquad \bar{a}_3=-\frac{a_3}{a_6\,(3a_3+a_6)}\;. \tag{18}$$

Clearly, the arising secant compliance and stiffness are no longer of the Valanis-type and actually include *all but* coefficients c_2, c_4 and e_2, e_4 for the tensor representations in terms of $\boldsymbol{\phi}$ and $\bar{\boldsymbol{\phi}}$, and *all but* coefficients c_1, c_2, c_9 and e_1, e_2, e_9 for the tensor representations in terms of $\mathbf{w}$ and $\bar{\mathbf{w}}$. Notice that to reduce the former terms, the isotropic square-root function $\sqrt{\boldsymbol{\phi}} \sim (\mathbf{I}, \boldsymbol{\phi}, \boldsymbol{\phi}^2)$, eqn (7), must be adopted, and similarly for $\bar{\boldsymbol{\phi}}$. This means that symmetric damage-effect tensors with either *all but* 'shear-like' coefficients a_2, a_4 and $\bar{a}_2, \bar{a}_4$, and *all but* coefficients a_1, a_2, a_9 and $\bar{a}_1, \bar{a}_2, \bar{a}_9$ should also be part of the solution set. The first possibility is considered much later in *Solution (9)*. The second instance missing 'shear-like' coefficients a_2, $\bar{a}_2$ and 'non-shear' coefficients a_1, a_9 and $\bar{a}_1, \bar{a}_9$ is considered much below in *Solution (3)*.

Solution (2). As an alternative to *Solution (0.2)*, by taking $\mathbf{w}^2$, $\bar{\mathbf{w}}^2$ in place of $\mathbf{w}$, $\bar{\mathbf{w}}$ in the tensor structure of eqn (11). Through Rivlin's tensorial identity (B.3), this corresponds to assume symmetric damage-effect tensors embedding the five-coefficients sets $(a_3, a_4, a_5, a_6, a_7)$ and $(\bar{a}_3, \bar{a}_4, \bar{a}_5, \bar{a}_6, \bar{a}_7)$ (lacking coefficients a_1, a_2, a_8, a_9 and $\bar{a}_1, \bar{a}_2, \bar{a}_8, \bar{a}_9$), namely:

$$\begin{aligned} \mathbb{A} &= a_4\,(\mathbf{w}\,\underline{\otimes}\,\mathbf{I} + \mathbf{I}\,\underline{\otimes}\,\mathbf{w}) + a_6\,\mathbf{w}\,\underline{\otimes}\,\mathbf{w} + a_3\,\mathbf{w}\otimes\mathbf{w} + a_5\,\mathbf{w}^2\otimes\mathbf{w}^2 + a_7\,(\mathbf{w}\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{w}); \\ \bar{\mathbb{A}} &= \bar{a}_4\,(\bar{\mathbf{w}}\,\underline{\otimes}\,\mathbf{I} + \mathbf{I}\,\underline{\otimes}\,\bar{\mathbf{w}}) + \bar{a}_6\,\bar{\mathbf{w}}\,\underline{\otimes}\,\bar{\mathbf{w}} + \bar{a}_3\,\bar{\mathbf{w}}\otimes\bar{\mathbf{w}} + \bar{a}_5\,\bar{\mathbf{w}}^2\otimes\bar{\mathbf{w}}^2 + \bar{a}_7\,(\bar{\mathbf{w}}\otimes\mathbf{I} + \mathbf{I}\otimes\bar{\mathbf{w}}), \end{aligned} \tag{19}$$

which form a complementary inverse pair provided that

$$a_3 = -a_6\ , \quad -a_7 = a_4 = -a_6\,\frac{{}^{\mathbf{w}}I_3}{{}^{\mathbf{w}}I_2}\ ; \qquad \bar{a}_3 = -\bar{a}_6\ , \quad -\bar{a}_7 = \bar{a}_4 = -\bar{a}_6\,\frac{{}^{\bar{\mathbf{w}}}I_3}{{}^{\bar{\mathbf{w}}}I_2}\ , \tag{20}$$

with

$$\bar{a}_6 = \frac{{}^{\mathbf{w}}I_1\,{}^{\mathbf{w}}I_2}{a_6\,{}^{\mathbf{w}}I_3}\ , \quad \bar{a}_5 = \frac{{}^{\mathbf{w}}I_2}{a_6}\,\frac{a_6 - 2a_5\,{}^{\mathbf{w}}I_2}{2a_6 - 3a_5\,{}^{\mathbf{w}}I_2}\ ; \quad \bar{a}_4 = \frac{1}{a_4}\ , \quad \bar{a}_3 = -\frac{{}^{\mathbf{w}}I_1\,{}^{\mathbf{w}}I_2}{a_6\,{}^{\mathbf{w}}I_3}\ , \quad \bar{a}_7 = \frac{{}^{\mathbf{w}}I_2}{a_6\,{}^{\mathbf{w}}I_3}\ , \tag{21}$$

where, similarly to ${}^{\mathbf{w}}I_1, {}^{\mathbf{w}}I_2, {}^{\mathbf{w}}I_3$, ${}^{\bar{\mathbf{w}}}I_1, {}^{\bar{\mathbf{w}}}I_2, {}^{\bar{\mathbf{w}}}I_3$ are the three principal invariants of $\bar{\mathbf{w}}{=}\mathbf{w}^{-1}$ (recall also that ${}^{\mathbf{w}}I_1{=}{}^{\bar{\mathbf{w}}}I_2/{}^{\bar{\mathbf{w}}}I_3$ and ${}^{\bar{\mathbf{w}}}I_1{=}{}^{\mathbf{w}}I_2/{}^{\mathbf{w}}I_3$). Notice the necessary constraints (20) to make up the complementary inverse pair (19). If $a_5{=}a_6/{}^{\mathbf{w}}I_2$, which corresponds strictly to assume $\mathbb{A}{=}a_5\,\mathbf{w}^2\,\underline{\otimes}\,\mathbf{w}^2$, $\bar{a}_5$ simplifies to: $\bar{a}_5{=}1/a_5{=}{}^{\mathbf{w}}I_2/a_6$. The coefficients in eqn (21) can be obtained as a particular case of the general solution treated later in Sections 4.2, 4.3, once structure (19) is inspected to belong to the solution set. In particular, coefficients $a_4, \bar{a}_4$ and $a_6, \bar{a}_6$ in eqn (20) are linked to each other consistently with *Solution (s.c)*, developed in Section 4.3 for the 'shear-like' coefficients. Recalling *Solution (0.2)*, eqns (19), (20) with $\bar{a}_5{=}1/a_5{=}{}^{\mathbf{w}}I_2/a_6$ (namely $\mathbb{A}{=}a_5\,\mathbf{w}^2\,\underline{\otimes}\,\mathbf{w}^2$, $\bar{\mathbb{A}}{=}\bar{a}_5\,\bar{\mathbf{w}}^2\,\underline{\otimes}\,\bar{\mathbf{w}}^2$) lead to compliance and stiffness with similar five-coefficients sets $(c_3, c_4, c_5, c_6, c_7)$ and $(e_3, e_4, e_5, e_6, e_7)$ in the $\boldsymbol{\phi}, \bar{\boldsymbol{\phi}}$ expansions (and all nine coefficients in the $\mathbf{w}, \bar{\mathbf{w}}$ expansions). The same structure (19) of the damage-effect tensors is also obtained by repeating the experiment of replacing $\mathbf{w}$, $\bar{\mathbf{w}}$ with $\mathbf{w}^2$, $\bar{\mathbf{w}}^2$ in Valanis-type damage-effect tensors (17), *Solution (1)*. Actually, this is precisely the case in (19), (20) with a_5 independent from a_6 and compliance and stiffness in the $\boldsymbol{\phi}, \bar{\boldsymbol{\phi}}$ expansions that embed also coefficients c_8 and e_8 and lack only coefficients c_1, c_2, c_9 and e_1, e_2, e_9. As remarked above, this means again that six-coefficients symmetric damage-effect tensors with *all but* coefficients a_1, a_2, a_9 and $\bar{a}_1, \bar{a}_2, \bar{a}_9$ should also be part of the solution set. Without considering constraints (20), damage-effect tensors (19) formally lead to compliance and stiffness embedding all tensor terms in both $\boldsymbol{\phi}, \bar{\boldsymbol{\phi}}$ and $\mathbf{w}, \bar{\mathbf{w}}$ expansions.

Solution (3). As signaled by the outcomes of the previous two solutions, by taking symmetric damage-effect tensors embedding the six-coefficients sets $(a_3, a_4, a_5, a_6, a_7, a_8)$ and $(\bar{a}_3, \bar{a}_4, \bar{a}_5, \bar{a}_6, \bar{a}_7, \bar{a}_8)$ (lacking coefficients a_1, a_2, a_9 and $\bar{a}_1, \bar{a}_2, \bar{a}_9$), namely:

$$\begin{aligned} \mathbb{A} &= a_4\ (\mathbf{w} \overline{\underline{\otimes}} \mathbf{I} + \mathbf{I} \overline{\underline{\otimes}} \mathbf{w}) + a_6\ \mathbf{w} \overline{\underline{\otimes}} \mathbf{w} + a_3\ \mathbf{w} \otimes \mathbf{w} + a_5\ \mathbf{w}^2 \otimes \mathbf{w}^2 + a_7\ (\mathbf{w} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{w}) \\ &\quad + a_8\ (\mathbf{w}^2 \otimes \mathbf{w} + \mathbf{w} \otimes \mathbf{w}^2); \\ \bar{\mathbb{A}} &= \bar{a}_4\ (\bar{\mathbf{w}} \overline{\underline{\otimes}} \mathbf{I} + \mathbf{I} \overline{\underline{\otimes}} \bar{\mathbf{w}}) + \bar{a}_6\ \bar{\mathbf{w}} \overline{\underline{\otimes}} \bar{\mathbf{w}} + \bar{a}_3\ \bar{\mathbf{w}} \otimes \bar{\mathbf{w}} + \bar{a}_5\ \bar{\mathbf{w}}^2 \otimes \bar{\mathbf{w}}^2 + \bar{a}_7\ (\bar{\mathbf{w}} \otimes \mathbf{I} + \mathbf{I} \otimes \bar{\mathbf{w}}) \\ &\quad + \bar{a}_8\ (\bar{\mathbf{w}}^2 \otimes \bar{\mathbf{w}} + \bar{\mathbf{w}} \otimes \bar{\mathbf{w}}^2), \end{aligned} \tag{22}$$

which form a complementary inverse pair provided that

$$\begin{aligned} a_3 &= -\frac{a_6(a_8 + a_5{}^{\mathbf{w}}I_1)^2 + a_8^2(a_6 - a_5{}^{\mathbf{w}}I_1{}^2)}{a_5^2\ {}^{\mathbf{w}}I_1{}^2}\ , \quad a_7 = \frac{a_6(a_8 + a_5{}^{\mathbf{w}}I_1){}^{\mathbf{w}}I_3}{a_5\ {}^{\mathbf{w}}I_1{}^{\mathbf{w}}I_2}\ , \quad a_4 = -a_6\frac{{}^{\mathbf{w}}I_3}{{}^{\mathbf{w}}I_2}\ ; \\ \bar{a}_3 &= -\frac{\bar{a}_6(\bar{a}_8 + \bar{a}_5{}^{\bar{\mathbf{w}}}I_1)^2 + \bar{a}_8^2(\bar{a}_6 - \bar{a}_5{}^{\bar{\mathbf{w}}}I_1{}^2)}{\bar{a}_5^2\ {}^{\bar{\mathbf{w}}}I_1{}^2}\ , \quad \bar{a}_7 = \frac{\bar{a}_6(\bar{a}_8 + \bar{a}_5{}^{\bar{\mathbf{w}}}I_1){}^{\bar{\mathbf{w}}}I_3}{\bar{a}_5\ {}^{\bar{\mathbf{w}}}I_1{}^{\bar{\mathbf{w}}}I_2}\ , \quad \bar{a}_4 = -\bar{a}_6\frac{{}^{\bar{\mathbf{w}}}I_3}{{}^{\bar{\mathbf{w}}}I_2}\ , \end{aligned} \tag{23}$$

with

$$\begin{aligned} \bar{a}_6 &= \frac{{}^{\mathbf{w}}I_1\,{}^{\mathbf{w}}I_2}{a_6\,{}^{\mathbf{w}}I_3}\ , \quad \bar{a}_5 = \frac{{}^{\mathbf{w}}I_2}{a_6}\,\frac{a_6 - 2a_5{}^{\mathbf{w}}I_2}{2a_6 - 3a_5{}^{\mathbf{w}}I_2}\ , \quad \bar{a}_8 = -\frac{a_8(a_6 - 2a_5{}^{\mathbf{w}}I_2){}^{\mathbf{w}}I_2{}^2}{a_6(3a_8 + a_5{}^{\mathbf{w}}I_1)(2a_6 - 3a_5{}^{\mathbf{w}}I_2){}^{\mathbf{w}}I_3}\ ; \\ \bar{a}_3 &= -\frac{(2a_6 - 3a_5{}^{\mathbf{w}}I_2)(5a_8^2 + 4a_5a_8{}^{\mathbf{w}}I_1 + a_5^2{}^{\mathbf{w}}I_1{}^2){}^{\mathbf{w}}I_1\,{}^{\mathbf{w}}I_3 - a_8^2(a_6 - 2a_5{}^{\mathbf{w}}I_2){}^{\mathbf{w}}I_2{}^2}{a_6(3a_8 + a_5{}^{\mathbf{w}}I_1)^2(2a_6 - 3a_5{}^{\mathbf{w}}I_2){}^{\mathbf{w}}I_3{}^2}\,{}^{\mathbf{w}}I_2\ , \\ \bar{a}_7 &= \frac{(2a_8 + a_5{}^{\mathbf{w}}I_1){}^{\mathbf{w}}I_2}{a_6(3a_8 + a_5{}^{\mathbf{w}}I_1){}^{\mathbf{w}}I_3}\ , \quad \bar{a}_4 = \frac{1}{a_4}\ . \end{aligned} \tag{24}$$

This solution comprises as a particular case previous *Solution (2)* when a_8=0, $\bar{a}_8$=0. In turn, it will be seen later in Section 4.4.2 that *Solution (3)* is also a particular case of *Solution (25)*, so that *Solutions (2), (3)* and *(25)*, form a family of progressively-enlarged solutions by adding a single supplemental coefficient. Furthermore, same as before, without accounting constraints (23), damage-effect tensors (22) formally lead to compliance and stiffness embedding all tensor terms. Notice that *Solutions (2)* and *(3)* are based on the relation $\bar{a}_6 = {}^{\mathbf{w}}I_1{}^{\mathbf{w}}I_2/(a_6{}^{\mathbf{w}}I_3)$ (and $\bar{a}_4 = 1/a_4$) for the 'shear-like' coefficients (Section 4.3), *Solutions (s.c)*, rather than on $\bar{a}_6 = 1/a_6$, *Solutions (s.b)*, which holds for *Solution (1)* and for most of the following solutions.

Solution (4). By switching the roles between coefficients with and without bars in non-symmetric *Solution (0.3)*, namely by taking the *'twin' solution* based on two-coefficients sets (a_6, a_{91}) in $\mathbb{A}$ and $(\bar{a}_6, \bar{a}_{92})$ in $\bar{\mathbb{A}}$:

$$\mathbb{A} = a_6\ \mathbf{w} \overline{\underline{\otimes}} \mathbf{w} + a_{91}\ \mathbf{w}^2 \otimes \mathbf{I}\ ; \quad \bar{\mathbb{A}} = \bar{a}_6\ \bar{\mathbf{w}} \overline{\underline{\otimes}} \bar{\mathbf{w}} + \bar{a}_{92}\ \mathbf{I} \otimes \bar{\mathbf{w}}^2\ , \tag{25}$$

with

$$\bar{a}_6 = \frac{1}{a_6}\ , \qquad \bar{a}_{92} = -\frac{a_{91}}{a_6\ (3a_{91} + a_6)}\ . \tag{26}$$

This leads to a form of secant compliance and stiffness different than that of Valanis-type and embedding respectively only five-coefficients sets $(c_1, c_3, c_6, c_7, c_9)$ and $(e_1, e_3, e_6, e_7, e_9)$ in the $\boldsymbol{\phi}, \bar{\boldsymbol{\phi}}$ expansions (missing coefficients c_2, c_4, c_5, c_8 and e_2, e_4, e_5, e_8), and *all but* coefficients c_2, c_8 and e_2, e_8 in the $\mathbf{w}, \bar{\mathbf{w}}$ expansions. However, solutions missing just a single 'shear-like' and a single 'non-shear' term turn out to be difficult to inspect as being part of the general solution and, most important, to be expressed simply by compact closed-form expressions (an example of such cases is given later in Section 4.4.2, *Solution (25)*). So, the possibility of a solution based on *all but* coefficients a_2, a_8 and $\bar{a}_2, \bar{a}_8$ is not further explored. The other suggested option leads instead to the solution considered next.

Solution (5). Since once again the latter structure of $\mathbb{C}$ and $\mathbb{E}$ in the $\boldsymbol{\phi}, \bar{\boldsymbol{\phi}}$ representations could be used by itself to express the damage-effect tensors, by taking symmetric damage-effect tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ with only the five-coefficients sets $(a_1, a_3, a_6, a_7, a_9)$ and $(\bar{a}_1, \bar{a}_3, \bar{a}_6, \bar{a}_7, \bar{a}_9)$ (lacking coefficients a_2, a_4, a_5, a_8 and $\bar{a}_2, \bar{a}_4, \bar{a}_5, \bar{a}_8$):

$$\begin{aligned} \mathbb{A} &= a_6\ \mathbf{w} \,\underline{\overline{\otimes}}\, \mathbf{w} + a_1\ \mathbf{I} \otimes \mathbf{I} + a_3\ \mathbf{w} \otimes \mathbf{w} + a_7\ (\mathbf{w} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{w}) + a_9\ (\mathbf{w}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{w}^2)\,; \\ \bar{\mathbb{A}} &= \bar{a}_6\ \bar{\mathbf{w}} \,\underline{\overline{\otimes}}\, \bar{\mathbf{w}} + \bar{a}_1\ \mathbf{I} \otimes \mathbf{I} + \bar{a}_3\ \bar{\mathbf{w}} \otimes \bar{\mathbf{w}} + \bar{a}_7\ (\bar{\mathbf{w}} \otimes \mathbf{I} + \mathbf{I} \otimes \bar{\mathbf{w}}) + \bar{a}_9\ (\bar{\mathbf{w}}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \bar{\mathbf{w}}^2)\,, \end{aligned} \tag{27}$$

provided that

$$\begin{aligned} a_6 a_7 &= a_3 a_9\, {}^{\mathbf{w}}\!I_1\,, \qquad a_6^2 a_1 = a_9^2 \left((a_3 + a_6)\, {}^{\mathbf{w}}\!I_1{}^2 - 2a_6\, {}^{\mathbf{w}}\!I_2 \right); \\ \bar{a}_6 \bar{a}_7 &= \bar{a}_3 \bar{a}_9\, {}^{\mathbf{w}}\!I_1\,, \qquad \bar{a}_6^2 \bar{a}_1 = \bar{a}_9^2 \left((\bar{a}_3 + \bar{a}_6)\, {}^{\mathbf{w}}\!I_1{}^2 - 2\bar{a}_6\, {}^{\mathbf{w}}\!I_2 \right), \end{aligned} \tag{28}$$

with

$$\begin{gathered} \bar{a}_6 = \frac{1}{a_6}, \quad \bar{a}_3 = -\frac{a_3}{a_6(3a_3 + a_6)}, \quad \bar{a}_9 = -\frac{a_9}{a_6(3a_9 + a_6)}; \quad \bar{a}_7 = \frac{a_3 a_9}{a_6(3a_3 + a_6)(3a_9 + a_6)}\, \frac{{}^{\mathbf{w}}\!I_2}{{}^{\mathbf{w}}\!I_3}, \\ \bar{a}_1 = a_9^2\ \frac{(2a_3 + a_6)\,{}^{\mathbf{w}}\!I_2{}^2 - 2(3a_3 + a_6)\,{}^{\mathbf{w}}\!I_1\,{}^{\mathbf{w}}\!I_3}{a_6(3a_3 + a_6)(3a_9 + a_6)^2\ {}^{\mathbf{w}}\!I_3{}^2}\,. \end{gathered} \tag{29}$$

Damage-effect tensors (27) formally lead to secant compliance and stiffness embedding *all but* coefficients c_2, c_4 and e_2, e_4 in the expansion of $\boldsymbol{\phi}, \bar{\boldsymbol{\phi}}$ and *all but* coefficients c_2 and e_2 in the expansion of $\mathbf{w}, \bar{\mathbf{w}}$. As already noticed in *Solution (1)* the first possibility is considered much below in *Solution (9)*. The other instance is not further explored since a solution missing only a 'shear-like' term is expected to be involved. This solution case with symmetric damage-effect tensors (27)-(29) displays strict similarities with non-symmetric *Solution (11)*, subsequently derived (Section 4.4.1).

Solution (6). Another solution instance arises as a particular case of *Solution (5)* above by setting $a_3{=}a_7{=}0$, which is compatible with the dual relations $\bar{a}_3{=}\bar{a}_7{=}0$, eqns (29b,d) and the constraints (28a,c), namely by taking symmetric damage-effect tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ with only the three-coefficients sets (a_1, a_6, a_9) and $(\bar{a}_1, \bar{a}_6, \bar{a}_9)$:

$$\begin{aligned} \mathbb{A} &= a_6\ \mathbf{w} \,\underline{\overline{\otimes}}\, \mathbf{w} + a_1\ \mathbf{I} \otimes \mathbf{I} + a_9\ (\mathbf{w}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{w}^2)\,; \\ \bar{\mathbb{A}} &= \bar{a}_6\ \bar{\mathbf{w}} \,\underline{\overline{\otimes}}\, \bar{\mathbf{w}} + \bar{a}_1\ \mathbf{I} \otimes \mathbf{I} + \bar{a}_9\ (\bar{\mathbf{w}}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \bar{\mathbf{w}}^2)\,, \end{aligned} \tag{30}$$

provided that

$$a_6 a_1 = a_9^2 \left({}^{\mathrm{w}}\!I_1{}^2 - 2\, {}^{\mathrm{w}}\!I_2 \right) \; ; \qquad \bar{a}_6 \bar{a}_1 = \bar{a}_9^2 \left({}^{\bar{\mathrm{w}}}\!I_1{}^2 - 2\, {}^{\bar{\mathrm{w}}}\!I_2 \right) , \tag{31}$$

with

$$\bar{a}_6 = \frac{1}{a_6} \, , \quad \bar{a}_9 = -\frac{a_9}{a_6(3a_9 + a_6)} \; ; \quad \bar{a}_1 = \frac{a_9^2 \left({}^{\mathrm{w}}\!I_2{}^2 - 2\, {}^{\mathrm{w}}\!I_1 {}^{\mathrm{w}}\!I_3 \right)}{a_6(3a_9 + a_6)^2 \; {}^{\mathrm{w}}\!I_3{}^2} \, . \tag{32}$$

Same as for *Solution (4)*, damage-effect tensors (30) formally lead to secant compliance and stiffness embedding only five-coefficients sets $(c_1, c_3, c_6, c_7, c_9)$ and $(e_1, e_3, e_6, e_7, e_9)$ in the $\boldsymbol{\phi}, \bar{\boldsymbol{\phi}}$ expansions, and *all but* coefficients c_2, c_8 and e_2, e_8 in the $\mathbf{w}, \bar{\mathbf{w}}$ expansions. Notice the necessary constraints (20), (23), (28), (31) on the coefficients of *Solutions (2)*, *(3)*, *(5)* and *(6)* to make up the complementary inverse pairs (19), (22), (27), (30).

Solution (7). Previous *Solution (6)* can in turn be enlarged by adding coefficients a_5, $\bar{a}_5$, that is by assuming symmetric damage-effect tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ with only the four-coefficients sets (a_1, a_5, a_6, a_9) and $(\bar{a}_1, \bar{a}_5, \bar{a}_6, \bar{a}_9)$:

$$\begin{aligned} \mathbb{A} &= a_6 \; \mathbf{w} \,\underline{\overline{\otimes}}\, \mathbf{w} + a_1 \; \mathbf{I} \otimes \mathbf{I} + a_5 \; \mathbf{w}^2 \otimes \mathbf{w}^2 + a_9 \left(\mathbf{w}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{w}^2 \right) ; \\ \bar{\mathbb{A}} &= \bar{a}_6 \; \bar{\mathbf{w}} \,\underline{\overline{\otimes}}\, \bar{\mathbf{w}} + \bar{a}_1 \; \mathbf{I} \otimes \mathbf{I} + \bar{a}_5 \; \bar{\mathbf{w}}^2 \otimes \bar{\mathbf{w}}^2 + \bar{a}_9 \left(\bar{\mathbf{w}}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \bar{\mathbf{w}}^2 \right) , \end{aligned} \tag{33}$$

with

$$\begin{aligned} \bar{a}_6 &= \frac{1}{a_6} \, , \quad \bar{a}_1 = \frac{\left(a_9{}^2 - a_1 a_5 \right) \left({}^{\mathrm{w}}\!I_2{}^2 - 2\, {}^{\mathrm{w}}\!I_1 \, {}^{\mathrm{w}}\!I_3 \right) - a_5 a_6 \, {}^{\mathrm{w}}\!I_3{}^2}{a_6 \; d} \, , \\ \bar{a}_5 &= -\frac{a_1 a_6 - \left(a_9{}^2 - a_1 a_5 \right) \left({}^{\mathrm{w}}\!I_1{}^2 - 2\, {}^{\mathrm{w}}\!I_2 \right)}{a_6 \; d} \, {}^{\mathrm{w}}\!I_3{}^2 \, , \quad \bar{a}_9 = -\frac{a_6 a_9 + 3(a_9{}^2 - a_1 a_5)}{a_6 \; d} \, {}^{\mathrm{w}}\!I_3{}^2 \, , \end{aligned} \tag{34}$$

where

$$\begin{aligned} d = &\Big(a_6(a_6 + 6a_9) + 9(a_9{}^2 - a_1 a_5) + a_5 a_6 \left({}^{\mathrm{w}}\!I_1{}^2 - 2\, {}^{\mathrm{w}}\!I_2 \right) \Big) \, {}^{\mathrm{w}}\!I_3{}^2 \\ &+ \Big(a_1 a_6 - (a_9{}^2 - a_1 a_5) \left({}^{\mathrm{w}}\!I_1{}^2 - 2\, {}^{\mathrm{w}}\!I_2 \right) \Big) \left({}^{\mathrm{w}}\!I_2{}^2 - 2\, {}^{\mathrm{w}}\!I_1 \, {}^{\mathrm{w}}\!I_3 \right) . \end{aligned} \tag{35}$$

Notice that there are *no constraints on the coefficients* for this solution case. It is apparent that *Solution (6)* is recovered as a particular case of *Solution (7)* by setting a_5=0, $\bar{a}_5$=0 in eqns (33)-(35). With respect to previous *Solutions (6)* and *(4)*, damage-effect tensors (33) lead to secant compliance and stiffness embedding the additional coefficients c_8, e_8 in both $\boldsymbol{\phi}, \bar{\boldsymbol{\phi}}$ and $\mathbf{w}, \bar{\mathbf{w}}$ expansions, that is comprising *all but* coefficients c_2, c_4, c_5 and e_2, e_4, e_5 in $\boldsymbol{\phi}, \bar{\boldsymbol{\phi}}$, and *all but* coefficients c_2, e_2 in $\mathbf{w}, \bar{\mathbf{w}}$. While the second option is not investigated further, the first eventuality is considered later in *Solution (10)*.

Solution (8). A further enlargement of previous *Solution (7)*, however based on constraints on the coefficients, can be provided by adding coefficients a_3 and $\bar{a}_3$ to *Solution (7)*, that is by assuming symmetric damage-effect tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ with only the five-coefficients sets $(a_1, a_3, a_5, a_6, a_9)$ and $(\bar{a}_1, \bar{a}_3, \bar{a}_5, \bar{a}_6, \bar{a}_9)$ (lacking coefficients a_2, a_4, a_7, a_8 and $\bar{a}_2, \bar{a}_4, \bar{a}_7, \bar{a}_8$):

$$\begin{aligned} \mathbb{A} &= a_6 \; \mathbf{w} \,\underline{\overline{\otimes}}\, \mathbf{w} + a_1 \; \mathbf{I} \otimes \mathbf{I} + a_3 \; \mathbf{w} \otimes \mathbf{w} + a_5 \; \mathbf{w}^2 \otimes \mathbf{w}^2 + a_9 \left(\mathbf{w}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{w}^2 \right) ; \\ \bar{\mathbb{A}} &= \bar{a}_6 \; \bar{\mathbf{w}} \,\underline{\overline{\otimes}}\, \bar{\mathbf{w}} + \bar{a}_1 \; \mathbf{I} \otimes \mathbf{I} + \bar{a}_3 \; \bar{\mathbf{w}} \otimes \bar{\mathbf{w}} + \bar{a}_5 \; \bar{\mathbf{w}}^2 \otimes \bar{\mathbf{w}}^2 + \bar{a}_9 \left(\bar{\mathbf{w}}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \bar{\mathbf{w}}^2 \right) , \end{aligned} \tag{36}$$

provided that

$$a_1 = -a_9\, \frac{{}^{\mathsf w}\!I_1\, {}^{\mathsf w}\!I_3}{{}^{\mathsf w}\!I_2}\,,\ a_5 = \frac{{a_9}^2}{a_1} = -a_9\, \frac{{}^{\mathsf w}\!I_2}{{}^{\mathsf w}\!I_1\, {}^{\mathsf w}\!I_3}\,;\ \ \bar a_1 = -\bar a_9\, \frac{{}^{\bar{\mathsf w}}\!I_1\, {}^{\bar{\mathsf w}}\!I_3}{{}^{\bar{\mathsf w}}\!I_2}\,,\ \bar a_5 = \frac{{\bar a_9}^2}{\bar a_1} = -\bar a_9\, \frac{{}^{\bar{\mathsf w}}\!I_2}{{}^{\bar{\mathsf w}}\!I_1\, {}^{\bar{\mathsf w}}\!I_3}\,, \quad (37)$$

with

$$\begin{gathered} \bar a_6 = \frac{1}{a_6}\,, \quad \bar a_3 = -\frac{a_3}{a_6\,(3a_3 + a_6)}\,, \quad \bar a_9 = -\frac{a_9\, {}^{\mathsf w}\!I_1\, {}^{\mathsf w}\!I_2\, {}^{\mathsf w}\!I_3}{a_6\; d'}\,; \\ \bar a_1 = \frac{a_9\, {{}^{\mathsf w}\!I_2}^2}{a_6\; d'}\,, \quad \bar a_5 = \frac{a_9\, {{}^{\mathsf w}\!I_1}^2\, {{}^{\mathsf w}\!I_3}^2}{a_6\; d'}\,, \end{gathered} \quad (38)$$

where

$$d' = a_6\, {}^{\mathsf w}\!I_1\, {}^{\mathsf w}\!I_2\, {}^{\mathsf w}\!I_3 - 2\, a_9 \left({{}^{\mathsf w}\!I_2}^2\, ({{}^{\mathsf w}\!I_1}^2 - {}^{\mathsf w}\!I_2) - {}^{\mathsf w}\!I_1\, {}^{\mathsf w}\!I_3\, ({{}^{\mathsf w}\!I_1}^2 + 3\, {}^{\mathsf w}\!I_2) \right)\,. \quad (39)$$

Notice that, when a_3=0, $\bar a_3$=0, eqn (38b) is consistently fulfilled, and *Solution (8)* degenerates only into a particular form of *Solution (7)* with constraints (37) holding on coefficients a_1, a_5 and $\bar a_1, \bar a_5$. Damage-effect tensors (36) formally lead to secant compliance and stiffness embedding *all but* coefficients c_2, c_4 and e_2, e_4 in the $\boldsymbol\phi, \bar{\boldsymbol\phi}$ expansions, and *all but* coefficients c_2, e_2 in the $\mathbf w, \bar{\mathbf w}$ representations. The first possibility leads to the quite general solution considered next.

Solution (9). As already noticed in *Solutions (1)*, *(5)*, and just now in *Solution (8)*, a full enlargement of previous *Solutions (5)*, *(8)* containing all the 'non-shear' terms (and thus without any constraint on the coefficients) can be obtained by assuming symmetric damage-effect tensors $\mathbb A$ and $\bar{\mathbb A}$ with the seven-coefficients sets $(a_1, a_3, a_5, a_6, a_7, a_8, a_9)$ and $(\bar a_1, \bar a_3, \bar a_5, \bar a_6, \bar a_7, \bar a_8, \bar a_9)$ (lacking only the two 'shear-like' coefficients a_2, a_4 and $\bar a_2, \bar a_4$):

$$\begin{aligned} \mathbb A &= a_6\ \mathbf w \,\overline{\underline{\otimes}}\, \mathbf w + a_1\ \mathbf I \otimes \mathbf I + a_3\ \mathbf w \otimes \mathbf w + a_5\ \mathbf w^2 \otimes \mathbf w^2 + a_7\ (\mathbf w \otimes \mathbf I + \mathbf I \otimes \mathbf w) \\ &\quad + a_8\ (\mathbf w^2 \otimes \mathbf w + \mathbf w \otimes \mathbf w^2) + a_9\ (\mathbf w^2 \otimes \mathbf I + \mathbf I \otimes \mathbf w^2)\,; \\ \bar{\mathbb A} &= \bar a_6\ \bar{\mathbf w} \,\overline{\underline{\otimes}}\, \bar{\mathbf w} + \bar a_1\ \mathbf I \otimes \mathbf I + \bar a_3\ \bar{\mathbf w} \otimes \bar{\mathbf w} + \bar a_5\ \bar{\mathbf w}^2 \otimes \bar{\mathbf w}^2 + \bar a_7\ (\bar{\mathbf w} \otimes \mathbf I + \mathbf I \otimes \bar{\mathbf w}) \\ &\quad + \bar a_8\ (\bar{\mathbf w}^2 \otimes \bar{\mathbf w} + \bar{\mathbf w} \otimes \bar{\mathbf w}^2) + \bar a_9\ (\bar{\mathbf w}^2 \otimes \mathbf I + \mathbf I \otimes \bar{\mathbf w}^2)\,, \end{aligned} \quad (40)$$

with

$$\begin{gathered} \bar a_6 = \frac{1}{a_6}\,, \quad \bar a_1 = -\frac{n_1^\star}{a_6\; d^\star}\,, \quad \bar a_3 = -\frac{n_3^\star}{a_6\; d^\star}\,, \quad \bar a_5 = \frac{n_5^\star\, {{}^{\mathsf w}\!I_3}^2}{a_6\; d^\star}\,, \\ \bar a_7 = -\frac{n_7^\star}{a_6\; d^\star}\,, \quad \bar a_8 = \frac{n_8^\star\, {}^{\mathsf w}\!I_3}{a_6\; d^\star}\,, \quad \bar a_9 = \frac{n_9^\star\, {}^{\mathsf w}\!I_3}{a_6\; d^\star}\,, \end{gathered} \quad (41)$$

where

$$\begin{aligned} d^\star = a_6 \Big(&-9\, a_1\, a_5 + 3\, a_3\, a_6 + (a_6 + 3\, a_9)^2 + 2\, a_6\, a_8\, {}^{\mathsf w}\!I_1 + a_5\, a_6\, ({{}^{\mathsf w}\!I_1}^2 - 2\, {}^{\mathsf w}\!I_2) \\ &+2\, (a_3\, a_5 - a_8^2)\, ({{}^{\mathsf w}\!I_1}^2 - 3\, {}^{\mathsf w}\!I_2) \Big)\, {{}^{\mathsf w}\!I_3}^2 \\ +a_6 \Big[&-2\, (a_1\, a_3 - a_7^2)\, {}^{\mathsf w}\!I_1 + 2\, \Big(a_6\, a_7 - 3\, (a_1\, a_8 - a_7\, a_9) \Big)\, {}^{\mathsf w}\!I_2 - 2\, (a_3\, a_9 - a_7\, a_8)\, ({}^{\mathsf w}\!I_1\, {}^{\mathsf w}\!I_2 - 9\, {}^{\mathsf w}\!I_3) \\ &+2\, (a_5\, a_7 - a_8\, a_9)\, \Big({}^{\mathsf w}\!I_2\, ({{}^{\mathsf w}\!I_1}^2 - 2\, {}^{\mathsf w}\!I_2) - 3\, {}^{\mathsf w}\!I_1\, {}^{\mathsf w}\!I_3 \Big) \Big]\, {}^{\mathsf w}\!I_3 \\ +a_6 \Big(&a_1 a_6 + 2(a_1 a_3 - a_7^2) + 2(a_1 a_8 - a_7 a_9) {}^{\mathsf w}\!I_1 + (a_1 a_5 - a_9^2)({{}^{\mathsf w}\!I_1}^2 - 2 {}^{\mathsf w}\!I_2) \Big) ({{}^{\mathsf w}\!I_2}^2 - 2 {}^{\mathsf w}\!I_1 {}^{\mathsf w}\!I_3) \\ -\Big(&a_1 a_8^2 - a_5 (a_1 a_3 - a_7^2) + a_9 (a_3 a_9 - 2 a_7 a_8) \Big) ({{}^{\mathsf w}\!I_1}^2 {{}^{\mathsf w}\!I_2}^2 - 4 {{}^{\mathsf w}\!I_2}^3 - 4 {{}^{\mathsf w}\!I_1}^3 {}^{\mathsf w}\!I_3 + 18 {}^{\mathsf w}\!I_1 {}^{\mathsf w}\!I_2 {}^{\mathsf w}\!I_3 - 27 {{}^{\mathsf w}\!I_3}^2)\,, \end{aligned} \quad (42)$$

$$n_1^\star = a_6\left[(a_1\,a_5-a_9^2)\,{}^{\rm w}I_1+2\,(a_5\,a_7-a_8\,a_9)\,{}^{\rm w}I_2+\Big(a_5\,(3\,a_3+a_6)-3\,a_8^2\Big)\,{}^{\rm w}I_3\right]{}^{\rm w}I_3$$
$$-\Big[2\,a_1\,a_8^2-a_5\,\Big(a_1\,(2\,a_3+a_6)-2\,a_7^2\Big)+a_9\,\Big(a_9\,(2\,a_3+a_6)-4\,a_7\,a_8\Big)\Big]\,({}^{\rm w}I_2{}^2-3\,{}^{\rm w}I_1\,{}^{\rm w}I_3)\;, \tag{43}$$

$$n_3^\star = a_6\,\Big(a_3\,a_6+6\,(a_3\,a_9-a_7\,a_8)+(a_3\,a_5-a_8^2)\,({}^{\rm w}I_1{}^2-2\,{}^{\rm w}I_2)\Big)\,{}^{\rm w}I_3{}^2$$
$$-\Big(a_1\,a_8^2-a_5\,(a_1\,a_3-a_7^2)+a_9\,(a_3\,a_9-2\,a_7\,a_8)\Big)\Big(({}^{\rm w}I_1{}^2-2\,{}^{\rm w}I_2)\,({}^{\rm w}I_2{}^2-2\,{}^{\rm w}I_1\,{}^{\rm w}I_3)-9\,{}^{\rm w}I_3{}^2\Big)$$
$$+a_6\,(a_1\,a_3-a_7^2)\,({}^{\rm w}I_2{}^2-2\,{}^{\rm w}I_1\,{}^{\rm w}I_3)\;, \tag{44}$$

$$\begin{aligned} n_5^\star &= a_6\,\Big(3\,a_7^2-a_1\,(3\,a_3+a_6)-2\,(a_1\,a_8-a_7\,a_9)\,{}^{\rm w}I_1-(a_1\,a_5-a_9^2)\,({}^{\rm w}I_1{}^2-2\,{}^{\rm w}I_2)\Big)\\ &\quad+2\,\Big(a_1\,a_8^2-a_5\,(a_1\,a_3-a_7^2)+a_9\,(a_3\,a_9-2\,a_7\,a_8)\Big)\,({}^{\rm w}I_1{}^2-3\,{}^{\rm w}I_2)\;, \end{aligned} \tag{45}$$

$$\begin{aligned} n_7^\star &= a_6\,a_8\,(a_6+3\,a_9+a_8\,{}^{\rm w}I_1)\,{}^{\rm w}I_3{}^2+a_6\,(a_1\,a_8-a_7\,a_9)\,({}^{\rm w}I_2{}^2-2\,{}^{\rm w}I_1\,{}^{\rm w}I_3)\\ &\quad+\Big(a_5\,a_7^2+a_9\,(a_3\,a_9-2\,a_7\,a_8)-a_1\,(a_3\,a_5-a_8^2)\Big)\Big({}^{\rm w}I_1\,({}^{\rm w}I_2{}^2-2\,{}^{\rm w}I_1\,{}^{\rm w}I_3)-3\,{}^{\rm w}I_2\,{}^{\rm w}I_3\Big)\\ &\quad-a_6\,{}^{\rm w}I_3\,\Big(a_7\,(3\,a_5\,{}^{\rm w}I_3-a_8\,{}^{\rm w}I_2)+a_3\,(a_5\,{}^{\rm w}I_1\,{}^{\rm w}I_3+a_9\,{}^{\rm w}I_2)\Big)\;, \end{aligned} \tag{46}$$

$$\begin{aligned} n_8^\star &= a_6\,(a_1\,a_3-a_7^2)\,{}^{\rm w}I_2\\ &\quad+a_6\,\Big[3\,a_1\,a_8-a_7\,(a_6+3\,a_9)-a_7\,\Big(a_8\,{}^{\rm w}I_1+a_5\,({}^{\rm w}I_1{}^2-2\,{}^{\rm w}I_2)\Big)\\ &\qquad+a_9\,\Big(a_3\,{}^{\rm w}I_1+a_8\,({}^{\rm w}I_1{}^2-2\,{}^{\rm w}I_2)\Big)\Big]\,{}^{\rm w}I_3\\ &\quad-\Big(a_1\,a_8^2-a_5\,(a_1\,a_3-a_7^2)+a_9\,(a_3\,a_9-2\,a_7\,a_8)\Big)\Big({}^{\rm w}I_2\,({}^{\rm w}I_1{}^2-2\,{}^{\rm w}I_2)-3\,{}^{\rm w}I_1\,{}^{\rm w}I_3\Big)\;, \end{aligned} \tag{47}$$

$$\begin{aligned} n_9^\star &= a_6\,(a_1\,a_8-a_7\,a_9)\,{}^{\rm w}I_2\\ &\quad+a_6\,\Big(3\,(a_1\,a_5+a_7\,a_8)-a_9\,(3\,a_3+a_6+3\,a_9)+(a_5\,a_7-a_8\,a_9)\,{}^{\rm w}I_1\Big)\,{}^{\rm w}I_3\\ &\quad+\Big(a_1\,a_8^2-a_5\,(a_1\,a_3-a_7^2)+a_9\,(a_3\,a_9-2\,a_7\,a_8)\Big)\,({}^{\rm w}I_1\,{}^{\rm w}I_2-9\,{}^{\rm w}I_3)\;. \end{aligned} \tag{48}$$

It can be seen that *Solution (7)*, eqns (33)-(35), which works as well with *no constraints on the coefficients*, can be recovered from *Solution (9)* by setting a_3=a_7=a_8=0 in eqns (40)-(48), consistently with $\bar{a}_3$=$\bar{a}_7$=$\bar{a}_8$=0. Damage-effect tensors (40) lead to secant compliance and stiffness embedding *all but* coefficients c_2, c_4 and e_2, e_4 in the $\boldsymbol{\phi}, \bar{\boldsymbol{\phi}}$ expansions, and *all but* coefficients c_2, e_2 in the $\mathbf{w}, \bar{\mathbf{w}}$ representations. Despite the fact that *Solution (9)* includes all the 'non-shear' coefficients, eqns (41)-(48) display a reasonable degree of complexity comparing to other solutions that are presented in the paper (and as well with respect to others that are not included here). This makes *Solution (9)* at the same time the more general form and much compact instance of the solution set that works without constraints on the coefficients and lacks only the two 'shear-like' terms missing in *Solution (s.b)*.

Solution (10). As suggested at the end of *Solution (7)*, another solution instance can be conceived either as a particular case of *Solution (9)* for a_5=0, $\bar{a}_5$=0, or as an enlargement of *Solution (5)* with the additional coefficients $a_8, \bar{a}_8$, that is by taking symmetric damage-effect tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ with the six-coefficients sets $(a_1, a_3, a_6, a_7, a_8, a_9)$ and

$(\bar{a}_1, \bar{a}_3, \bar{a}_6, \bar{a}_7, \bar{a}_8, \bar{a}_9)$ (lacking only coefficients a_2, a_4, a_5 and $\bar{a}_2, \bar{a}_4, \bar{a}_5$):

$$
\begin{aligned}
\mathbb{A} &= a_6\, \mathbf{w} \,\underline{\overline{\otimes}}\, \mathbf{w} + a_1\, \mathbf{I} \otimes \mathbf{I} + a_3\, \mathbf{w} \otimes \mathbf{w} + a_7\, (\mathbf{w} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{w}) + a_8\, (\mathbf{w}^2 \otimes \mathbf{w} + \mathbf{w} \otimes \mathbf{w}^2) \\
&\quad + a_9\, (\mathbf{w}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{w}^2)\,; \\
\bar{\mathbb{A}} &= \bar{a}_6\, \bar{\mathbf{w}} \,\underline{\overline{\otimes}}\, \bar{\mathbf{w}} + \bar{a}_1\, \mathbf{I} \otimes \mathbf{I} + \bar{a}_3\, \bar{\mathbf{w}} \otimes \bar{\mathbf{w}} + \bar{a}_7\, (\bar{\mathbf{w}} \otimes \mathbf{I} + \mathbf{I} \otimes \bar{\mathbf{w}}) + \bar{a}_8\, (\bar{\mathbf{w}}^2 \otimes \bar{\mathbf{w}} + \bar{\mathbf{w}} \otimes \bar{\mathbf{w}}^2) \\
&\quad + \bar{a}_9\, (\bar{\mathbf{w}}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \bar{\mathbf{w}}^2)\,,
\end{aligned}
\tag{49}
$$

provided that

$$
\begin{aligned}
a_1 &= \frac{2\, a_9\, (a_3\, a_9 - 2\, a_7\, a_8)\, ({}^{\mathrm{w}}I_1{}^2 - 3\, {}^{\mathrm{w}}I_2) + a_6\, \Big(a_7\, (3\, a_7 + 2\, a_9 {}^{\mathrm{w}}I_1) + a_9^2\, ({}^{\mathrm{w}}I_1{}^2 - 2\, {}^{\mathrm{w}}I_2)\Big)}{a_6\, (3\, a_3 + a_6) + 2\, a_8\, \Big(a_6\, {}^{\mathrm{w}}I_1 - a_8\, ({}^{\mathrm{w}}I_1{}^2 - 3\, {}^{\mathrm{w}}I_2)\Big)}\,; \\
\bar{a}_1 &= \frac{2\, \bar{a}_9\, (\bar{a}_3\, \bar{a}_9 - 2\, \bar{a}_7\, \bar{a}_8)\, ({}^{\bar{\mathrm{w}}}I_1{}^2 - 3\, {}^{\bar{\mathrm{w}}}I_2) + \bar{a}_6\, \Big(\bar{a}_7\, (3\, \bar{a}_7 + 2\, \bar{a}_9 {}^{\bar{\mathrm{w}}}I_1) + \bar{a}_9^2\, ({}^{\bar{\mathrm{w}}}I_1{}^2 - 2\, {}^{\bar{\mathrm{w}}}I_2)\Big)}{\bar{a}_6\, (3\, \bar{a}_3 + \bar{a}_6) + 2\, \bar{a}_8\, \Big(\bar{a}_6\, {}^{\bar{\mathrm{w}}}I_1 - \bar{a}_8\, ({}^{\bar{\mathrm{w}}}I_1{}^2 - 3\, {}^{\bar{\mathrm{w}}}I_2)\Big)}\,,
\end{aligned}
\tag{50}
$$

with

$$
\begin{gathered}
\bar{a}_6 = \frac{1}{a_6}\,, \quad \bar{a}_1 = -\frac{\check{n}_1}{a_6\, \check{d}^2}\,, \quad \bar{a}_3 = -\frac{\check{n}_3}{a_6\, \check{d}^2}\,, \quad \bar{a}_7 = -\frac{\check{n}_7}{a_6\, \check{d}^2}\,, \\
\bar{a}_8 = \frac{-a_6\, a_7 + (a_3\, a_9 - a_7\, a_8)\, {}^{\mathrm{w}}I_1 + a_8\, a_9\, ({}^{\mathrm{w}}I_1{}^2 - 2 {}^{\mathrm{w}}I_2)}{a_6\, \check{d}}\, {}^{\mathrm{w}}I_3\,, \\
\bar{a}_9 = \frac{3\, a_7\, a_8 - a_9\, (3\, a_3 + a_6 + a_8\, {}^{\mathrm{w}}I_1)}{a_6\, \check{d}}\, {}^{\mathrm{w}}I_3\,,
\end{gathered}
\tag{51}
$$

where

$$
\begin{aligned}
\check{d} &= a_6\, a_7\, {}^{\mathrm{w}}I_2 + a_6\, (3\, a_3 + a_6 + 3\, a_9 + 2\, a_8\, {}^{\mathrm{w}}I_1)\, {}^{\mathrm{w}}I_3 - 2\, a_8^2\, ({}^{\mathrm{w}}I_1{}^2 - 3\, {}^{\mathrm{w}}I_2)\, {}^{\mathrm{w}}I_3 \\
&\quad - a_8\, a_9\, \Big(({}^{\mathrm{w}}I_1{}^2 - 2\, {}^{\mathrm{w}}I_2)\, {}^{\mathrm{w}}I_2 - 3\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3\Big) - (a_3\, a_9 - a_7\, a_8)\, ({}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2 - 9\, {}^{\mathrm{w}}I_3)\,,
\end{aligned}
\tag{52}
$$

$$
\begin{gathered}
\check{n}_1 = 2\, a_8^3\, (3\, a_8\, {}^{\mathrm{w}}I_3 + 2\, a_9\, {}^{\mathrm{w}}I_2)\, ({}^{\mathrm{w}}I_1{}^2 - 3 {}^{\mathrm{w}}I_2)\, {}^{\mathrm{w}}I_3 + 2\, a_8^2\, a_9^2\, ({}^{\mathrm{w}}I_1{}^3\, {}^{\mathrm{w}}I_3 - {}^{\mathrm{w}}I_2{}^3) \\
-\Big(3\, a_7\, a_8 - a_9\, (3\, a_3 + a_6)\Big)\Big(2\, a_7\, a_8 - a_9 (3\, a_3 + a_6 + 2\, a_8\, {}^{\mathrm{w}}I_1)\Big)\, ({}^{\mathrm{w}}I_2{}^2 - 3\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3) \\
+(3a_3 + a_6 + 2a_8 {}^{\mathrm{w}}I_1)\Big(a_9 (a_9 a_3 - a_7 a_8)({}^{\mathrm{w}}I_2{}^2 - 3 {}^{\mathrm{w}}I_1 {}^{\mathrm{w}}I_3) - a_6 {}^{\mathrm{w}}I_3 (a_9^2 {}^{\mathrm{w}}I_1 + 2 a_8 a_9 {}^{\mathrm{w}}I_2 + 3 a_8^2 {}^{\mathrm{w}}I_3)\Big)\,,
\end{gathered}
\tag{53}
$$

$$
\begin{gathered}
\check{n}_3 = 2\, a_8^4\, ({}^{\mathrm{w}}I_1{}^2 - 3\, {}^{\mathrm{w}}I_2)\, ({}^{\mathrm{w}}I_1{}^2 - 2\, {}^{\mathrm{w}}I_2)\, {}^{\mathrm{w}}I_3{}^2 - a_7^2\, a_8^2\, ({}^{\mathrm{w}}I_1{}^2\, {}^{\mathrm{w}}I_2{}^2 - 2\, {}^{\mathrm{w}}I_1{}^3\, {}^{\mathrm{w}}I_3 - 27\, {}^{\mathrm{w}}I_3{}^2) \\
-a_6^2\, \Big(6\, a_7\, a_8\, {}^{\mathrm{w}}I_3{}^2 + a_8^2\, ({}^{\mathrm{w}}I_1{}^2 - 2\, {}^{\mathrm{w}}I_2)\, {}^{\mathrm{w}}I_3{}^2 + a_7^2\, ({}^{\mathrm{w}}I_2{}^2 - 2\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3)\Big) \\
+a_8^2\, a_9^2\, ({}^{\mathrm{w}}I_1{}^2 - 2\, {}^{\mathrm{w}}I_2)\, \Big(9\, {}^{\mathrm{w}}I_3{}^2 - ({}^{\mathrm{w}}I_1{}^2 - 2\, {}^{\mathrm{w}}I_2)\, ({}^{\mathrm{w}}I_2{}^2 - 2\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3)\Big) \\
+a_3^2\, \Big(3\, a_6^2\, {}^{\mathrm{w}}I_3{}^2 + 18\, a_6\, a_9\, {}^{\mathrm{w}}I_3{}^2 - a_9^2\, ({}^{\mathrm{w}}I_1{}^2\, {}^{\mathrm{w}}I_2{}^2 - 2\, {}^{\mathrm{w}}I_1{}^3\, {}^{\mathrm{w}}I_3 - 27\, {}^{\mathrm{w}}I_3{}^2)\Big) \\
+2\, a_7\, a_8^2\, \Big[6\, a_8\, ({}^{\mathrm{w}}I_1{}^2 - 3\, {}^{\mathrm{w}}I_2)\, {}^{\mathrm{w}}I_3{}^2 - a_9\, {}^{\mathrm{w}}I_1\, \Big(9\, {}^{\mathrm{w}}I_3{}^2 - ({}^{\mathrm{w}}I_1{}^2 - 2\, {}^{\mathrm{w}}I_2)\, ({}^{\mathrm{w}}I_2{}^2 - 2\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3)\Big)\Big] \\
-2\, a_6\, a_8\, \Big\{a_8^2\, {}^{\mathrm{w}}I_1\, ({}^{\mathrm{w}}I_1{}^2 - 2\, {}^{\mathrm{w}}I_2)\, {}^{\mathrm{w}}I_3{}^2 + a_7^2\, {}^{\mathrm{w}}I_1\, ({}^{\mathrm{w}}I_2{}^2 - 2\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3) \\
+a_7\, \Big[6\, a_8\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3{}^2 + a_9\, \Big(9\, {}^{\mathrm{w}}I_3{}^2 - ({}^{\mathrm{w}}I_1{}^2 - 2\, {}^{\mathrm{w}}I_2)\, ({}^{\mathrm{w}}I_2{}^2 - 2\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3)\Big)\Big]\Big\} \\
+a_3\, \Big\{a_6^3\, {}^{\mathrm{w}}I_3{}^2 + 2\, a_6^2\, (3\, a_9 + a_8\, {}^{\mathrm{w}}I_1)\, {}^{\mathrm{w}}I_3{}^2 - 12\, a_8^2\, a_9\, ({}^{\mathrm{w}}I_1{}^2 - 3\, {}^{\mathrm{w}}I_2)\, {}^{\mathrm{w}}I_3{}^2 \\
+2 a_7 a_8 a_9 ({}^{\mathrm{w}}I_1{}^2 {}^{\mathrm{w}}I_2{}^2 - 2 {}^{\mathrm{w}}I_1{}^3 {}^{\mathrm{w}}I_3 - 27 {}^{\mathrm{w}}I_3{}^2) + 2 a_8 a_9^2 {}^{\mathrm{w}}I_1 \Big(9 {}^{\mathrm{w}}I_3{}^2 - ({}^{\mathrm{w}}I_1{}^2 - 2 {}^{\mathrm{w}}I_2)({}^{\mathrm{w}}I_2{}^2 - 2 {}^{\mathrm{w}}I_1 {}^{\mathrm{w}}I_3)\Big) \\
+a_6 \Big[\Big(9 a_9^2 + 12 a_8 a_9 {}^{\mathrm{w}}I_1 - a_8^2 (5 {}^{\mathrm{w}}I_1{}^2 - 12 {}^{\mathrm{w}}I_2)\Big) {}^{\mathrm{w}}I_3{}^2 - a_7 \Big(18 a_8 {}^{\mathrm{w}}I_3{}^2 - 2 a_9 {}^{\mathrm{w}}I_1 ({}^{\mathrm{w}}I_2{}^2 - 2 {}^{\mathrm{w}}I_1 {}^{\mathrm{w}}I_3)\Big)\Big]\Big\}\,,
\end{gathered}
\tag{54}
$$

$$
\begin{aligned}
\check{n}_7 = {} & a_6^3\, a_8\, {}^{\mathrm{w}}I_3^{\,2} - 6\, a_8^3\, a_9\, ({}^{\mathrm{w}}I_1^{\,2} - 3\, {}^{\mathrm{w}}I_2)\, {}^{\mathrm{w}}I_3^{\,2} - 2\, a_8^4\, ({}^{\mathrm{w}}I_1^{\,3} - 3\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2)\, {}^{\mathrm{w}}I_3^{\,2} \\
& - a_8^2 \Big(3\, a_7^2 + a_9^2\, ({}^{\mathrm{w}}I_1^{\,2} - 2\, {}^{\mathrm{w}}I_2)\Big) \Big(3\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3 - {}^{\mathrm{w}}I_1\, ({}^{\mathrm{w}}I_2^{\,2} - 2\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3)\Big) \\
& - 3\, a_3^2\, a_9\, \Big(a_6\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3 - a_9\, ({}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2^{\,2} - 2\, {}^{\mathrm{w}}I_1^{\,2}\, {}^{\mathrm{w}}I_3 - 3\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3)\Big) \\
& - 2\, a_7\, a_8^2\, \Big(a_8\, ({}^{\mathrm{w}}I_1^{\,2} - 3\, {}^{\mathrm{w}}I_2)\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3 + a_9\, (2\, {}^{\mathrm{w}}I_1^{\,2}\, {}^{\mathrm{w}}I_2^{\,2} - 3\, {}^{\mathrm{w}}I_2^{\,3} - 4\, {}^{\mathrm{w}}I_1^{\,3}\, {}^{\mathrm{w}}I_3 + 3\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3)\Big) \\
& + a_6^2\, \Big[3\, a_8\, (a_9 + a_8\, {}^{\mathrm{w}}I_1)\, {}^{\mathrm{w}}I_3^{\,2} + a_7\, \Big(a_8\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3 - a_9\, ({}^{\mathrm{w}}I_2^{\,2} - 2\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3)\Big)\Big] \\
& + a_6\, a_8\, \Big[6\, a_8 (a_9\, {}^{\mathrm{w}}I_1 + a_8\, {}^{\mathrm{w}}I_2) {}^{\mathrm{w}}I_3^{\,2} + \Big(3\, a_7^2 + a_9^2\, ({}^{\mathrm{w}}I_1^{\,2} - 2\, {}^{\mathrm{w}}I_2)\Big)\, ({}^{\mathrm{w}}I_2^{\,2} - 2\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3) \\
& \quad + 2\, a_7\, \Big(a_8\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3 - a_9\, ({}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2^{\,2} - 2\, {}^{\mathrm{w}}I_1^{\,2}\, {}^{\mathrm{w}}I_3 - 3\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3)\Big)\Big] \\
& + a_3\, \Big\{2\, a_8^2\, a_9\, ({}^{\mathrm{w}}I_1^{\,2} - 3\, {}^{\mathrm{w}}I_2)\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3 - a_6^2\, {}^{\mathrm{w}}I_3\, (a_9\, {}^{\mathrm{w}}I_2 - 3\, a_8\, {}^{\mathrm{w}}I_3) \\
& \quad + 2\, a_8\, a_9^2\, (2\, {}^{\mathrm{w}}I_1^{\,2}\, {}^{\mathrm{w}}I_2^{\,2} - 3\, {}^{\mathrm{w}}I_2^{\,3} - 4\, {}^{\mathrm{w}}I_1^{\,3}\, {}^{\mathrm{w}}I_3 + 3\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3) \\
& \quad + 6\, a_7\, a_8\, a_9\, \Big(3\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3 - {}^{\mathrm{w}}I_1\, ({}^{\mathrm{w}}I_2^{\,2} - 2\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3)\Big) \\
& \quad - a_6\, \Big[a_8\, a_9\, (2\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2 - 9\, {}^{\mathrm{w}}I_3)\, {}^{\mathrm{w}}I_3 - 3\, a_8^2\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3^{\,2} - a_9^2\, ({}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2^{\,2} - 2\, {}^{\mathrm{w}}I_1^{\,2}\, {}^{\mathrm{w}}I_3 - 3\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3) \\
& \qquad - 3\, a_7\, \Big(a_8\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3 - a_9\, ({}^{\mathrm{w}}I_2^{\,2} - 2\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3)\Big)\Big]\Big\}\, .
\end{aligned}
\tag{55}
$$

Notice that, when a_8=0, $\bar{a}_8$=0 and constraints (28a,c) hold, constraints (50) degenerate into constraints (28b,d). Damage-effect tensors (49) lead to secant compliance and stiffness embedding *all but* coefficients c_2, c_4 and e_2, e_4 in the $\boldsymbol{\phi}, \bar{\boldsymbol{\phi}}$ expansions, and *all but* coefficients c_2, e_2 in the $\mathbf{w}, \bar{\mathbf{w}}$ representations. Despite the fact that *Solution (10)* is just a particular case of *Solution (9)*, it turns out to display a higher degree of complexity in the expressions of the dual coefficients. Other particular instances of *Solution (9)* based on six-coefficients sets could be obtained as well as alternatives of *Solution (10)* by setting to zero, instead of a_5, one of the remaining 'non-shear' coefficients and getting a corresponding constraint on the parameters. However, these solutions turn out to be even more complex than *Solution (10)* and are not presented here.

The solutions already advanced here are now framed by a more general treatment.

4.2 General framework

The general solution of the problem under consideration is in principle formally simple to obtain. Consider the **L**-2**M** matrix representation of $\mathbb{A}$, Appendix A, eqn (A.7), with six 'Lamé's-like' parameters l_{ij}, $i{\neq}j$=1-3, three 'shear-like' parameters m_{ij}, $i{<}j$=1-3, and three diagonal parameters l_{ii}+2m_{ii}, i=1-3, expressed in eqns (A.4), (A.6) in terms of twelve coefficients a_i, i=1-6; a_{i1}, a_{i2}, i=7-9, and principal values w_i, i=1-3. Consider as well a similar dual $\bar{\mathbf{L}}$-2$\bar{\mathbf{M}}$ decomposition of $\bar{\mathbb{A}}$ and dual relations for $\bar{l}_{ij}$, $\bar{m}_{ij}$ and $\bar{l}_{ii}$+2$\bar{m}_{ii}$ in terms of twelve dual coefficients $\bar{a}_i, \bar{a}_{i1}, \bar{a}_{i2}$, and principal values $\bar{w}_i$=1/w_i.

For the problem at hand, we want to impose that 3×3 submatrices **L**, $\bar{\mathbf{L}}$ and 2**M**, 2$\bar{\mathbf{M}}$ correspond to each other through an inversion operation (notice that here, with the present notation privileging the duality structure of 2**M** and 2$\bar{\mathbf{M}}$, $\bar{\mathbf{M}}$=$\mathbf{M}^{-1}/4{\neq}\mathbf{M}^{-1}$). Working like that it is possible to express formally the complementary coefficients $\bar{a}_i, \bar{a}_{i1}, \bar{a}_{i2}$ of the sought solution set in terms of the given a_i, a_{i1}, a_{i2} and principal values w_i or invariants of **w** (and viceversa). In this respect, note that the solution of the three coefficients of the 'shear terms' with symmetrized dyadic products '$\underline{\otimes}$' solves independently from the others; this solution is also considerably simpler due to the diagonal nature of matrices $2\mathbf{M}$ and $2\bar{\mathbf{M}}$. However, the problem is not fully decoupled in the sense that these latter coefficients also enter the final expression of the solution of the coefficients of the 'non-shear' rank-one update terms with dyadic products '$\otimes$'. This is due to the fact that the 'shear-like' parameters affect as well the upper three-diagonal entries of $\mathbf{L}$ and $\bar{\mathbf{L}}$ in the matrix representations of the damage-effect tensors (Appendix A).

The full problem solution could be achieved in at least two ways: (a) directly, by first inverting formally $3{\times}3$ (full) matrix $\mathbf{L}$ and (diagonal) matrix $2\mathbf{M}$ (expressed in terms of l_{ij}, m_{ij}, a_i, a_{i1}, a_{i2}, w_i), subsequently imposing entry-by-entry the equality to $\bar{\mathbf{L}}$ and $2\bar{\mathbf{M}}$ (expressed in terms of $\bar{l}_{ij}$, $\bar{m}_{ij}$, $\bar{a}_i$, $\bar{a}_{i1}, \bar{a}_{i2}$, $\bar{w}_i{=}1/w_i$) and finally solving the obtained equations in terms of the coefficients $\bar{a}_i$, $\bar{a}_{i1}, \bar{a}_{i2}$ (this last phase involves the solution of a $9{\times}9$ and a $3{\times}3$ system); (b) indirectly, by first preparing inverse relations (A.8)-(A.13), written in terms of coefficients and parameters with bars (which imply the solution of a $9{\times}9$ and a $3{\times}3$ system at this stage), and replacing in them 'Lamé's-like' and 'shear-like' parameters with bars coming from the entries of the formal inverses of $\mathbf{L}$ and $2\mathbf{M}$ (expressed in terms of a_i, a_{i1}, a_{i2}).

What just explained here should become clearer now with the solution of the 'shear-like' coefficients, which can be done in a straight-forward manner and is reported first below. However, although the complete solution of the other nine 'non-shear' coefficients can be determined rapidly by using any mathematical symbolic software, it finally results rather involved and too lengthy to be reported in closed form. Also, and most important, it is difficult to locate in this general solution the convenient forms of complementary damage-effect tensors that possess just a reduced number of corresponding non-zero coefficients and relevant tensor terms. Thus, instead of trying to enumerate all the possible special solutions, we will be content to collect a set of particular solutions that, as done in Section 4.1, are guessed or determined through alternative procedures and are then verified to belong to the general solution discussed above.

4.3 'Shear-like' coefficients

The solution related to the inversion of the three 'shear-like' parameters is quite straight-forward. Since diagonal $3{\times}3$ matrices $2\mathbf{M}$ and $2\bar{\mathbf{M}}$ must be the inverse of each other, the three 'shear-like' parameters m_{ij} and $\bar{m}_{ij}$ must satisfy the three relations $4m_{ij}\bar{m}_{ij}{=}1$, $i{<}j{=}1\text{-}3$. As noted above, this can be done in at least two ways: (a) take eqns (A.4b) for $2\bar{m}_{ij}$ in terms of $\bar{a}_2, \bar{a}_4, \bar{a}_6$ and $\bar{w}_i$; pose $\bar{w}_i{=}1/w_i$ and $2\bar{m}_{ij}{=}1/(2m_{ij})$; invert the system of the three obtained equations and express the three $2m_{ij}$ by relations (A.4b) to get $\bar{a}_2, \bar{a}_4, \bar{a}_6$ in terms of a_2, a_4, a_6 and eigenvalues w_i; (b) take already inverse relations (A.13) for the coefficients and parameters with bars and eigenvalues $\bar{w}_i{=}1/w_i$; replace in them $2\bar{m}_{ij}{=}1/(2m_{ij})$, with $2m_{ij}$ given by eqn (A.4b).

Solution (s). The general 'shear-like' solution arising from either of the two approaches above can be expressed in compact closed form as follows:

$$
\begin{cases}
\bar{a}_2 = \dfrac{a_2^2 + a_4(a_2\,{}^{\mathrm{w}}I_1 + a_4\,{}^{\mathrm{w}}I_2 + a_6\,{}^{\mathrm{w}}I_3)}{d_s} \,, \\[2ex]
\bar{a}_4 = \dfrac{a_2\,a_6 - a_4^2}{d_s}\ {}^{\mathrm{w}}I_3 \,, \\[2ex]
\bar{a}_6 = \dfrac{a_4(a_2 + a_4\,{}^{\mathrm{w}}I_1 + a_6\,{}^{\mathrm{w}}I_2) + a_6^2\ {}^{\mathrm{w}}I_3}{d_s}\ {}^{\mathrm{w}}I_3 \,,
\end{cases}
\tag{56}
$$

where the denominator factor

$$
\begin{aligned}
d_s &= 2m_{12}\,2m_{23}\,2m_{13} \\
&= {a_2}^3 + {a_2}^2\,(2\,a_4\,{}^{\mathrm{w}}I_1 + a_6\,{}^{\mathrm{w}}I_2) + {a_4}^3\,({}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2 - {}^{\mathrm{w}}I_3) + 2\,a_4\,{a_6}^2\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3 + {a_6}^3\,{{}^{\mathrm{w}}I_3}^2 \\
&\quad + {a_4}^2\,a_6({{}^{\mathrm{w}}I_2}^2 + {}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3) + a_2\left({a_4}^2\,({{}^{\mathrm{w}}I_1}^2 + {}^{\mathrm{w}}I_2) + {a_6}^2\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3 + a_4\,a_6\,({}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2 + 3\,{}^{\mathrm{w}}I_3)\right)
\end{aligned}
\tag{57}
$$

is obtained in invariant form from the expressions of the 'shear-like' parameters $2m_{12}$, $2m_{23}$, $2m_{13}$ given in eqn (A.4b) in terms of coefficients a_2, a_4, a_6 and principal values w_1, w_2, w_3. Notice that, when the natural constraint (8a) is satisfied by a_2, a_4, a_6 for $\mathbf{w}{=}\mathbf{I}$, namely $[a_2{+}2a_4{+}a_6](\mathbf{I}){=}1$, the analogous constraint for $\bar{a}_2, \bar{a}_4, \bar{a}_6$ is automatically granted. Indeed, for $\mathbf{w}{=}\mathbf{I}$, from eqns (56), (57) one obtains $d_s(\mathbf{I}){=}([a_2{+}2a_4{+}a_6](\mathbf{I}))^3$ and $[\bar{a}_2{+}2\bar{a}_4{+}\bar{a}_6](\mathbf{I}){=}1/([a_2{+}2a_4{+}a_6](\mathbf{I}))$.

Particular instances of general 'shear-like' solution (56) with *a single non-zero 'shear-like' coefficient* (both with coefficient a_4 equal to zero) are the following:

Solution (s.a): $a_4{=}a_6{=}0$, $a_2{\neq}0$ and $\bar{a}_4{=}\bar{a}_6{=}0$, $\bar{a}_2{=}1/a_2$ (leading to possibly-constant 'shear-like' terms $2m_{ij}{=}a_2$);

Solution (s.b): $a_2{=}a_4{=}0$, $a_6{\neq}0$ and $\bar{a}_2{=}\bar{a}_4{=}0$, $\bar{a}_6{=}1/a_6$ (leading to Valanis-type 'shear-like' terms $2m_{ij}{=}a_6 w_i w_j$).

These two cases, when not accompanied by other non-zero coefficients of the 'non-shear' terms, *do form* by themselves instances of the solution set and render respectively known *Solution (0.1)* of isotropic damage based on a single scalar damage variable (or even of no damage if $a_2{=}1$), and *Solution (0.2)* of Valanis-type damage. Examples of the solution set based on *Solution (s.a)* will be provided in Section 4.4, while examples of the solution set based on *Solution (s.b)* have been given in Section 4.1 and additional ones will be reported in Section 4.4.

Particular cases of general solution (56) with *two non-zero 'shear-like' coefficients* (both pivoting on non-zero coefficient a_4, with common feature $\bar{a}_4{=}1/a_4$) are the following:

Solution (s.c): $a_2{=}0$, $a_6{=}{-}a_4{}^{\mathrm{w}}I_2/{}^{\mathrm{w}}I_3{=}{-}a_4{}^{\bar{\mathrm{w}}}I_1$ and $\bar{a}_2{=}0$, $\bar{a}_4{=}1/a_4$, $\bar{a}_6{=}{-}\bar{a}_4{}^{\bar{\mathrm{w}}}I_2/{}^{\bar{\mathrm{w}}}I_3{=}$ ${-}\bar{a}_4{}^{\mathrm{w}}I_1{=}\ {}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2/(a_6{}^{\mathrm{w}}I_3)$;

Solution (s.d): $a_6{=}0$, $a_2{=}{-}a_4{}^{\bar{\mathrm{w}}}I_2/{}^{\bar{\mathrm{w}}}I_3{=}{-}a_4{}^{\mathrm{w}}I_1$ and $\bar{a}_6{=}0$, $\bar{a}_4{=}1/a_4$, $\bar{a}_2{=}{-}\bar{a}_4{}^{\mathrm{w}}I_2/{}^{\mathrm{w}}I_3{=}$ ${-}\bar{a}_4{}^{\bar{\mathrm{w}}}I_1{=}{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2/(a_2{}^{\mathrm{w}}I_3)$.

These two cases, when not accompanied by other non-zero coefficients of the 'non-shear' terms, *do not form* by themselves instances of the solution set: the inversion of damage-effect tensors based only on 'shear-like' terms *(s.c)* and *(s.d)* spreads over all the 'non-shear' terms of the dual base. *Solution (s.c)* has been recognized as being part of *Solutions (2)* and *(3)* and used there to get the expressions of the other 'non-shear' coefficients through the inspection of the general solution. Additional solution instances based on *Solution (s.c)* and generalizing *Solutions (2)-(3)* will be given in Section 4.4.2, *Solutions (25)* and *(26)*. Specific new solution instances adding 'non-shear' coefficients to *Solution (s.d)* have not been suggested by the additional foreseen solutions advanced in Section 4.1. Obviously this does not imply that such solutions do not exist. However, without any specific hint, the inspection of the general solution for this case turns out to be particularly difficult. Actually, this holds as well for guesses that may arise from the known solution instances. For example, it is possible to verify that a counterpart of Solution *(3)* lacking only coefficients (a_1, a_6, a_9), $(\bar{a}_1, \bar{a}_6, \bar{a}_9)$ is indeed a solution instance but expressed by relations exceeding by far in complexity that of Solution *(3)*, and rather resembling more that of *Solution (25)*. The same holds as well for a counterpart of *Solution (25)* itself, which would lack only coefficients (a_1, a_6), $(\bar{a}_1, \bar{a}_6)$: this is indeed a solution instance but expressed in a form even more involved than that already given in *Solution (25)*. Then, to simplify the solution search on the 'non-shear' coefficients, the possibility to formalize instances based on *Solution (s.d)* is not further explored. The same is adopted as well for solutions that would be characterized by the full set of 'shear-like' coefficients a_2, a_4, a_6, $\bar{a}_2, \bar{a}_4, \bar{a}_6$ in *Solution (s)*. Then, the sequel of the paper is devoted to explore only new solution instances based just on subset 'shear-like' *Solutions (s.a)*, *(s.b)* and *(s.c)*, the first one characterized by the feature $a_2{\neq}0$, $\bar{a}_2{\neq}0$, the second two by the common feature $a_2{=}0$, $\bar{a}_2{=}0$.

4.4 'Non-shear' coefficients

As commented above, the general solution concerning the 'non-shear' coefficients is rather involved and does not make easy neither the complete enumeration of all particular cases nor the location of single specific subsets possessing the complementary structure. However, a guided search for additional particular solutions rendered at least the set of *new sixteen complementary solutions* that is listed below. *Ten* of these solutions are non-symmetric and have been derived through the considerations reported later in Section 5. The remaining *six* solutions are symmetric and have been obtained either on the base of the new non-symmetric solutions, or as enlargements or alternatives of the additional foreseen solutions already advanced in Section 4.1.

Focusing only on solutions embedding either *Solutions (s.a)*, *(s.b)* or *(s.c)*, in the following of the paper it is always assumed that: either $a_4{=}a_6{=}0$, $\bar{a}_4{=}\bar{a}_6{=}0$, with 'orthogonal' parameters $a_2{=}1/\bar{a}_2$, $\bar{a}_2{=}1/a_2$; or $a_2{=}a_4{=}0$, $\bar{a}_2{=}\bar{a}_4{=}0$, with 'orthogonal' parameters $a_6{=}1/\bar{a}_6$, $\bar{a}_6{=}1/a_6$; or $a_2{=}0$, $\bar{a}_2{=}0$ with 'orthogonal' parameters $a_4{=}1/\bar{a}_4{=}-a_6\,{}^{\mathrm{w}}I_3/{}^{\mathrm{w}}I_2$, $\bar{a}_4{=}1/a_4{=}-\bar{a}_6\,{}^{\mathrm{w}}I_3/{}^{\mathrm{w}}I_2$.

4.4.1 Non-symmetric solutions

Solution (11). Four-coefficients sets $(a_6, a_3, a_{92}, a_{72})$ and $(\bar{a}_6, \bar{a}_3, \bar{a}_{91}, \bar{a}_{71})$ give rise to the non-symmetric complementary inverse pair:

$$
\begin{aligned}
\mathbb{A} &= a_6 \ \mathbf{w} \,\underline{\overline{\otimes}}\, \mathbf{w} + a_3 \ \mathbf{w} \otimes \mathbf{w} + a_{92} \ \mathbf{I} \otimes \mathbf{w}^2 + a_{72} \ \mathbf{I} \otimes \mathbf{w} \ ; \\
\bar{\mathbb{A}} &= \bar{a}_6 \ \bar{\mathbf{w}} \,\underline{\overline{\otimes}}\, \bar{\mathbf{w}} + \bar{a}_3 \ \bar{\mathbf{w}} \otimes \bar{\mathbf{w}} + \bar{a}_{91} \ \bar{\mathbf{w}}^2 \otimes \mathbf{I} + \bar{a}_{71} \ \bar{\mathbf{w}} \otimes \mathbf{I} \ ,
\end{aligned}
\tag{58}
$$

provided that

$$
a_6 a_{72} = a_3 a_{92} \, \mathrm{tr} \, \mathbf{w} \ ; \qquad \bar{a}_6 \bar{a}_{71} = \bar{a}_3 \bar{a}_{91} \, \mathrm{tr} \, \bar{\mathbf{w}} \ , \tag{59}
$$

with

$$
\bar{a}_6 = \frac{1}{a_6} \ , \quad \bar{a}_3 = -\frac{a_3}{a_6(3a_3 + a_6)} \ , \quad \bar{a}_{91} = -\frac{a_{92}}{a_6(3a_{92} + a_6)} \ ; \quad \bar{a}_{71} = \frac{a_3 a_{92} \ \mathrm{tr} \, \bar{\mathbf{w}}}{a_6(3a_3 + a_6)(3a_{92} + a_6)} \ , \tag{60}
$$

where $\mathrm{tr}\,\bar{\mathbf{w}} = {}^{\mathrm{w}}I_1 = {}^{\mathrm{w}}I_2 / {}^{\mathrm{w}}I_3$.

Eqns (58)-(60) originate damage-effect tensors homogeneous of degree two in $\mathbf{w}$ and $\bar{\mathbf{w}}$ based e.g. on the three free coefficients (a_6, a_3, a_{92}) and $(\bar{a}_6, \bar{a}_3, \bar{a}_{91})$:

$$
\begin{aligned}
\mathbb{A} &= a_6 \ \mathbf{w} \,\underline{\overline{\otimes}}\, \mathbf{w} + a_3 \ \mathbf{w} \otimes \mathbf{w} + a_{92} \ \mathbf{I} \otimes \mathbf{w}^2 + \frac{a_3 a_{92}}{a_6} \ \mathrm{tr} \, \mathbf{w} \ \ \mathbf{I} \otimes \mathbf{w} \ ; \\
\bar{\mathbb{A}} &= \bar{a}_6 \ \bar{\mathbf{w}} \,\underline{\overline{\otimes}}\, \bar{\mathbf{w}} + \bar{a}_3 \ \bar{\mathbf{w}} \otimes \bar{\mathbf{w}} + \bar{a}_{91} \ \bar{\mathbf{w}}^2 \otimes \mathbf{I} + \frac{\bar{a}_3 \bar{a}_{91}}{\bar{a}_6} \ \mathrm{tr} \, \bar{\mathbf{w}} \ \ \bar{\mathbf{w}} \otimes \mathbf{I} \ ,
\end{aligned}
\tag{61}
$$

with the natural constraints (8) at zero damage, that is $a_6(\mathbf{I})=1$, $[a_3+a_{92}+3a_3a_{92}](\mathbf{I})=0$ and $\bar{a}_6(\mathbf{I})=1$, $[\bar{a}_3+\bar{a}_{91}+3\bar{a}_3\bar{a}_{91}](\mathbf{I})=0$ still to be imposed on the coefficients.

A possible simple one-parameter instance of eqn (61) satisfying directly the natural constraints and based e.g. on the single coefficient $a_3 \neq 1/3$ is obtained by taking $a_6=\bar{a}_6=1$, $a_{92}=-a_3/(1+3a_3)$, $\bar{a}_{91}=-\bar{a}_3/(1+3\bar{a}_3)$ and $\bar{a}_3=-a_3/(1+3a_3)$:

$$
\begin{aligned}
\mathbb{A} &= \mathbf{w} \,\underline{\overline{\otimes}}\, \mathbf{w} + a_3 \ \mathbf{w} \otimes \mathbf{w} - \frac{a_3}{1+3a_3} \ \mathbf{I} \otimes \mathbf{w}^2 - \frac{a_3^2}{1+3a_3} \ \mathrm{tr} \, \mathbf{w} \ \ \mathbf{I} \otimes \mathbf{w} \ ; \\
\bar{\mathbb{A}} &= \bar{\mathbf{w}} \,\underline{\overline{\otimes}}\, \bar{\mathbf{w}} + \bar{a}_3 \ \bar{\mathbf{w}} \otimes \bar{\mathbf{w}} - \frac{\bar{a}_3}{1+3\bar{a}_3} \ \bar{\mathbf{w}}^2 \otimes \mathbf{I} - \frac{\bar{a}_3^2}{1+3\bar{a}_3} \ \mathrm{tr} \, \bar{\mathbf{w}} \ \ \bar{\mathbf{w}} \otimes \mathbf{I} \ .
\end{aligned}
\tag{62}
$$

For $a_3=\bar{a}_3=0$, this solution reduces right-away to Valanis-type damage, *Solution (0.2)*.

Notice that the constraints (59) on the coefficients are necessary to make up the complementary inverse pair (58) (see also explanations given later in Section 5). The only reduced particular instances of *Solution (11)* are the previous two-coefficients non-symmetric *Solution (0.3)*, 'extended' formulation (with $a_3=a_{72}=0$, $\bar{a}_3=\bar{a}_{71}=0$), and symmetric *Solution (1)*, Valanis-type damage-effect tensors (with $a_{72}=a_{92}=0$, $\bar{a}_{71}=\bar{a}_{91}=0$). So, as shown by constraints (59), the three-coefficients particular case with just $a_{72}=0$, $\bar{a}_{71}=0$ is *not* part of the solution set, *nor* are, as apparent from eqn (61), the other three-coefficients particular cases with just $a_3=0$, $\bar{a}_3=0$ and with just $a_{92}=0$, $\bar{a}_{91}=0$. Also, note that, despite the similarity to symmetric *Solution (5)*, this non-symmetric solution instance *cannot* be enlarged by including also the terms $\mathbf{I}\otimes\mathbf{I}$. Indeed, an attempt to do that by introducing $a_1\,\mathbf{I}\otimes\mathbf{I}$ in (58a), is immediately contradicted by the condition $\bar{a}_1{=}0$ required by the inspection of the general solution of the 'non-shear' coefficients discussed in Section 4.2. So, a non-symmetric counterpart of *Solution (5)*, eqn (27), that might be expected to arise as an extension of eqn (58), is actually *not* a case of the sought solution set. Same as for *Solution (1)*, compliance and stiffness obtained formally from damage-effect tensors (58) include *all but* coefficients c_2, c_4 and e_2, e_4 for the $\boldsymbol{\phi}, \bar{\boldsymbol{\phi}}$ representations, and *all but* coefficients c_1, c_2, c_9 and e_1, e_2, e_9 for the $\mathbf{w}, \bar{\mathbf{w}}$ representations.

Solution (12). A sort of 'reciprocal' of the previous solution is obtained just by replacing a_{72}, $\bar{a}_{71}$ with a_{82}, $\bar{a}_{81}$: four-coefficients sets $(a_6, a_3, a_{92}, a_{82})$ and $(\bar{a}_6, \bar{a}_3, \bar{a}_{91}, \bar{a}_{81})$ give rise to the non-symmetric complementary inverse pair:

$$\begin{aligned} \mathbb{A} &= a_6\ \mathbf{w}\,\underline{\otimes}\,\mathbf{w} + a_3\ \mathbf{w}\otimes\mathbf{w} + a_{92}\ \mathbf{I}\otimes\mathbf{w}^2 + a_{82}\ \mathbf{w}\otimes\mathbf{w}^2\ ; \\ \bar{\mathbb{A}} &= \bar{a}_6\ \bar{\mathbf{w}}\,\underline{\otimes}\,\bar{\mathbf{w}} + \bar{a}_3\ \bar{\mathbf{w}}\otimes\bar{\mathbf{w}} + \bar{a}_{91}\ \bar{\mathbf{w}}^2\otimes\mathbf{I} + \bar{a}_{81}\ \bar{\mathbf{w}}^2\otimes\bar{\mathbf{w}}\ , \end{aligned} \tag{63}$$

provided that

$$a_6 a_{82} = a_3 a_{92}\,\mathrm{tr}\,\bar{\mathbf{w}}\ ; \qquad \bar{a}_6\bar{a}_{81} = \bar{a}_3\bar{a}_{91}\,\mathrm{tr}\,\mathbf{w}\ , \tag{64}$$

where $\mathrm{tr}\ \bar{\mathbf{w}}{=}{}^{\bar{\mathbf{w}}}I_1{=}{}^{\mathbf{w}}I_2/{}^{\mathbf{w}}I_3$, $\mathrm{tr}\ \mathbf{w}{=}{}^{\mathbf{w}}I_1{=}{}^{\bar{\mathbf{w}}}I_2/{}^{\bar{\mathbf{w}}}I_3$, with

$$\bar{a}_6 = \frac{1}{a_6}\,,\quad \bar{a}_3 = -\frac{a_3}{a_6(3a_3+a_6)}\,,\quad \bar{a}_{91} = -\frac{a_{92}}{a_6(3a_{92}+a_6)}\,;\quad \bar{a}_{81} = \frac{a_3 a_{92}\ \mathrm{tr}\,\mathbf{w}}{a_6(3a_3+a_6)(3a_{92}+a_6)}\,. \tag{65}$$

Notice the interchange of roles between the first invariants ${}^{\mathbf{w}}I_1{=}\mathrm{tr}\,\mathbf{w}$ and ${}^{\bar{\mathbf{w}}}I_1{=}\mathrm{tr}\,\bar{\mathbf{w}}$ in eqns (59)-(60) and (64)-(65). Same as for *Solution (11)*, compliance and stiffness obtained formally from damage-effect tensors (63) include *all but* coefficients c_2, c_4 and e_2, e_4 for the $\boldsymbol{\phi}, \bar{\boldsymbol{\phi}}$ representations, and lack coefficients c_1, c_2 and e_1, e_2 for the $\mathbf{w}, \bar{\mathbf{w}}$ representations. However, coefficients a_{82}, $\bar{a}_{81}$ bring in the additional terms attached to coefficients c_9, e_9. This suggests that a solution with *all but* coefficients a_2, a_1 and $\bar{a}_1, \bar{a}_2$ should also be part of the solution set. Such solution instance, which further enlarges *Solutions (2)* and *(3)*, is considered later in *Solution (25)*, Section 4.4.2. The same remark outlined above for *Solution (11)* is also valid here: the terms $a_1\,\mathbf{I}\otimes\mathbf{I}$ and $\bar{a}_1\,\mathbf{I}\otimes\mathbf{I}$ *cannot* be added to the tensor terms in eqn (63) without self-contradiction.

Solution (13). Five-coefficients sets $(a_6, a_3, a_{92}, a_{72}, a_{82})$ and $(\bar{a}_6, \bar{a}_3, \bar{a}_{91}, \bar{a}_{71}, \bar{a}_{81})$ give rise to the non-symmetric complementary inverse pair:

$$\begin{aligned} \mathbb{A} &= a_6\ \mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w} + a_3\ \mathbf{w}\otimes\mathbf{w} + a_{92}\ \mathbf{I}\otimes\mathbf{w}^2 + a_{72}\ \mathbf{I}\otimes\mathbf{w} + a_{82}\ \mathbf{w}\otimes\mathbf{w}^2\ ; \\ \bar{\mathbb{A}} &= \bar{a}_6\ \bar{\mathbf{w}}\,\overline{\underline{\otimes}}\,\bar{\mathbf{w}} + \bar{a}_3\ \bar{\mathbf{w}}\otimes\bar{\mathbf{w}} + \bar{a}_{91}\ \bar{\mathbf{w}}^2\otimes\mathbf{I} + \bar{a}_{71}\ \bar{\mathbf{w}}\otimes\mathbf{I} + \bar{a}_{81}\ \bar{\mathbf{w}}^2\otimes\bar{\mathbf{w}}\ , \end{aligned} \tag{66}$$

with

$$\begin{aligned} &\bar{a}_6 = \frac{1}{a_6}, \quad \bar{a}_3 = -\frac{a_6 a_3 + 3(a_3 a_{92} - a_{72}a_{82})}{a_6\ d_2}, \quad \bar{a}_{71} = -\frac{a_6 a_{82} - \mathrm{tr}\,\bar{\mathbf{w}}(a_3 a_{92} - a_{72}a_{82})}{a_6\ d_2}, \\ &\qquad\qquad \bar{a}_{91} = -\frac{a_6 a_{92} + 3(a_3 a_{92} - a_{72}a_{82})}{a_6\ d_2}, \quad \bar{a}_{81} = -\frac{a_6 a_{72} - \mathrm{tr}\,\mathbf{w}(a_3 a_{92} - a_{72}a_{82})}{a_6\ d_2}, \end{aligned} \tag{67}$$

and

$$
\begin{aligned}
d_2 &= (a_6 + 3a_3 + a_{82}\mathrm{tr}\,\mathbf{w})(a_6 + 3a_{92} + a_{72}\mathrm{tr}\,\bar{\mathbf{w}}) - (3a_{72} + a_{92}\mathrm{tr}\,\mathbf{w})(3a_{82} + a_3\ \mathrm{tr}\,\bar{\mathbf{w}}) \\
&= (a_6 + 3a_3 + a_{72}\mathrm{tr}\,\bar{\mathbf{w}})(a_6 + 3a_{92} + a_{82}\mathrm{tr}\,\mathbf{w}) - (3a_{72} + a_3\ \mathrm{tr}\,\mathbf{w})(3a_{82} + a_{92}\mathrm{tr}\,\bar{\mathbf{w}}) \quad (68)\\
&= a_6(a_6 + 3a_3 + 3a_{92} + a_{72}\mathrm{tr}\,\bar{\mathbf{w}} + a_{82}\mathrm{tr}\,\mathbf{w}) + (a_3a_{92} - a_{72}a_{82})(9 - \mathrm{tr}\,\mathbf{w}\ \mathrm{tr}\,\bar{\mathbf{w}})\ ,
\end{aligned}
$$

where tr $\bar{\mathbf{w}}={}^{\mathrm{w}}I_1={}^{\mathrm{w}}I_2/{}^{\mathrm{w}}I_3$. There are *no constraints on the coefficients* for this solution case. *Solutions (11)* and *(12)* are contained as the only particular cases of *Solution (13)* (except for those already cited and included in *Solutions (11)* and *(12)* themselves). *Solution (11)* is recovered by setting a_{82}=0 in (66a) and imposing $\bar{a}_{81}$=0 in (67e), leading to the constraint (59a). Also, d_2, eqn (68), simplifies to $d_2=(3a_3+a_6)(3a_{92}+a_6)$, while coefficients $\bar{a}_3, \bar{a}_{91}, \bar{a}_{71}$ in (67) reduce to the expressions in (60) and satisfy constraint (59b). Similarly, *Solution (12)* is obtained by setting a_{72}=0, $\bar{a}_{71}$=0. Same as for *Solutions (11)* and *(12)*, the terms $a_1\ \mathbf{I}\otimes\mathbf{I}$ and $\bar{a}_1\ \mathbf{I}\otimes\mathbf{I}$ *cannot* be added to eqn (66) without self-contradiction. As for *Solution (12)*, compliance and stiffness obtained from damage-effect tensors (66) include *all but* coefficients c_2, c_4 and e_2, e_4 for the $\boldsymbol{\phi}, \bar{\boldsymbol{\phi}}$ representations, and *all but* coefficients c_1, c_2 and e_1, e_2 for the $\mathbf{w}, \bar{\mathbf{w}}$ representations.

Three additional non-symmetric solution cases can also be obtained as *'twins' of Solutions (11)-(13)* just by inverting the roles between coefficients with and without bars. Then, the counterparts of *Solutions (11)-(13)* can readily be listed below for the sake of completeness:

Solution (14). Four-coefficients sets $(a_6, a_3, a_{91}, a_{71})$ and $(\bar{a}_6, \bar{a}_3, \bar{a}_{92}, \bar{a}_{72})$ give rise to the non-symmetric complementary inverse pair:

$$
\begin{aligned}
\mathbb{A} &= a_6\ \mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w} + a_3\ \mathbf{w}\otimes\mathbf{w} + a_{91}\ \mathbf{w}^2\otimes\mathbf{I} + a_{71}\ \mathbf{w}\otimes\mathbf{I}\ ;\\
\bar{\mathbb{A}} &= \bar{a}_6\ \bar{\mathbf{w}}\,\overline{\underline{\otimes}}\,\bar{\mathbf{w}} + \bar{a}_3\ \bar{\mathbf{w}}\otimes\bar{\mathbf{w}} + \bar{a}_{92}\ \mathbf{I}\otimes\bar{\mathbf{w}}^2 + \bar{a}_{72}\ \mathbf{I}\otimes\bar{\mathbf{w}}\ ,
\end{aligned}
\tag{69}
$$

provided that

$$
a_6a_{71} = a_3a_{91}\ \mathrm{tr}\ \mathbf{w}\ ; \qquad \bar{a}_6\bar{a}_{72} = \bar{a}_3\bar{a}_{92}\ \mathrm{tr}\ \bar{\mathbf{w}}\ , \tag{70}
$$

with

$$
\bar{a}_6 = \frac{1}{a_6}\ ,\ \ \bar{a}_3 = -\frac{a_3}{a_6(3a_3+a_6)}\ ,\ \ \bar{a}_{92} = -\frac{a_{91}}{a_6(3a_{91}+a_6)}\ ;\ \ \bar{a}_{72} = \frac{a_3a_{91}\ \mathrm{tr}\ \bar{\mathbf{w}}}{a_6(3a_3+a_6)(3a_{91}+a_6)}\ . \tag{71}
$$

Solution (15). Four-coefficients sets $(a_6, a_3, a_{91}, a_{81})$ and $(\bar{a}_6, \bar{a}_3, \bar{a}_{92}, \bar{a}_{82})$ give rise to the non-symmetric complementary inverse pair:

$$
\begin{aligned}
\mathbb{A} &= a_6\ \mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w} + a_3\ \mathbf{w}\otimes\mathbf{w} + a_{91}\ \mathbf{w}^2\otimes\mathbf{I} + a_{81}\ \mathbf{w}^2\otimes\mathbf{w}\ ;\\
\bar{\mathbb{A}} &= \bar{a}_6\ \bar{\mathbf{w}}\,\overline{\underline{\otimes}}\,\bar{\mathbf{w}} + \bar{a}_3\ \bar{\mathbf{w}}\otimes\bar{\mathbf{w}} + \bar{a}_{92}\ \mathbf{I}\otimes\bar{\mathbf{w}}^2 + \bar{a}_{82}\ \bar{\mathbf{w}}\otimes\bar{\mathbf{w}}^2\ ,
\end{aligned}
\tag{72}
$$

provided that

$$
a_6a_{81} = a_3a_{91}\ \mathrm{tr}\ \bar{\mathbf{w}}\ ; \qquad \bar{a}_6\bar{a}_{82} = \bar{a}_3\bar{a}_{92}\ \mathrm{tr}\ \mathbf{w}\ , \tag{73}
$$

with

$$
\bar{a}_6 = \frac{1}{a_6}\ ,\ \ \bar{a}_3 = -\frac{a_3}{a_6(3a_3+a_6)}\ ,\ \ \bar{a}_{92} = -\frac{a_{91}}{a_6(3a_{91}+a_6)}\ ;\ \ \bar{a}_{82} = \frac{a_3a_{91}\ \mathrm{tr}\ \mathbf{w}}{a_6(3a_3+a_6)(3a_{91}+a_6)}\ . \tag{74}
$$

Solution (16). Five-coefficients sets $(a_6, a_3, a_{91}, a_{71}, a_{81})$ and $(\bar{a}_6, \bar{a}_3, \bar{a}_{92}, \bar{a}_{72}, \bar{a}_{82})$ give rise to the non-symmetric complementary inverse pair:

$$\begin{aligned} \mathbb{A} &= a_6\ \mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w} + a_3\ \mathbf{w}\otimes\mathbf{w} + a_{91}\ \mathbf{w}^2\otimes\mathbf{I} + a_{71}\ \mathbf{w}\otimes\mathbf{I} + a_{81}\ \mathbf{w}^2\otimes\mathbf{w}\ ; \\ \bar{\mathbb{A}} &= \bar{a}_6\ \bar{\mathbf{w}}\,\underline{\overline{\otimes}}\,\bar{\mathbf{w}} + \bar{a}_3\ \bar{\mathbf{w}}\otimes\bar{\mathbf{w}} + \bar{a}_{92}\ \mathbf{I}\otimes\bar{\mathbf{w}}^2 + \bar{a}_{72}\ \mathbf{I}\otimes\bar{\mathbf{w}} + \bar{a}_{82}\ \bar{\mathbf{w}}\otimes\bar{\mathbf{w}}^2\ , \end{aligned} \tag{75}$$

with

$$\begin{aligned} \bar{a}_6 = \frac{1}{a_6},\quad \bar{a}_3 &= -\frac{a_6a_3 + 3(a_3a_{91} - a_{71}a_{81})}{a_6\ d_1},\quad \bar{a}_{72} = -\frac{a_6a_{81} - \operatorname{tr}\bar{\mathbf{w}}(a_3a_{91} - a_{71}a_{81})}{a_6\ d_1}, \\ \bar{a}_{92} &= -\frac{a_6a_{91} + 3(a_3a_{91} - a_{71}a_{81})}{a_6\ d_1},\quad \bar{a}_{82} = -\frac{a_6a_{71} - \operatorname{tr}\mathbf{w}(a_3a_{91} - a_{71}a_{81})}{a_6\ d_1}, \end{aligned} \tag{76}$$

and

$$\begin{aligned} d_1 &= (a_6 + 3a_3 + a_{81}\operatorname{tr}\mathbf{w})(a_6 + 3a_{91} + a_{71}\operatorname{tr}\bar{\mathbf{w}}) - (3a_{71} + a_{91}\operatorname{tr}\mathbf{w})(3a_{81} + a_3\ \operatorname{tr}\bar{\mathbf{w}}) \\ &= (a_6 + 3a_3 + a_{71}\operatorname{tr}\bar{\mathbf{w}})(a_6 + 3a_{91} + a_{81}\operatorname{tr}\mathbf{w}) - (3a_{71} + a_3\ \operatorname{tr}\mathbf{w})(3a_{81} + a_{91}\operatorname{tr}\bar{\mathbf{w}}) \\ &= a_6(a_6 + 3a_3 + 3a_{91} + a_{71}\operatorname{tr}\bar{\mathbf{w}} + a_{81}\operatorname{tr}\mathbf{w}) + (a_3a_{91} - a_{71}a_{81})(9 - \operatorname{tr}\mathbf{w}\ \operatorname{tr}\bar{\mathbf{w}})\ . \end{aligned} \tag{77}$$

There are *no constraints on the coefficients* for this solution case. *Solutions (14)-(16)* formally lead to compliance and stiffness embedding *all but* coefficients c_2, c_4 and e_2, e_4 in the $\boldsymbol{\phi}, \bar{\boldsymbol{\phi}}$ representation, and *all but* coefficient c_2 and e_2 in the $\mathbf{w}, \bar{\mathbf{w}}$ expansion.

A new family of *four* non-symmetric solutions is listed below, which is based on damage-effect tensors obtained as single rank-one updates of the symmetric identity terms $a_2\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}$, $\bar{a}_2\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}$, *Solution (s.a)* (see Section 5.2).

Solution (17). Four-coefficients sets $(a_2, a_3, a_{71}, a_{82})$ and $(\bar{a}_2, \bar{a}_3, \bar{a}_{72}, \bar{a}_{81})$ give rise to the non-symmetric complementary inverse pair:

$$\begin{aligned} \mathbb{A} &= a_2\ \mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I} + a_3\ \mathbf{w}\otimes\mathbf{w} + a_{71}\ \mathbf{w}\otimes\mathbf{I} + a_{82}\ \mathbf{w}\otimes\mathbf{w}^2\ ; \\ \bar{\mathbb{A}} &= \bar{a}_2\ \mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I} + \bar{a}_3\ \bar{\mathbf{w}}\otimes\bar{\mathbf{w}} + \bar{a}_{72}\ \mathbf{I}\otimes\bar{\mathbf{w}} + \bar{a}_{81}\ \bar{\mathbf{w}}^2\otimes\bar{\mathbf{w}}\ , \end{aligned} \tag{78}$$

provided that

$$a_{71} = -\frac{a_3}{{}^{\mathrm{w}}I_1}\,{}^{\mathrm{w}}I_2\ ,\quad a_{82} = \frac{a_{71}}{{}^{\mathrm{w}}I_2} = -\frac{a_3}{{}^{\mathrm{w}}I_1}\ ;\qquad \bar{a}_{72} = -\frac{\bar{a}_3}{{}^{\mathrm{w}}I_1}\,{}^{\mathrm{w}}I_2\ ,\quad \bar{a}_{81} = \frac{\bar{a}_{72}}{{}^{\mathrm{w}}I_2} = -\frac{\bar{a}_3}{{}^{\mathrm{w}}I_1}\ , \tag{79}$$

with

$$\begin{aligned} \bar{a}_2 = \frac{1}{a_2}\ ,\quad \bar{a}_3 = -\frac{a_3\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3}{a_2(a_2\,{}^{\mathrm{w}}I_1 - 3a_3\,{}^{\mathrm{w}}I_3)}\ ; \\ \bar{a}_{72} = \frac{a_3\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3}{a_2(a_2\,{}^{\mathrm{w}}I_1 - 3a_3\,{}^{\mathrm{w}}I_3)}\ ,\quad \bar{a}_{81} = \frac{a_3\,{}^{\mathrm{w}}I_3{}^2}{a_2(a_2\,{}^{\mathrm{w}}I_1 - 3a_3\,{}^{\mathrm{w}}I_3)}\ . \end{aligned} \tag{80}$$

Notice that this solution displays 'mixed' coefficients a_{71}, a_{82} (i.e. with different second index) and dual ones $\bar{a}_{72}, \bar{a}_{81}$. *Solution (17)* corresponds to take the dual damage-effect tensors $\mathbb{A}{=}a_2\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}{+}\alpha\,\mathbf{w}{\otimes}\bar{\mathbf{w}}$, $\bar{\mathbb{A}}{=}\bar{a}_2\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}{+}\bar{\alpha}\,\mathbf{w}{\otimes}\bar{\mathbf{w}}$, where $\bar{a}_2{=}1/a_2$ and $\bar{\alpha}{=}{-}\alpha/(a_2(3\alpha{+}a_2))$, and conversely $a_2{=}1/\bar{a}_2$ and $\alpha{=}{-}\bar{\alpha}/(\bar{a}_2(3\bar{\alpha}{+}\bar{a}_2))$. The link with expressions (78) goes with relations $\alpha{=}-a_3{}^{\mathbf{w}}I_3/{}^{\mathbf{w}}I_1$ and $\bar{\alpha}{=}-\bar{a}_3{}^{\bar{\mathbf{w}}}I_3/{}^{\bar{\mathbf{w}}}I_1$. Compliance and stiffness obtained formally from damage-effect tensors (78) include *all but* coefficients c_4, c_6 and e_4, e_6 in the $\mathbf{w}, \bar{\mathbf{w}}$ representations. This would lead to a more general solution instance of dual damage-effect tensors embedding *all but* coefficients a_4, a_6 and $\bar{a}_4, \bar{a}_6$, that is the counterpart of *Solution (9)* but based on a_2, $\bar{a}_2$ instead of a_6, $\bar{a}_6$. This solution would represent the more general form of solutions based on *Solution (s.a)* and would comprise as particular cases *Solution (17)* and *Solutions (18)-(24)* presented next. However, this solution instance results more involved than *Solution (9)* and even of *Solution (10)*, thus it is not presented here.

Solution (18). A *'twin'* of previous *Solution (17)* is obtained by taking the four-coefficients sets $(a_2, a_3, a_{72}, a_{81})$ and $(\bar{a}_2, \bar{a}_3, \bar{a}_{71}, \bar{a}_{82})$, which give rise to the non-symmetric complementary inverse pair:

$$\begin{aligned} \mathbb{A} &= a_2\, \mathbf{I} \underline{\overline{\otimes}} \mathbf{I} + a_3\, \mathbf{w} \otimes \mathbf{w} + a_{72}\, \mathbf{I} \otimes \mathbf{w} + a_{81}\, \mathbf{w}^2 \otimes \mathbf{w}\ ; \\ \bar{\mathbb{A}} &= \bar{a}_2\, \mathbf{I} \underline{\overline{\otimes}} \mathbf{I} + \bar{a}_3\, \bar{\mathbf{w}} \otimes \bar{\mathbf{w}} + \bar{a}_{71}\, \bar{\mathbf{w}} \otimes \mathbf{I} + \bar{a}_{82}\, \bar{\mathbf{w}} \otimes \bar{\mathbf{w}}^2\ , \end{aligned} \tag{81}$$

provided that

$$a_{72} = -\frac{a_3}{{}^{\mathbf{w}}I_1}\, {}^{\mathbf{w}}I_2\ , \quad a_{81} = \frac{a_{72}}{{}^{\mathbf{w}}I_2} = -\frac{a_3}{{}^{\mathbf{w}}I_1}\ ; \quad \bar{a}_{71} = -\frac{\bar{a}_3}{{}^{\bar{\mathbf{w}}}I_1}\, {}^{\bar{\mathbf{w}}}I_2\ , \quad \bar{a}_{82} = \frac{\bar{a}_{71}}{{}^{\bar{\mathbf{w}}}I_2} = -\frac{\bar{a}_3}{{}^{\bar{\mathbf{w}}}I_1}\ , \tag{82}$$

with

$$\begin{aligned} \bar{a}_2 &= \frac{1}{a_2}\ , \quad \bar{a}_3 = -\frac{a_3\, {}^{\mathbf{w}}I_2\, {}^{\mathbf{w}}I_3}{a_2(a_2\, {}^{\mathbf{w}}I_1 - 3a_3\, {}^{\mathbf{w}}I_3)}\ ; \\ \bar{a}_{71} &= \frac{a_3\, {}^{\mathbf{w}}I_1\, {}^{\mathbf{w}}I_3}{a_2(a_2\, {}^{\mathbf{w}}I_1 - 3a_3\, {}^{\mathbf{w}}I_3)}\ , \quad \bar{a}_{82} = \frac{a_3\, {}^{\mathbf{w}}I_3{}^2}{a_2(a_2\, {}^{\mathbf{w}}I_1 - 3a_3\, {}^{\mathbf{w}}I_3)}\ . \end{aligned} \tag{83}$$

Again, this solution instance displays 'mixed' coefficients a_{72}, a_{81}, and dual ones $\bar{a}_{71}, \bar{a}_{82}$. It corresponds to assume the 'twin' dual damage-effect tensors $\mathbb{A}{=}a_2\, \mathbf{I} \underline{\overline{\otimes}} \mathbf{I}{+}\alpha'\, \bar{\mathbf{w}}{\otimes}\mathbf{w}$, $\bar{\mathbb{A}}{=}\bar{a}_2\, \mathbf{I} \underline{\overline{\otimes}} \mathbf{I}{+}\bar{\alpha}'\, \bar{\mathbf{w}}{\otimes}\mathbf{w}$, where, similarly as above, $\bar{a}_2{=}1/a_2$, $\bar{\alpha}'{=}-\alpha'/(a_2(3\alpha'{+}a_2))$; $a_2{=}1/\bar{a}_2$, $\alpha'{=}-\bar{\alpha}'/(\bar{a}_2(3\bar{\alpha}'{+}\bar{a}_2))$. The link with expressions (81) goes with relations $\alpha'{=}-a_3{}^{\mathbf{w}}I_3/{}^{\mathbf{w}}I_1$ and $\bar{\alpha}'{=}-\bar{a}_3{}^{\bar{\mathbf{w}}}I_3/{}^{\bar{\mathbf{w}}}I_1$. As a new important feature with respect to 'twin' *Solution (17)*, compliance and stiffness obtained formally from damage-effect tensors (81) include only coefficients c_1, c_2, c_3, c_7, c_8 and e_1, e_2, e_3, e_7, e_8 in the $\mathbf{w}, \bar{\mathbf{w}}$ representations. This leads to the new instance of dual symmetric damage-effect tensors based on five-coefficients sets $(a_1, a_2, a_3, a_7, a_8)$ and $(\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_7, \bar{a}_8)$ that is given later in Section 4.4.2, *Solution (21)*.

Solution (19). Four-coefficients sets $(a_2, a_5, a_{81}, a_{91})$ and $(\bar{a}_2, \bar{a}_5, \bar{a}_{82}, \bar{a}_{92})$ give rise to the non-symmetric complementary inverse pair:

$$\begin{aligned} \mathbb{A} &= a_2\, \mathbf{I} \underline{\overline{\otimes}} \mathbf{I} + a_5\, \mathbf{w}^2 \otimes \mathbf{w}^2 + a_{81}\, \mathbf{w}^2 \otimes \mathbf{w} + a_{91}\, \mathbf{w}^2 \otimes \mathbf{I}\ ; \\ \bar{\mathbb{A}} &= \bar{a}_2\, \mathbf{I} \underline{\overline{\otimes}} \mathbf{I} + \bar{a}_5\, \bar{\mathbf{w}}^2 \otimes \bar{\mathbf{w}}^2 + \bar{a}_{82}\, \bar{\mathbf{w}} \otimes \bar{\mathbf{w}}^2 + \bar{a}_{92}\, \mathbf{I} \otimes \bar{\mathbf{w}}^2\ , \end{aligned} \tag{84}$$

provided that

$$\begin{aligned} a_{81} &= a_5\, \frac{{}^{\mathbf{w}}I_3 - {}^{\mathbf{w}}I_1\, {}^{\mathbf{w}}I_2}{{}^{\mathbf{w}}I_2}\ , \qquad a_{91} = a_5\, \frac{{}^{\mathbf{w}}I_2{}^2 - {}^{\mathbf{w}}I_1\, {}^{\mathbf{w}}I_3}{{}^{\mathbf{w}}I_2}\ ; \\ \bar{a}_{82} &= \bar{a}_5\, \frac{{}^{\bar{\mathbf{w}}}I_3 - {}^{\bar{\mathbf{w}}}I_1\, {}^{\bar{\mathbf{w}}}I_2}{{}^{\bar{\mathbf{w}}}I_2}\ , \qquad \bar{a}_{92} = \bar{a}_5\, \frac{{}^{\bar{\mathbf{w}}}I_2{}^2 - {}^{\bar{\mathbf{w}}}I_1\, {}^{\bar{\mathbf{w}}}I_3}{{}^{\bar{\mathbf{w}}}I_2}\ , \end{aligned} \tag{85}$$

with

$$
\begin{aligned}
\bar{a}_2 = \frac{1}{a_2}\ , \quad \bar{a}_5 = -\frac{a_5\, {}^{\mathrm{w}}\!I_1\, {}^{\mathrm{w}}\!I_3{}^3}{a_2(a_2\, {}^{\mathrm{w}}\!I_2 + 3a_5\, {}^{\mathrm{w}}\!I_3{}^2)}\ ; \\
\bar{a}_{82} = -\frac{a_5\, {}^{\mathrm{w}}\!I_3{}^2\, ({}^{\mathrm{w}}\!I_3 - {}^{\mathrm{w}}\!I_1\, {}^{\mathrm{w}}\!I_2)}{a_2(a_2\, {}^{\mathrm{w}}\!I_2 + 3a_5\, {}^{\mathrm{w}}\!I_3{}^2)}\ , \quad \bar{a}_{92} = -\frac{a_5\, {}^{\mathrm{w}}\!I_3{}^2\, ({}^{\mathrm{w}}\!I_1{}^2 - {}^{\mathrm{w}}\!I_2)}{a_2(a_2\, {}^{\mathrm{w}}\!I_2 + 3a_5\, {}^{\mathrm{w}}\!I_3{}^2)}\ .
\end{aligned}
\tag{86}
$$

This solution corresponds to take the dual damage-effect tensors $\mathbb{A}{=}a_2\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}{+}\beta\,\mathbf{w}^2{\otimes}\bar{\mathbf{w}}^2$, $\bar{\mathbb{A}}{=}\bar{a}_2\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}{+}\bar{\beta}\,\mathbf{w}^2{\otimes}\bar{\mathbf{w}}^2$, where $\bar{a}_2{=}1/a_2$ and $\bar{\beta}{=}{-}\beta/(a_2(3\beta{+}a_2))$, and conversely $a_2{=}1/\bar{a}_2$ and $\beta{=}{-}\bar{\beta}/(\bar{a}_2(3\bar{\beta}{+}\bar{a}_2))$. The link with expressions (84) goes with relations $\beta{=}a_5{}^{\mathrm{w}}\!I_3{}^2/{}^{\mathrm{w}}\!I_2$ and $\bar{\beta}{=}\bar{a}_5{}^{\bar{\mathrm{w}}}\!I_3{}^2/{}^{\bar{\mathrm{w}}}\!I_2$. Compliance and stiffness obtained formally from damage-effect tensors (84) include *all but* coefficients c_4, c_6 and e_4, e_6 in the $\mathbf{w}, \bar{\mathbf{w}}$ representations.

Solution (20). A *'twin'* of previous *Solution (19)* is obtained by taking the four-coefficients sets $(a_2, a_5, a_{82}, a_{92})$ and $(\bar{a}_2, \bar{a}_5, \bar{a}_{81}, \bar{a}_{91})$, which give rise to the non-symmetric complementary inverse pair:

$$
\begin{aligned}
\mathbb{A} &= a_2\ \mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I} + a_5\ \mathbf{w}^2 \otimes \mathbf{w}^2 + a_{82}\ \mathbf{w} \otimes \mathbf{w}^2 + a_{92}\ \mathbf{I} \otimes \mathbf{w}^2\ ; \\
\bar{\mathbb{A}} &= \bar{a}_2\ \mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I} + \bar{a}_5\ \bar{\mathbf{w}}^2 \otimes \bar{\mathbf{w}}^2 + \bar{a}_{81}\ \bar{\mathbf{w}}^2 \otimes \bar{\mathbf{w}} + \bar{a}_{91}\ \bar{\mathbf{w}}^2 \otimes \mathbf{I}\ ,
\end{aligned}
\tag{87}
$$

provided that

$$
\begin{aligned}
a_{82} = a_5\, \frac{{}^{\mathrm{w}}\!I_3 - {}^{\mathrm{w}}\!I_1\, {}^{\mathrm{w}}\!I_2}{{}^{\mathrm{w}}\!I_2}\ , \qquad a_{92} = a_5\, \frac{{}^{\mathrm{w}}\!I_2{}^2 - {}^{\mathrm{w}}\!I_1\, {}^{\mathrm{w}}\!I_3}{{}^{\mathrm{w}}\!I_2}\ ; \\
\bar{a}_{81} = \bar{a}_5\, \frac{{}^{\bar{\mathrm{w}}}\!I_3 - {}^{\bar{\mathrm{w}}}\!I_1\, {}^{\bar{\mathrm{w}}}\!I_2}{{}^{\bar{\mathrm{w}}}\!I_2}\ , \qquad \bar{a}_{91} = \bar{a}_5\, \frac{{}^{\bar{\mathrm{w}}}\!I_2{}^2 - {}^{\bar{\mathrm{w}}}\!I_1\, {}^{\bar{\mathrm{w}}}\!I_3}{{}^{\bar{\mathrm{w}}}\!I_2}\ ,
\end{aligned}
\tag{88}
$$

with

$$
\begin{aligned}
\bar{a}_2 = \frac{1}{a_2}\ , \quad \bar{a}_5 = -\frac{a_5\, {}^{\mathrm{w}}\!I_1\, {}^{\mathrm{w}}\!I_3{}^3}{a_2(a_2\, {}^{\mathrm{w}}\!I_2 + 3a_5\, {}^{\mathrm{w}}\!I_3{}^2)}\ ; \\
\bar{a}_{81} = -\frac{a_5\, {}^{\mathrm{w}}\!I_3{}^2\, ({}^{\mathrm{w}}\!I_3 - {}^{\mathrm{w}}\!I_1\, {}^{\mathrm{w}}\!I_2)}{a_2(a_2\, {}^{\mathrm{w}}\!I_2 + 3a_5\, {}^{\mathrm{w}}\!I_3{}^2)}\ , \quad \bar{a}_{91} = -\frac{a_5\, {}^{\mathrm{w}}\!I_3{}^2\, ({}^{\mathrm{w}}\!I_1{}^2 - {}^{\mathrm{w}}\!I_2)}{a_2(a_2\, {}^{\mathrm{w}}\!I_2 + 3a_5\, {}^{\mathrm{w}}\!I_3{}^2)}\ .
\end{aligned}
\tag{89}
$$

This solution corresponds to take 'twin' dual damage-effect tensors $\mathbb{A}{=}a_2\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}{+}\beta'\,\bar{\mathbf{w}}^2{\otimes}\mathbf{w}^2$, $\bar{\mathbb{A}}{=}\bar{a}_2\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}{+}\bar{\beta}'\,\bar{\mathbf{w}}^2{\otimes}\mathbf{w}^2$, where once again $\bar{a}_2{=}1/a_2$, $\bar{\beta}'{=}{-}\beta'/(a_2(3\beta'{+}a_2))$ and $a_2{=}1/\bar{a}_2$, $\beta'{=}\bar{\beta}'/(\bar{a}_2(3\bar{\beta}'{+}\bar{a}_2))$. The link with expressions (87) goes with relations $\beta'{=}a_5{}^{\mathrm{w}}\!I_3{}^2/{}^{\mathrm{w}}\!I_2$ and $\bar{\beta}'{=}\bar{a}_5{}^{\bar{\mathrm{w}}}\!I_3{}^2/{}^{\bar{\mathrm{w}}}\!I_2$. As a new important feature with respect to 'twin' *Solution (19)*, compliance and stiffness obtained formally from damage-effect tensors (87) include only coefficients c_1, c_2, c_5, c_8, c_9 and e_1, e_2, e_5, e_8, e_9 in the $\mathbf{w}, \bar{\mathbf{w}}$ representations. This leads to the new instance of dual symmetric damage-effect tensors based on five-coefficients sets $(a_1, a_2, a_5, a_8, a_9)$ and $(\bar{a}_1, \bar{a}_2, \bar{a}_5, \bar{a}_8, \bar{a}_9)$ that is given in the next section, *Solution (22)*.

4.4.2 Symmetric solutions

Four new symmetric solutions belonging to the family generated by *Solution (s.a)* are listed below.

Solution (21). As signaled by the outcomes of previous non-symmetric *Solution (18)*, the five-coefficients sets $(a_1, a_2, a_3, a_7, a_8)$ and $(\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_7, \bar{a}_8)$ give rise to the symmetric complementary inverse pair (lacking coefficients a_4, a_5, a_6, a_9 and $\bar{a}_4, \bar{a}_5, \bar{a}_6, \bar{a}_9$):

$$
\begin{aligned}
\mathbb{A} &= a_2\, \mathbf{I} \overline{\underline{\otimes}} \mathbf{I} + a_1\, \mathbf{I} \otimes \mathbf{I} + a_3\, \mathbf{w} \otimes \mathbf{w} + a_7\, (\mathbf{w} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{w}) + a_8\, (\mathbf{w}^2 \otimes \mathbf{w} + \mathbf{w} \otimes \mathbf{w}^2)\,; \\
\bar{\mathbb{A}} &= \bar{a}_2\, \mathbf{I} \overline{\underline{\otimes}} \mathbf{I} + \bar{a}_1\, \mathbf{I} \otimes \mathbf{I} + \bar{a}_3\, \bar{\mathbf{w}} \otimes \bar{\mathbf{w}} + \bar{a}_7\, (\bar{\mathbf{w}} \otimes \mathbf{I} + \mathbf{I} \otimes \bar{\mathbf{w}}) + \bar{a}_8\, (\bar{\mathbf{w}}^2 \otimes \bar{\mathbf{w}} + \bar{\mathbf{w}} \otimes \bar{\mathbf{w}}^2)\,,
\end{aligned}
\tag{90}
$$

provided that

$$
\begin{aligned}
a_1 &= \frac{a_2\, (a_7 - a_8\, {}^{\mathrm{w}}I_2)}{a_8\, {}^{\mathrm{w}}I_2}\,, \qquad a_3 = -a_8\, \frac{2\, {}^{\mathrm{w}}I_1\, (a_2 + a_8\, {}^{\mathrm{w}}I_3) - a_7\, {}^{\mathrm{w}}I_2}{a_2}\,; \\
\bar{a}_1 &= \frac{\bar{a}_2\, (\bar{a}_7 - \bar{a}_8\, {}^{\bar{\mathrm{w}}}I_2)}{\bar{a}_8\, {}^{\bar{\mathrm{w}}}I_2}\,, \qquad \bar{a}_3 = -\bar{a}_8\, \frac{2\, {}^{\bar{\mathrm{w}}}I_1\, (\bar{a}_2 + \bar{a}_8\, {}^{\bar{\mathrm{w}}}I_3) - \bar{a}_7\, {}^{\bar{\mathrm{w}}}I_2}{\bar{a}_2}\,,
\end{aligned}
\tag{91}
$$

with

$$
\begin{aligned}
\bar{a}_2 = \frac{1}{a_2}\,,\quad \bar{a}_7 &= -\frac{a_8\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3\, (2\, a_7 - a_8\, {}^{\mathrm{w}}I_2)}{a_2\, (a_2 + 3\, a_8\, {}^{\mathrm{w}}I_3)\, (3\, a_7 - 2 a_8\, {}^{\mathrm{w}}I_2)}\,,\quad \bar{a}_8 = -\frac{a_8\, {}^{\mathrm{w}}I_3^{\,2}}{a_2\, (a_2 + 3\, a_8\, {}^{\mathrm{w}}I_3)}\,; \\
\bar{a}_1 &= -\frac{a_7 - a_8\, {}^{\mathrm{w}}I_2}{a_2\, (3\, a_7 - 2\, a_8\, {}^{\mathrm{w}}I_2)}\,, \\
\bar{a}_3 = a_8\, {}^{\mathrm{w}}I_3\, & \frac{2\, {}^{\mathrm{w}}I_2\, (a_2 + 2\, a_8\, {}^{\mathrm{w}}I_3)\, (3\, a_7 - 2\, a_8\, {}^{\mathrm{w}}I_2) + a_8\, {}^{\mathrm{w}}I_1^{\,2}\, {}^{\mathrm{w}}I_3\, (2\, a_7 - a_8\, {}^{\mathrm{w}}I_2)}{a_2\, (a_2 + 3\, a_8\, {}^{\mathrm{w}}I_3)^2\, (3\, a_7 - 2 a_8\, {}^{\mathrm{w}}I_2)}\,.
\end{aligned}
\tag{92}
$$

Damage-effect tensors (90) formally lead to compliance and stiffness embedding *all but* coefficients c_4, c_6 and e_4, e_6 in the $\mathbf{w}, \bar{\mathbf{w}}$ representations.

Solution (22). As signaled by the outcomes of previous non-symmetric *Solution (20)*, the five-coefficients sets $(a_1, a_2, a_5, a_8, a_9)$ and $(\bar{a}_1, \bar{a}_2, \bar{a}_5, \bar{a}_8, \bar{a}_9)$ give rise to the symmetric complementary inverse pair (lacking coefficients a_3, a_4, a_6, a_7 and $\bar{a}_3, \bar{a}_4, \bar{a}_6, \bar{a}_7$):

$$
\begin{aligned}
\mathbb{A} &= a_2\, \mathbf{I} \overline{\underline{\otimes}} \mathbf{I} + a_1\, \mathbf{I} \otimes \mathbf{I} + a_5\, \mathbf{w}^2 \otimes \mathbf{w}^2 + a_8\, (\mathbf{w}^2 \otimes \mathbf{w} + \mathbf{w} \otimes \mathbf{w}^2) + a_9\, (\mathbf{w}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{w}^2)\,; \\
\bar{\mathbb{A}} &= \bar{a}_2\, \mathbf{I} \overline{\underline{\otimes}} \mathbf{I} + \bar{a}_1\, \mathbf{I} \otimes \mathbf{I} + \bar{a}_3\, \bar{\mathbf{w}}^2 \otimes \bar{\mathbf{w}}^2 + \bar{a}_8\, (\bar{\mathbf{w}}^2 \otimes \bar{\mathbf{w}} + \bar{\mathbf{w}} \otimes \bar{\mathbf{w}}^2) + \bar{a}_9\, (\bar{\mathbf{w}}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \bar{\mathbf{w}}^2)\,,
\end{aligned}
\tag{93}
$$

provided that

$$
\begin{aligned}
a_5 &= -a_8\, \frac{n}{a_2^{\,2}\, ({}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2 - {}^{\mathrm{w}}I_3)^2}\,, \quad a_9 = -a_8\, \frac{a_1\, ({}^{\mathrm{w}}I_2^{\,2} - 2\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3) + a_2\, ({}^{\mathrm{w}}I_2^{\,2} - {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3)}{a_2\, ({}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2 - {}^{\mathrm{w}}I_3)}\,; \\
\bar{a}_5 &= -\bar{a}_8\, \frac{\bar{n}}{\bar{a}_2^{\,2}\, ({}^{\bar{\mathrm{w}}}I_1\, {}^{\bar{\mathrm{w}}}I_2 - {}^{\bar{\mathrm{w}}}I_3)^2}\,, \quad \bar{a}_9 = -\bar{a}_8\, \frac{\bar{a}_1\, ({}^{\bar{\mathrm{w}}}I_2^{\,2} - 2\, {}^{\bar{\mathrm{w}}}I_1\, {}^{\bar{\mathrm{w}}}I_3) + \bar{a}_2\, ({}^{\bar{\mathrm{w}}}I_2^{\,2} - {}^{\bar{\mathrm{w}}}I_1\, {}^{\bar{\mathrm{w}}}I_3)}{\bar{a}_2\, ({}^{\bar{\mathrm{w}}}I_1\, {}^{\bar{\mathrm{w}}}I_2 - {}^{\bar{\mathrm{w}}}I_3)}\,,
\end{aligned}
\tag{94}
$$

where

$$
\begin{aligned}
n &= 2\, a_2^{\,2}\, {}^{\mathrm{w}}I_2\, ({}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2 - {}^{\mathrm{w}}I_3) - a_1\, a_8\, ({}^{\mathrm{w}}I_2^{\,2} - 2\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3)^2 \\
&\quad - a_2\, a_8\, \left(2\, {}^{\mathrm{w}}I_3^{\,2}\, ({}^{\mathrm{w}}I_1^{\,2} + 2\, {}^{\mathrm{w}}I_2) + {}^{\mathrm{w}}I_2^{\,2}\, ({}^{\mathrm{w}}I_2^{\,2} - 4\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3)\right)\,; \\
\bar{n} &= 2\, \bar{a}_2^{\,2}\, {}^{\bar{\mathrm{w}}}I_2\, ({}^{\bar{\mathrm{w}}}I_1\, {}^{\bar{\mathrm{w}}}I_2 - {}^{\bar{\mathrm{w}}}I_3) - \bar{a}_1\, \bar{a}_8\, ({}^{\bar{\mathrm{w}}}I_2^{\,2} - 2\, {}^{\bar{\mathrm{w}}}I_1\, {}^{\bar{\mathrm{w}}}I_3)^2 \\
&\quad - \bar{a}_2\, \bar{a}_8\, \left(2\, {}^{\bar{\mathrm{w}}}I_3^{\,2}\, ({}^{\bar{\mathrm{w}}}I_1^{\,2} + 2\, {}^{\bar{\mathrm{w}}}I_2) + {}^{\bar{\mathrm{w}}}I_2^{\,2}\, ({}^{\bar{\mathrm{w}}}I_2^{\,2} - 4\, {}^{\bar{\mathrm{w}}}I_1\, {}^{\bar{\mathrm{w}}}I_3)\right)\,,
\end{aligned}
\tag{95}
$$

with

$$\bar a_2 = \frac{1}{a_2}\ ,\quad \bar a_1 = -\frac{a_1}{a_2\,(3a_1+a_2)}\ ,\quad \bar a_8 = -\frac{a_8\,{}^{\mathrm w}\!I_3{}^2\ ({}^{\mathrm w}\!I_1\,{}^{\mathrm w}\!I_2-{}^{\mathrm w}\!I_3)}{a_2\ \Big(a_2\ ({}^{\mathrm w}\!I_1\,{}^{\mathrm w}\!I_2-{}^{\mathrm w}\!I_3)-3\,a_8\,{}^{\mathrm w}\!I_3{}^2\Big)}\ ;$$

$$\bar a_5 = \frac{a_8\,{}^{\mathrm w}\!I_3{}^3\,\bar n'}{a_2\,(3\,a_1+a_2)\Big(a_2\,({}^{\mathrm w}\!I_1\,{}^{\mathrm w}\!I_2-{}^{\mathrm w}\!I_3)-3\,a_8\,{}^{\mathrm w}\!I_3{}^2\Big)^2}\ , \tag{96}$$

$$\bar a_9 = \frac{a_8\,{}^{\mathrm w}\!I_3{}^2\ \Big(a_1\,(2\,{}^{\mathrm w}\!I_1{}^2-{}^{\mathrm w}\!I_2)+a_2\,({}^{\mathrm w}\!I_1{}^2-{}^{\mathrm w}\!I_2)\Big)}{a_2\,(3\,a_1+a_2)\ \Big(a_2\ ({}^{\mathrm w}\!I_1\,{}^{\mathrm w}\!I_2-{}^{\mathrm w}\!I_3)-3\,a_8\,{}^{\mathrm w}\!I_3{}^2\Big)}\ ,$$

where

$$\begin{aligned}\bar n' = {} & (3\,a_1+a_2)\Big[\,2\,a_2\,{}^{\mathrm w}\!I_1\,({}^{\mathrm w}\!I_1\,{}^{\mathrm w}\!I_2-{}^{\mathrm w}\!I_3)+a_8\,{}^{\mathrm w}\!I_3\,\Big({}^{\mathrm w}\!I_1{}^2\,({}^{\mathrm w}\!I_1{}^2-4\,{}^{\mathrm w}\!I_2)+2\,({}^{\mathrm w}\!I_2{}^2-{}^{\mathrm w}\!I_1\,{}^{\mathrm w}\!I_3)\Big)\Big]\\ & -\,a_1\,a_8\,({}^{\mathrm w}\!I_1{}^2-2\,{}^{\mathrm w}\!I_2)^2\,{}^{\mathrm w}\!I_3\ .\end{aligned} \tag{97}$$

Damage-effect tensors (93) formally lead to compliance and stiffness embedding *all but* coefficients c_4, c_6 and e_4, e_6 in the $\mathbf{w}, \bar{\mathbf{w}}$ representations.

Solution (23). From relation (96b) it is apparent that a new solution instance can be obtained consistently just as a particular case of *Solution (22)* for a_1=0, $\bar a_1$=0. Indeed, the four-coefficients sets (a_2, a_5, a_8, a_9) and $(\bar a_2, \bar a_5, \bar a_8, \bar a_9)$ give rise to the symmetric complementary inverse pair:

$$\begin{aligned}\mathbb{A} &= a_2\,\mathbf{I}\,\underline{\otimes}\,\mathbf{I}+a_5\,\mathbf{w}^2\otimes\mathbf{w}^2+a_8\,(\mathbf{w}^2\otimes\mathbf{w}+\mathbf{w}\otimes\mathbf{w}^2)+a_9\,(\mathbf{w}^2\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w}^2)\,;\\ \bar{\mathbb{A}} &= \bar a_2\,\mathbf{I}\,\underline{\otimes}\,\mathbf{I}+\bar a_5\,\bar{\mathbf{w}}^2\otimes\bar{\mathbf{w}}^2+\bar a_8\,(\bar{\mathbf{w}}^2\otimes\bar{\mathbf{w}}+\bar{\mathbf{w}}\otimes\bar{\mathbf{w}}^2)+\bar a_9\,(\bar{\mathbf{w}}^2\otimes\mathbf{I}+\mathbf{I}\otimes\bar{\mathbf{w}}^2)\,,\end{aligned} \tag{98}$$

provided that

$$\begin{aligned} a_5 &= -a_8\ \frac{n_1}{a_2\,({}^{\mathrm w}\!I_1\,{}^{\mathrm w}\!I_2-{}^{\mathrm w}\!I_3)^2}\ ,\quad a_9 = -a_8\ \frac{{}^{\mathrm w}\!I_2{}^2-{}^{\mathrm w}\!I_1\,{}^{\mathrm w}\!I_3}{{}^{\mathrm w}\!I_1\,{}^{\mathrm w}\!I_2-{}^{\mathrm w}\!I_3}\ ;\\ \bar a_5 &= -\bar a_8\ \frac{\bar n_1}{\bar a_2\,({}^{\bar{\mathrm w}}\!I_1\,{}^{\bar{\mathrm w}}\!I_2-{}^{\bar{\mathrm w}}\!I_3)^2}\ ,\quad \bar a_9 = -\bar a_8\ \frac{{}^{\bar{\mathrm w}}\!I_2{}^2-{}^{\bar{\mathrm w}}\!I_1\,{}^{\bar{\mathrm w}}\!I_3}{{}^{\bar{\mathrm w}}\!I_1\,{}^{\bar{\mathrm w}}\!I_2-{}^{\bar{\mathrm w}}\!I_3}\ ,\end{aligned} \tag{99}$$

where n_1=$n(a_8$=$0)/a_2$, $\bar n_1$=$\bar n(\bar a_8$=$0)/\bar a_2$, eqn (95), namely

$$\begin{aligned} n_1 &= 2\,a_2\,{}^{\mathrm w}\!I_2\,({}^{\mathrm w}\!I_1\,{}^{\mathrm w}\!I_2-{}^{\mathrm w}\!I_3)-\ a_8\ \Big(2\,{}^{\mathrm w}\!I_3{}^2\,({}^{\mathrm w}\!I_1{}^2+2\,{}^{\mathrm w}\!I_2)+{}^{\mathrm w}\!I_2{}^2\,({}^{\mathrm w}\!I_2{}^2-4\,{}^{\mathrm w}\!I_1\,{}^{\mathrm w}\!I_3)\Big)\ ;\\ \bar n_1 &= 2\,\bar a_2\,{}^{\bar{\mathrm w}}\!I_2\,({}^{\bar{\mathrm w}}\!I_1\,{}^{\bar{\mathrm w}}\!I_2-{}^{\bar{\mathrm w}}\!I_3)-\bar a_8\ \Big(2\,{}^{\bar{\mathrm w}}\!I_3{}^2\,({}^{\bar{\mathrm w}}\!I_1{}^2+2\,{}^{\bar{\mathrm w}}\!I_2)+{}^{\bar{\mathrm w}}\!I_2{}^2\,({}^{\bar{\mathrm w}}\!I_2{}^2-4\,{}^{\bar{\mathrm w}}\!I_1\,{}^{\bar{\mathrm w}}\!I_3)\Big)\ ,\end{aligned} \tag{100}$$

with

$$\bar a_2 = \frac{1}{a_2}\ ,\quad \bar a_8 = -\frac{a_8\,{}^{\mathrm w}\!I_3{}^2\ ({}^{\mathrm w}\!I_1\,{}^{\mathrm w}\!I_2-{}^{\mathrm w}\!I_3)}{a_2\ \Big(a_2\ ({}^{\mathrm w}\!I_1\,{}^{\mathrm w}\!I_2-{}^{\mathrm w}\!I_3)-3\,a_8\,{}^{\mathrm w}\!I_3{}^2\Big)}\ ;$$

$$\bar a_5 = a_8\,{}^{\mathrm w}\!I_3{}^3\ \frac{2\,a_2\,{}^{\mathrm w}\!I_1\,({}^{\mathrm w}\!I_1\,{}^{\mathrm w}\!I_2-{}^{\mathrm w}\!I_3)+a_8\,{}^{\mathrm w}\!I_3\,\Big({}^{\mathrm w}\!I_1{}^2\,({}^{\mathrm w}\!I_1{}^2-4\,{}^{\mathrm w}\!I_2)+2\,({}^{\mathrm w}\!I_2{}^2-{}^{\mathrm w}\!I_1\,{}^{\mathrm w}\!I_3)\Big)}{a_2\ \Big(a_2\,({}^{\mathrm w}\!I_1\,{}^{\mathrm w}\!I_2-{}^{\mathrm w}\!I_3)-3\,a_8\,{}^{\mathrm w}\!I_3{}^2\Big)^2}\ , \tag{101}$$

$$\bar a_9 = \frac{a_8\,{}^{\mathrm w}\!I_3{}^2\ ({}^{\mathrm w}\!I_1{}^2-{}^{\mathrm w}\!I_2)}{a_2\ \Big(a_2\ ({}^{\mathrm w}\!I_1\,{}^{\mathrm w}\!I_2-{}^{\mathrm w}\!I_3)-3\,a_8\,{}^{\mathrm w}\!I_3{}^2\Big)}\ .$$

Damage-effect tensors (98) formally lead to compliance and stiffness embedding *all but* coefficients c_4, c_6 and e_4, e_6 in the $\mathbf{w}, \bar{\mathbf{w}}$ representations.

Solution (24). An alternative of previous *Solution (23)* is obtained just by replacing coefficients a_9, $\bar{a}_9$ with a_3, $\bar{a}_3$: the four-coefficients sets (a_2, a_3, a_5, a_8) and $(\bar{a}_2, \bar{a}_3, \bar{a}_5, \bar{a}_8)$ give rise to the symmetric complementary inverse pair:

$$\begin{aligned} \mathbb{A} &= a_2\, \mathbf{I} \underline{\otimes} \mathbf{I} + a_3\, \mathbf{w} \otimes \mathbf{w} + a_5\, \mathbf{w}^2 \otimes \mathbf{w}^2 + a_8\, (\mathbf{w}^2 \otimes \mathbf{w} + \mathbf{w} \otimes \mathbf{w}^2)\,; \\ \bar{\mathbb{A}} &= \bar{a}_2\, \mathbf{I} \underline{\otimes} \mathbf{I} + \bar{a}_3\, \bar{\mathbf{w}} \otimes \bar{\mathbf{w}} + \bar{a}_5\, \bar{\mathbf{w}}^2 \otimes \bar{\mathbf{w}}^2 + \bar{a}_8\, (\bar{\mathbf{w}}^2 \otimes \bar{\mathbf{w}} + \bar{\mathbf{w}} \otimes \bar{\mathbf{w}}^2)\,, \end{aligned} \tag{102}$$

provided that

$$\begin{aligned} a_3 &= -a_8\, \frac{{}^{\mathrm{w}}I_1{}^2 - {}^{\mathrm{w}}I_2}{{}^{\mathrm{w}}I_1}\,, \quad a_5 = \frac{a_8^2}{a_3} = -a_8\, \frac{{}^{\mathrm{w}}I_1}{{}^{\mathrm{w}}I_1{}^2 - {}^{\mathrm{w}}I_2}\,; \\ \bar{a}_3 &= -\bar{a}_8\, \frac{{}^{\bar{\mathrm{w}}}I_1{}^2 - {}^{\bar{\mathrm{w}}}I_2}{{}^{\bar{\mathrm{w}}}I_1}\,, \quad \bar{a}_5 = \frac{\bar{a}_8^2}{\bar{a}_3} = -\bar{a}_8\, \frac{{}^{\bar{\mathrm{w}}}I_1}{{}^{\bar{\mathrm{w}}}I_1{}^2 - {}^{\bar{\mathrm{w}}}I_2}\,, \end{aligned} \tag{103}$$

with

$$\begin{aligned} \bar{a}_2 &= \frac{1}{a_2}\,, \quad \bar{a}_8 = -a_8\, \frac{{}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3\, ({}^{\mathrm{w}}I_2{}^2 - {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3)}{a_2\, d^{\diamond}}\,; \\ \bar{a}_3 &= a_8\, \frac{({}^{\mathrm{w}}I_2{}^2 - {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3)^2}{a_2\, d^{\diamond}}\,, \quad \bar{a}_5 = a_8\, \frac{{}^{\mathrm{w}}I_2{}^2\, {}^{\mathrm{w}}I_3{}^2}{a_2\, d^{\diamond}}\,, \end{aligned} \tag{104}$$

where

$$d^{\diamond} = a_2 \left({}^{\mathrm{w}}I_1\, ({}^{\mathrm{w}}I_1{}^2 - {}^{\mathrm{w}}I_2) \right) - a_8 \left(2\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3 + ({}^{\mathrm{w}}I_1{}^2 - 2\, {}^{\mathrm{w}}I_2)\, ({}^{\mathrm{w}}I_2{}^2 - 2\, {}^{\mathrm{w}}I_1 {}^{\mathrm{w}}I_3) \right). \tag{105}$$

Damage-effect tensors (102) formally lead to compliance and stiffness embedding *all but* coefficients c_4, c_6 and e_4, e_6 in the $\mathbf{w}, \bar{\mathbf{w}}$ representations.

Two additional symmetric solutions are now considered, which are based on generator *Solution (s.c)* with two 'shear-like' coefficients.

Solution (25). As commented in *Solution (12)*, a further enlargement of *Solutions (2)* and *(3)*, based on *Solution (s.c)*, can be obtained by taking symmetric damage-effect tensors with seven-coefficients sets $(a_3, a_4, a_5, a_6, a_7, a_8, a_9)$ and $(\bar{a}_3, \bar{a}_4, \bar{a}_5, \bar{a}_6, \bar{a}_7, \bar{a}_8, \bar{a}_9)$ (lacking only coefficients a_1, a_2 and $\bar{a}_1, \bar{a}_2$), namely:

$$\begin{aligned} \mathbb{A} &= a_4\, (\mathbf{w} \underline{\otimes} \mathbf{I} + \mathbf{I} \underline{\otimes} \mathbf{w}) + a_6\, \mathbf{w} \underline{\otimes} \mathbf{w} + a_3\, \mathbf{w} \otimes \mathbf{w} + a_5\, \mathbf{w}^2 \otimes \mathbf{w}^2 + a_7\, (\mathbf{w} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{w}) \\ &\quad + a_8\, (\mathbf{w}^2 \otimes \mathbf{w} + \mathbf{w} \otimes \mathbf{w}^2) + a_9\, (\mathbf{w}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{w}^2); \\ \bar{\mathbb{A}} &= \bar{a}_4\, (\bar{\mathbf{w}} \underline{\otimes} \mathbf{I} + \mathbf{I} \underline{\otimes} \bar{\mathbf{w}}) + \bar{a}_6\, \bar{\mathbf{w}} \underline{\otimes} \bar{\mathbf{w}} + \bar{a}_3\, \bar{\mathbf{w}} \otimes \bar{\mathbf{w}} + \bar{a}_5\, \bar{\mathbf{w}}^2 \otimes \bar{\mathbf{w}}^2 + \bar{a}_7\, (\bar{\mathbf{w}} \otimes \mathbf{I} + \mathbf{I} \otimes \bar{\mathbf{w}}) \\ &\quad + \bar{a}_8\, (\bar{\mathbf{w}}^2 \otimes \bar{\mathbf{w}} + \bar{\mathbf{w}} \otimes \bar{\mathbf{w}}^2) + \bar{a}_9\, (\bar{\mathbf{w}}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \bar{\mathbf{w}}^2), \end{aligned} \tag{106}$$

which form a complementary inverse pair provided that

$$a_3 = -\frac{n_3}{d_3}\,, \quad a_4 = -a_6 \frac{{}^{\mathrm{w}}I_3}{{}^{\mathrm{w}}I_2}\,; \qquad \bar{a}_3 = -\frac{\bar{n}_3}{\bar{d}_3}\,, \quad \bar{a}_4 = -\bar{a}_6 \frac{{}^{\bar{\mathrm{w}}}I_3}{{}^{\bar{\mathrm{w}}}I_2}\,, \tag{107}$$

where

$$
\begin{aligned}
n_3 \;=\;& 2a_6^3\,{}^{\mathrm{w}}\!I_3{}^2 + a_6^2\,{}^{\mathrm{w}}\!I_3\Big({}^{\mathrm{w}}\!I_3\Big(a_5(2\,{}^{\mathrm{w}}\!I_1{}^2 - 3\,{}^{\mathrm{w}}\!I_2) + 4a_8\,{}^{\mathrm{w}}\!I_1\Big) - 4a_7\,{}^{\mathrm{w}}\!I_2\Big) \\
&+ a_6\,{}^{\mathrm{w}}\!I_2\Big(3a_8^2\,{}^{\mathrm{w}}\!I_3{}^2 + 2a_8a_9\,{}^{\mathrm{w}}\!I_2(2\,{}^{\mathrm{w}}\!I_1{}^2 - 3\,{}^{\mathrm{w}}\!I_2) \\
&\qquad + a_9^2\Big({}^{\mathrm{w}}\!I_2{}^2 + 2\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_3 + 2\,{}^{\mathrm{w}}\!I_1({}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2 - {}^{\mathrm{w}}\!I_3) - 4\,{}^{\mathrm{w}}\!I_2{}^2\Big)\Big) \\
&+ 2a_6a_7\,{}^{\mathrm{w}}\!I_2\Big(- {}^{\mathrm{w}}\!I_3\Big(a_5(2\,{}^{\mathrm{w}}\!I_1{}^2 - 3\,{}^{\mathrm{w}}\!I_2) + 2a_8\,{}^{\mathrm{w}}\!I_1\Big) + 2a_9\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2\Big) \\
&+ 2a_7(a_5a_7 - 2a_8a_9)\,{}^{\mathrm{w}}\!I_2{}^2({}^{\mathrm{w}}\!I_1{}^2 - 3\,{}^{\mathrm{w}}\!I_2)\ , \\[4pt]
d_3 \;=\;& a_6\,{}^{\mathrm{w}}\!I_3{}^2(2a_6 - 3a_5\,{}^{\mathrm{w}}\!I_2) + 4a_6a_9\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3 + 2a_9^2\,{}^{\mathrm{w}}\!I_2{}^2({}^{\mathrm{w}}\!I_1{}^2 - 3\,{}^{\mathrm{w}}\!I_2)\ ,
\end{aligned}
\tag{108}
$$

and

$$
\begin{aligned}
\bar{n}_3 \;=\;& 2\bar{a}_6^3\,{}^{\bar{\mathrm{w}}}\!I_3{}^2 + \bar{a}_6^2\,{}^{\bar{\mathrm{w}}}\!I_3\Big({}^{\bar{\mathrm{w}}}\!I_3\Big(\bar{a}_5(2\,{}^{\bar{\mathrm{w}}}\!I_1{}^2 - 3\,{}^{\bar{\mathrm{w}}}\!I_2) + 4\bar{a}_8\,{}^{\bar{\mathrm{w}}}\!I_1\Big) - 4\bar{a}_7\,{}^{\bar{\mathrm{w}}}\!I_2\Big) \\
&+ \bar{a}_6\,{}^{\bar{\mathrm{w}}}\!I_2\Big(3\bar{a}_8^2\,{}^{\bar{\mathrm{w}}}\!I_3{}^2 + 2\bar{a}_8\bar{a}_9\,{}^{\bar{\mathrm{w}}}\!I_2(2\,{}^{\bar{\mathrm{w}}}\!I_1{}^2 - 3\,{}^{\bar{\mathrm{w}}}\!I_2) \\
&\qquad + \bar{a}_9^2\Big({}^{\bar{\mathrm{w}}}\!I_2{}^2 + 2\,{}^{\bar{\mathrm{w}}}\!I_1\,{}^{\bar{\mathrm{w}}}\!I_3 + 2\,{}^{\bar{\mathrm{w}}}\!I_1({}^{\bar{\mathrm{w}}}\!I_1\,{}^{\bar{\mathrm{w}}}\!I_2 - {}^{\bar{\mathrm{w}}}\!I_3) - 4\,{}^{\bar{\mathrm{w}}}\!I_2{}^2\Big)\Big) \\
&+ 2\bar{a}_6\bar{a}_7\,{}^{\bar{\mathrm{w}}}\!I_2\Big(- {}^{\bar{\mathrm{w}}}\!I_3\Big(\bar{a}_5(2\,{}^{\bar{\mathrm{w}}}\!I_1{}^2 - 3\,{}^{\bar{\mathrm{w}}}\!I_2) + 2\bar{a}_8\,{}^{\bar{\mathrm{w}}}\!I_1\Big) + 2\bar{a}_9\,{}^{\bar{\mathrm{w}}}\!I_1\,{}^{\bar{\mathrm{w}}}\!I_2\Big) \\
&+ 2\bar{a}_7(\bar{a}_5\bar{a}_7 - 2\bar{a}_8\bar{a}_9)\,{}^{\bar{\mathrm{w}}}\!I_2{}^2({}^{\bar{\mathrm{w}}}\!I_1{}^2 - 3\,{}^{\bar{\mathrm{w}}}\!I_2)\ , \\[4pt]
\bar{d}_3 \;=\;& \bar{a}_6\,{}^{\bar{\mathrm{w}}}\!I_3{}^2(2\bar{a}_6 - 3\bar{a}_5\,{}^{\bar{\mathrm{w}}}\!I_2) + 4\bar{a}_6\bar{a}_9\,{}^{\bar{\mathrm{w}}}\!I_1\,{}^{\bar{\mathrm{w}}}\!I_2\,{}^{\bar{\mathrm{w}}}\!I_3 + 2\bar{a}_9^2\,{}^{\bar{\mathrm{w}}}\!I_2{}^2({}^{\bar{\mathrm{w}}}\!I_1{}^2 - 3\,{}^{\bar{\mathrm{w}}}\!I_2)\ ,
\end{aligned}
\tag{109}
$$

with

$$
\begin{gathered}
\bar{a}_6 = \frac{{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2}{a_6\,{}^{\mathrm{w}}\!I_3}\ ,\quad \bar{a}_5 = -\frac{n_5\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3}{a_6\,d_9{}^2}\ ,\quad \bar{a}_7 = \frac{n_7\,{}^{\mathrm{w}}\!I_2}{a_6\,d_9\,{}^{\mathrm{w}}\!I_3}\ ,\quad \bar{a}_8 = \frac{n_8\,{}^{\mathrm{w}}\!I_2{}^2}{a_6\,d_9{}^2}\ ,\quad \bar{a}_9 = -\frac{n_9\,{}^{\mathrm{w}}\!I_2{}^2}{a_6\,d_9}\ ; \\
\bar{a}_3 = -\frac{\tilde{n}_3\,{}^{\mathrm{w}}\!I_2}{a_6\,d_9{}^2\,{}^{\mathrm{w}}\!I_3}\ ,\quad \bar{a}_4 = \frac{1}{a_4}\ ,
\end{gathered}
\tag{110}
$$

where

$$
\begin{aligned}
d_9 = {}& a_6\,{}^{\mathrm{w}}\!I_3{}^2\,\Big(a_8\,{}^{\mathrm{w}}\!I_2{}^2 + a_5\,{}^{\mathrm{w}}\!I_2\,({}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2 - 6\,{}^{\mathrm{w}}\!I_3) + 4\,a_6\,{}^{\mathrm{w}}\!I_3\Big) \\
&+ a_9^2\,{}^{\mathrm{w}}\!I_2{}^2\,\Big({}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2{}^2 - 2\,{}^{\mathrm{w}}\!I_1{}^2\,{}^{\mathrm{w}}\!I_3 - 3\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3\Big) \\
&+ a_9\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3\,\Big(a_8\,{}^{\mathrm{w}}\!I_2\,({}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2 - 9\,{}^{\mathrm{w}}\!I_3) - a_6\,({}^{\mathrm{w}}\!I_2{}^2 - 2\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_3)\Big) \\
&+ a_7\,{}^{\mathrm{w}}\!I_2\,\Big(2\,a_9\,({}^{\mathrm{w}}\!I_2{}^3 - 3\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3) + {}^{\mathrm{w}}\!I_3\,\Big(- 6\,a_6\,{}^{\mathrm{w}}\!I_3 + a_5\,{}^{\mathrm{w}}\!I_2\,(-{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2 + 9\,{}^{\mathrm{w}}\!I_3)\Big)\Big),
\end{aligned}
\tag{111}
$$

$$
\begin{aligned}
n_7 = {}& a_6\,{}^{\mathrm{w}}\!I_3{}^2\,\Big(a_8\,{}^{\mathrm{w}}\!I_2{}^2 + a_5\,{}^{\mathrm{w}}\!I_2\,({}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2 - 3\,{}^{\mathrm{w}}\!I_3) + 2\,a_6\,{}^{\mathrm{w}}\!I_3\Big) \\
&+ a_9\,{}^{\mathrm{w}}\!I_2{}^2\,{}^{\mathrm{w}}\!I_3\,\Big(- a_6\,{}^{\mathrm{w}}\!I_2 + a_8\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2 - 6\,a_8\,{}^{\mathrm{w}}\!I_3\Big) + a_9^2\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2{}^2\,\Big({}^{\mathrm{w}}\!I_2{}^2 - 2\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_3\Big) \\
&+ a_7\,{}^{\mathrm{w}}\!I_2\,\Big(2\,a_9\,\Big({}^{\mathrm{w}}\!I_2{}^3 - 2\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3\Big) + {}^{\mathrm{w}}\!I_3\,\Big(- 4\,a_6\,{}^{\mathrm{w}}\!I_3 + a_5\,{}^{\mathrm{w}}\!I_2\,(-{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2 + 6\,{}^{\mathrm{w}}\!I_3)\Big)\Big),
\end{aligned}
\tag{112}
$$

$$
\begin{aligned}
n_9 = {}& a_9^2\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2{}^2 + a_9\,(-a_6 + a_8\,{}^{\mathrm{w}}\!I_1)\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3 + a_6\,(a_8 + a_5\,{}^{\mathrm{w}}\!I_1)\,{}^{\mathrm{w}}\!I_3{}^2 \\
&+ a_7\,{}^{\mathrm{w}}\!I_2\,(2\,a_9\,{}^{\mathrm{w}}\!I_2 - a_5\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_3)\,.
\end{aligned}
\tag{113}
$$

The other factors $\tilde{n}_3, n_5, n_8$ entering eqn (110) are given by longer expressions and are reported in Appendix E for the sake of completeness. Factors $\bar{n}_3$, $\bar{d}_3$ and $\tilde{n}_3$, d_9 appearing in the expression of $\bar{a}_3$ in eqns (107c), (110f) are linked by the relations

$$\bar{n}_3 = \frac{1}{{}^{\mathrm{w}}I_1} \left(\frac{{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2}{a_6\,{}^{\mathrm{w}}I_3}\right)^3 \frac{f^2\,d_3}{{d_9}^4}\,\tilde{n}_3\ ; \qquad \bar{d}_3 = \left(\frac{{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2}{a_6\,{}^{\mathrm{w}}I_3}\right)^2 \frac{f^2\,d_3}{{d_9}^2}\ , \tag{114}$$

where d_3 is already given in eqn (108b) and factor f is expressed by

$$f = -2\,a_6\,{}^{\mathrm{w}}I_3 + 3\,a_7\,{}^{\mathrm{w}}I_2 + a_9\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2\ . \tag{115}$$

Solution (25) is the instance of the solution set found here that embeds the larger number of coefficients and relevant tensorial terms of the general orthotropic representation (only two coefficients are lacking, the 'non-shear' $a_1, \bar{a}_1$ and the 'shear-like' $a_2, \bar{a}_2$). This solution forms with *Solutions (2)* and *(3)* a family of progressively-enlarged solutions and comprises them as particular cases: *Solution (3)*, when a_9=0, $\bar{a}_9$=0; *Solution (2)* when a_8=a_9=0, $\bar{a}_8$=$\bar{a}_9$=0. Damage-effect tensors (106) formally lead to compliance and stiffness embedding all tensor terms.

Solution (26). Another particular case of *Solution (25)*, alternative in a sense to *Solution (3)*, is obtained by setting a_7=0, $\bar{a}_7$=0 in *Solution (25)*, instead of a_9=0, $\bar{a}_9$=0 as in *Solution (3)*, thus by taking symmetric damage-effect tensors embedding the six-coefficients sets $(a_3, a_4, a_5, a_6, a_8, a_9)$ and $(\bar{a}_3, \bar{a}_4, \bar{a}_5, \bar{a}_6, \bar{a}_8, \bar{a}_9)$ (lacking only coefficients a_1, a_2, a_7 and $\bar{a}_1, \bar{a}_2, \bar{a}_7$), namely:

$$\begin{aligned}
\mathbb{A} &= a_4\,(\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{I} + \mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{w}) + a_6\,\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w} + a_3\,\mathbf{w}\otimes\mathbf{w} + a_5\,\mathbf{w}^2\otimes\mathbf{w}^2 \\
&\quad + a_8\,(\mathbf{w}^2\otimes\mathbf{w} + \mathbf{w}\otimes\mathbf{w}^2) + a_9\,(\mathbf{w}^2\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{w}^2); \\
\bar{\mathbb{A}} &= \bar{a}_4\,(\bar{\mathbf{w}}\,\underline{\overline{\otimes}}\,\mathbf{I} + \mathbf{I}\,\underline{\overline{\otimes}}\,\bar{\mathbf{w}}) + \bar{a}_6\,\bar{\mathbf{w}}\,\underline{\overline{\otimes}}\,\bar{\mathbf{w}} + \bar{a}_3\,\bar{\mathbf{w}}\otimes\bar{\mathbf{w}} + \bar{a}_5\,\bar{\mathbf{w}}^2\otimes\bar{\mathbf{w}}^2 \\
&\quad + \bar{a}_8\,(\bar{\mathbf{w}}^2\otimes\bar{\mathbf{w}} + \bar{\mathbf{w}}\otimes\bar{\mathbf{w}}^2) + \bar{a}_9\,(\bar{\mathbf{w}}^2\otimes\mathbf{I} + \mathbf{I}\otimes\bar{\mathbf{w}}^2),
\end{aligned} \tag{116}$$

which form a complementary inverse pair provided that

$$\begin{aligned}
a_3 &= -\frac{a_6\,n'_3}{{{}^{\mathrm{w}}I_2}^3\,{d_8}^2}\ , \quad a_8 = \frac{n'_8}{{{}^{\mathrm{w}}I_2}^2\,{}^{\mathrm{w}}I_3\,d_8}\ , \quad a_4 = -a_6\frac{{}^{\mathrm{w}}I_3}{{}^{\mathrm{w}}I_2}\ ; \\
\bar{a}_3 &= -\frac{\bar{a}_6\,\bar{n}'_3}{{{}^{\bar{\mathrm{w}}}I_2}^3\,{\bar{d}_8}^2}\ , \quad \bar{a}_8 = \frac{\bar{n}'_8}{{{}^{\bar{\mathrm{w}}}I_2}^2\,{}^{\bar{\mathrm{w}}}I_3\,\bar{d}_8}\ , \quad \bar{a}_4 = -\bar{a}_6\frac{{}^{\bar{\mathrm{w}}}I_3}{{}^{\bar{\mathrm{w}}}I_2}\ ,
\end{aligned} \tag{117}$$

where

$$d_8 = a_6\,{}^{\mathrm{w}}I_3 + a_9\,({}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2 - 6\,{}^{\mathrm{w}}I_3)\ , \tag{118}$$

$$\begin{aligned}
n'_3 &= 2\,a_6\,a_9\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3\,\Big(-6\,{{}^{\mathrm{w}}I_2}^2\,{}^{\mathrm{w}}I_3 + {}^{\mathrm{w}}I_1\,({{}^{\mathrm{w}}I_2}^3 + 6\,{{}^{\mathrm{w}}I_3}^2)\Big) \\
&\quad + a_6\,{{}^{\mathrm{w}}I_3}^2\,\Big(-a_5\,{}^{\mathrm{w}}I_2\,({}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2 - 3\,{}^{\mathrm{w}}I_3)^2 + a_6\,({{}^{\mathrm{w}}I_2}^3 - 4\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3 + 6\,{{}^{\mathrm{w}}I_3}^2)\Big) \\
&\quad + {a_9}^2\,{{}^{\mathrm{w}}I_2}^2\,\Big(4\,{{}^{\mathrm{w}}I_1}^3\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3 + 18\,{}^{\mathrm{w}}I_2\,{{}^{\mathrm{w}}I_3}^2 - {{}^{\mathrm{w}}I_1}^2\,({{}^{\mathrm{w}}I_2}^3 + 18\,{{}^{\mathrm{w}}I_3}^2)\Big)\ ,
\end{aligned} \tag{119}$$

$$\begin{aligned}
n'_8 &= a_6\,a_9\,{{}^{\mathrm{w}}I_2}^3\,{}^{\mathrm{w}}I_3 - a_6\,{{}^{\mathrm{w}}I_3}^2\,\Big(a_5\,{}^{\mathrm{w}}I_2\,({}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2 - 3\,{}^{\mathrm{w}}I_3) + 2\,a_6\,{}^{\mathrm{w}}I_3\Big) \\
&\quad + {a_9}^2\,{}^{\mathrm{w}}I_1\,{{}^{\mathrm{w}}I_2}^2\,\Big(-{{}^{\mathrm{w}}I_2}^2 + 2\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3\Big)\ ,
\end{aligned} \tag{120}$$

and

$$\bar{d}_8 = \bar{a}_6\,{}^{\bar{\mathrm{w}}}I_3 + \bar{a}_9\,({}^{\bar{\mathrm{w}}}I_1\,{}^{\bar{\mathrm{w}}}I_2 - 6\,{}^{\bar{\mathrm{w}}}I_3)\ , \tag{121}$$

$$
\begin{aligned}
\bar{n}'_3 = {} & 2\,\bar{a}_6\,\bar{a}_9\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3\,\left(-6\,{}^{\mathrm{w}}I_2^{\,2}\,{}^{\mathrm{w}}I_3 + {}^{\mathrm{w}}I_1\,({}^{\mathrm{w}}I_2^{\,3} + 6\,{}^{\mathrm{w}}I_3^{\,2})\right) \\
& + \bar{a}_6\,{}^{\mathrm{w}}I_3^{\,2}\left(-\,\bar{a}_5\,{}^{\mathrm{w}}I_2\,({}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2 - 3\,{}^{\mathrm{w}}I_3)^2 + \bar{a}_6\,({}^{\mathrm{w}}I_2^{\,3} - 4\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3 + 6\,{}^{\mathrm{w}}I_3^{\,2})\right) \\
& + \bar{a}_9^{\,2}\,{}^{\mathrm{w}}I_2^{\,2}\,\left(4\,{}^{\mathrm{w}}I_1^{\,3}\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3 + 18\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3^{\,2} - {}^{\mathrm{w}}I_1^{\,2}\,({}^{\mathrm{w}}I_2^{\,3} + 18\,{}^{\mathrm{w}}I_3^{\,2})\right)\ ,
\end{aligned}
\tag{122}
$$

$$
\begin{aligned}
\bar{n}'_8 = {} & \bar{a}_6\,\bar{a}_9\,{}^{\mathrm{w}}I_2^{\,3}\,{}^{\mathrm{w}}I_3 - \bar{a}_6\,{}^{\mathrm{w}}I_3^{\,2}\left(\bar{a}_5\,{}^{\mathrm{w}}I_2\,\left({}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2 - 3\,{}^{\mathrm{w}}I_3\right) + 2\,\bar{a}_6\,{}^{\mathrm{w}}I_3\right) \\
& + \bar{a}_9^{\,2}\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2^{\,2}\,(-{}^{\mathrm{w}}I_2^{\,2} + 2\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3)\ ,
\end{aligned}
\tag{123}
$$

with

$$
\begin{aligned}
& \bar{a}_6 = \frac{{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2}{a_6\,{}^{\mathrm{w}}I_3}\ ,\quad \bar{a}_5 = -\frac{\hat{n}_5}{a_6(a_6-3a_9)^2\,d_3\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3^{\,2}}\ ,\quad \bar{a}_9 = \frac{a_6\,{}^{\mathrm{w}}I_3 - a_9\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2}{a_6(a_6-3a_9)\,{}^{\mathrm{w}}I_3}\ ; \\
& \bar{a}_3 = \frac{\hat{n}_3\,{}^{\mathrm{w}}I_2}{a_6\,d_3\,{}^{\mathrm{w}}I_3}\ ,\quad \bar{a}_8 = \frac{\hat{n}_8}{a_6(a_6-3a_9)\,d_3\,{}^{\mathrm{w}}I_3^{\,2}}\ ,\quad \bar{a}_4 = \frac{1}{a_4}\ ,
\end{aligned}
\tag{124}
$$

where d_3 is already given in eqn (108b) and

$$
\begin{aligned}
\hat{n}_3 = {} & 2\,a_6\,a_9\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3\,\left({}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2^{\,2} - 2\,{}^{\mathrm{w}}I_1^{\,2}\,{}^{\mathrm{w}}I_3 + 2\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3\right) \\
& + a_6\,{}^{\mathrm{w}}I_3^{\,2}\,\left(-2\,a_5\,{}^{\mathrm{w}}I_2^{\,3} + 3\,a_5\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3 + a_6\,({}^{\mathrm{w}}I_2^{\,2} - 2\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3)\right) \\
& + a_9^{\,2}\,{}^{\mathrm{w}}I_2^{\,2}\,\left({}^{\mathrm{w}}I_1^{\,2}\,{}^{\mathrm{w}}I_2^{\,2} - 2\,{}^{\mathrm{w}}I_1^{\,3}\,{}^{\mathrm{w}}I_3 + 10\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3 - 4\,({}^{\mathrm{w}}I_2^{\,3} + 3\,{}^{\mathrm{w}}I_3^{\,2})\right)\ ,
\end{aligned}
\tag{125}
$$

$$
\begin{aligned}
\hat{n}_5 = {} & 2\,a_6^{\,3}\,{}^{\mathrm{w}}I_3^{\,4}\left(-\,2\,a_6\,{}^{\mathrm{w}}I_2^{\,2} + 4\,a_5\,{}^{\mathrm{w}}I_2^{\,3} + 2\,a_6\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3 - 3\,a_5\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3\right) \\
& + a_9^{\,4}\,{}^{\mathrm{w}}I_2^{\,4}\left(-{}^{\mathrm{w}}I_1^{\,4}\,{}^{\mathrm{w}}I_2^{\,2} + 4\,{}^{\mathrm{w}}I_1^{\,2}\,{}^{\mathrm{w}}I_2^{\,3} + 4\,{}^{\mathrm{w}}I_1^{\,5}\,{}^{\mathrm{w}}I_3 - 24\,{}^{\mathrm{w}}I_1^{\,3}\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3 + 24\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2^{\,2}\,{}^{\mathrm{w}}I_3 + 36\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3^{\,2}\right) \\
& -\,4\,a_6^{\,2}\,a_9\,{}^{\mathrm{w}}I_2^{\,2}\,{}^{\mathrm{w}}I_3^{\,3}\,\left(a_6\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2 + a_5\,(2\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2^{\,2} - 3\,{}^{\mathrm{w}}I_1^{\,2}\,{}^{\mathrm{w}}I_3 + 6\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3)\right) \\
& + 2\,a_6\,a_9^{\,3}\,{}^{\mathrm{w}}I_2^{\,3}\,{}^{\mathrm{w}}I_3\,\left({}^{\mathrm{w}}I_1^{\,3}\,{}^{\mathrm{w}}I_2^{\,2} + 18\,{}^{\mathrm{w}}I_1^{\,2}\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3 - 24\,{}^{\mathrm{w}}I_2^{\,2}\,{}^{\mathrm{w}}I_3 - 4\,{}^{\mathrm{w}}I_1\,(2\,{}^{\mathrm{w}}I_2^{\,3} + 9\,{}^{\mathrm{w}}I_3^{\,2})\right) \\
& + a_6\,a_9^{\,2}\,{}^{\mathrm{w}}I_2^{\,2}\,{}^{\mathrm{w}}I_3^{\,2}\left(2\,a_5\,{}^{\mathrm{w}}I_2\,({}^{\mathrm{w}}I_1^{\,2}\,{}^{\mathrm{w}}I_2^{\,2} - 3\,{}^{\mathrm{w}}I_1^{\,3}\,{}^{\mathrm{w}}I_3 + 6\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3 + 9\,{}^{\mathrm{w}}I_3^{\,2}) \right. \\
& \left. + a_6\,(3\,{}^{\mathrm{w}}I_1^{\,2}\,{}^{\mathrm{w}}I_2^{\,2} + 16\,{}^{\mathrm{w}}I_2^{\,3} - 8\,{}^{\mathrm{w}}I_1^{\,3}\,{}^{\mathrm{w}}I_3 + 12\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3 + 36\,{}^{\mathrm{w}}I_3^{\,2})\right)\ ,
\end{aligned}
\tag{126}
$$

$$
\begin{aligned}
\hat{n}_8 = {} & a_6^{\,2}\,{}^{\mathrm{w}}I_3^{\,3}\left(-\,2\,a_6\,{}^{\mathrm{w}}I_2^{\,2} + 4\,a_5\,{}^{\mathrm{w}}I_2^{\,3} + 2\,a_6\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3 - 3\,a_5\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3\right) \\
& -\,2\,a_6\,a_9^{\,2}\,{}^{\mathrm{w}}I_2^{\,2}\,{}^{\mathrm{w}}I_3\,\left(-4\,{}^{\mathrm{w}}I_2^{\,3} + {}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3 + ({}^{\mathrm{w}}I_1^{\,3} - 12\,{}^{\mathrm{w}}I_3)\,{}^{\mathrm{w}}I_3\right) \\
& - a_6 a_9\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3^{\,2}\left(a_6(3\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2^{\,2} - 2\,{}^{\mathrm{w}}I_1^{\,2}\,{}^{\mathrm{w}}I_3 + 4\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3) + a_5\,{}^{\mathrm{w}}I_2\,(2\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2^{\,2} - 3\,{}^{\mathrm{w}}I_1^{\,2}\,{}^{\mathrm{w}}I_3 + 6\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3)\right) \\
& + a_9^{\,3}\,{}^{\mathrm{w}}I_2^{\,3}\,\left({}^{\mathrm{w}}I_1^{\,3}\,{}^{\mathrm{w}}I_2^{\,2} - 2\,{}^{\mathrm{w}}I_1^{\,4}\,{}^{\mathrm{w}}I_3 + 14\,{}^{\mathrm{w}}I_1^{\,2}\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3 - 12\,{}^{\mathrm{w}}I_2^{\,2}\,{}^{\mathrm{w}}I_3 - 4\,{}^{\mathrm{w}}I_1\,({}^{\mathrm{w}}I_2^{\,3} + 3\,{}^{\mathrm{w}}I_3^{\,2})\right)\ .
\end{aligned}
\tag{127}
$$

Damage-effect tensors (116) formally lead to compliance and stiffness embedding all tensor terms.

5 Derivation of the solution set

In the present section the reasoning that has led to non-symmetric *Solutions (11)-(13)* (and 'twin' ones *(14)-(16)*) and *(17)-(20)* is briefly presented. Two different approaches have been followed, which make use, respectively, of multiplication tables of tensor addends pertaining to the general orthotropic representation, and of Sherman-Morrison's formulas allowing the inversion of multiple rank-one updates of a given non-singular fourth-order tensor.

5.1 Derivation based on multiplication tables of orthotropic tensor addends

Trying to enlarge known *Solutions (0.2)* ('basic' Valanis-damage, founded on single coefficient a_6) and *(0.3)* ('extended' damage, based on coefficients set (a_6, a_{92})), and foreseen *Solution (1)* (Valanis-type damage-effect tensor, based on coefficients set (a_6, a_3)), one is hinted first to attempt keeping the terms attached to the three coefficients a_6, a_3, a_{92} all together, and then, to start with, to try adding further terms associated to the other 'non-symmetric' coefficients a_{72}, a_{82}.

This way of thinking is corroborated also by the independent consideration that, by exploring a possible multiplicative family of solutions, one finds out that not only the 'extended' structure of *Solution (0.3)* can be decomposed in the formed product of the isotropic structure *(0.1)* and the 'basic' structure *(0.2)*, symbolically $(a_2, a_1):(a_6) \rightarrow (a_6, a_{92})$, but also that the structure of *Solution (0.3)* multiplied by the isotropic *(0.1)* is 'eingenbased' (in the sense that it spans the same tensor products space), symbolically $(a_2, a_1):(a_6, a_{92}) \rightarrow (a_6, a_{92})$. Also, the multiplication of Valanis-type damage-effect tensor *Solution (1)* with the isotropic *(0.1)* spans the tensor terms attached to the new coefficients set $(a_6, a_3, a_{92}, a_{72})$, symbolically $(a_2, a_1):(a_6, a_3) \rightarrow (a_6, a_3, a_{92}, a_{72})$. So, this appears to be a candidate for the solution set. Also, this set is in turn 'eingenbased' since a further multiplication with isotropic *Solution (0.1)* renders the same tensor terms, symbolically $(a_2, a_1):(a_6, a_3, a_{92}, a_{72}) \rightarrow (a_6, a_3, a_{92}, a_{72})$. On the other hand, a new structure based only on (a_6, a_3, a_{92}) (no a_{72}) does not arise from the multiplication of the previous solutions and at the same time turns out not to be 'eingenbased' after multiplication with the isotropic one, symbolically $(a_2, a_1):(a_6, a_3, a_{92}) \rightarrow (a_6, a_3, a_{92}, a_{72})$. So, one is led to conjecture that only the full set $(a_6, a_3, a_{92}, a_{72})$ should be a solution case, while (a_6, a_3, a_{92}) might not. Then, as an additional step, although this set is not generated from the previous products, one attempts as well to add the term attached to the coefficient a_{82} and discovers that this new set of terms is also 'eingenbased' after multiplication with the isotropic structure *(0.1)*, symbolically $(a_2, a_1):(a_6, a_3, a_{92}, a_{72}, a_{82}) \rightarrow (a_6, a_3, a_{92}, a_{72}, a_{82})$. Although this does not prove necessarily that this structure is of dual type, since this last case contains as well the previous, one is led to inquire if the set $(a_6, a_3, a_{92}, a_{72}, a_{82})$ and the relevant particular cases do belong or not to the sought solution set, and to attempt to work-out the complete expression of the dual coefficients set $(\bar{a}_6, \bar{a}_3, \bar{a}_{91}, \bar{a}_{71}, \bar{a}_{81})$.

One way to do that is forming the product between trial damage-effect tensors $\mathbb{A}$ with coefficients set $(a_6, a_3, a_{92}, a_{72}, a_{82})$ and dual damage-effect tensors $\bar{\mathbb{A}}$ with coefficients set $(\bar{a}_6, \bar{a}_3, \bar{a}_{91}, \bar{a}_{71}, \bar{a}_{81})$, and posing the equality $\mathbb{A}{:}\bar{\mathbb{A}} = \mathbf{I} \underline{\overline{\otimes}} \mathbf{I}$ for any damage state $\mathbf{w}$, $\bar{\mathbf{w}}$. To build up such product it is convenient to prepare a multiplication table with all the tensor terms involved in the operation. The multiplication table pivoting on the coefficients sets $(a_6, a_3, a_{92}, a_{72}, a_{82})$ and $(\bar{a}_6, \bar{a}_3, \bar{a}_{91}, \bar{a}_{71}, \bar{a}_{81})$ is reported in Table 1, Appendix D.

Table 1 shows that, besides the identity $\mathbf{I} \underline{\overline{\otimes}} \mathbf{I}$, four distinct tensor terms fill the table entries (with 'complementary symmetry' with respect to the table diagonal): $\mathbf{w} \otimes \bar{\mathbf{w}}$, $\mathbf{I} \otimes \mathbf{I}$, $\mathbf{w} \otimes \mathbf{I}$, $\mathbf{I} \otimes \bar{\mathbf{w}}$. These terms are already generated at the level of three-coefficients sets (a_6, a_3, a_{92}) and $(\bar{a}_6, \bar{a}_3, \bar{a}_{91})$. Since the coefficient $\bar{a}_6$ is fixed by the 'orthogonality' condition $\bar{a}_6 = 1/a_6$, which is necessary to make up the identity $\mathbf{I} \underline{\overline{\otimes}} \mathbf{I}$ (see also 'shear-like' *Solution (s.b)*), from Table 1 it appears that: *i)* it should be possible to determine the four coefficients $\bar{a}_3, \bar{a}_{91}, \bar{a}_{71}, \bar{a}_{81}$ by setting to zero the four factors pre-multiplying the four extra terms exceeding the identity in the formed product $\mathbb{A}{:}\bar{\mathbb{A}}$; *ii)* it might be possible to determine the reduced cases with either $a_{72}{=}0$ or $a_{82}{=}0$ if an additional constraint on the remaining four unbarred coefficients holds; *iii)* it might not be possible to determine the reduced case with $a_{72}{=}a_{82}{=}0$ unless if two additional constraints on the remaining three unbarred coefficients hold; *iv)* there appear clearly the reduced particular cases already known as *Solutions (0.3)* and *(1)*, which both span a two-dimensional base (see the two internal frames depicted in Table 1).

By exploring further the possibility *i)* remarked above one is left with the solution of a linear system of two pairs of equations, which is governed by a 2×2 coefficient matrix and can be written as follows:

$$\begin{bmatrix} a_6 + 3a_3 + a_{82}\mathrm{tr}\,\mathbf{w} & a_3\mathrm{tr}\,\bar{\mathbf{w}} + 3a_{82} \\ a_{92}\mathrm{tr}\,\mathbf{w} + 3a_{72} & a_6 + 3a_{92} + a_{72}\mathrm{tr}\,\bar{\mathbf{w}} \end{bmatrix} \cdot \begin{Bmatrix} \bar{a}_3 \;,\; \bar{a}_{71} \\ \bar{a}_{81} \,,\; \bar{a}_{91} \end{Bmatrix} = -\frac{1}{a_6} \begin{Bmatrix} a_3 \;,\; a_{82} \\ a_{72} \,,\; a_{92} \end{Bmatrix} . \quad (128)$$

The solution of this system of equations in the unknowns $\bar{a}_3$, $\bar{a}_{81}$; $\bar{a}_{71}$, $\bar{a}_{91}$ is straightforward and leads precisely to the expressions of the coefficients given in *Solution (13)*, eqns (67)-(68). Notice that the determinant of the 2×2 coefficient matrix in the linear system (128) is given by d_2, eqns (68a,c).

At this stage, it is also possible to check that possibilities *ii)* listed above do work through the constraints (59), (64) and do originate *Solutions (11)* and *(12)*. This can be checked as either a particular case of the previous solution or directly by applying the same procedure above for imposing that the candidate dual damage-effect tensors are indeed inverse of each other. On the other hand, it is confirmed that possibility *iii)* listed above is not a solution instance, namely coefficients sets (a_6, a_3, a_{92}) and $(\bar{a}_6, \bar{a}_3, \bar{a}_{91})$ do not form a dual base. Indeed the four requirements on the coefficients are mutually excluding the possibility to have at the same time $a_3{\neq}0$ and $a_{92}{\neq}0$, and this leaves open only the reduced options of the two particular cases already known, namely *Solutions (0.3)* and *(1)*. By further inspecting *Solution (13)* it is also possibile to exclude any other reduced case different from those already outlined.

As already noticed in Section 4.4.1, despite the similarity between *Solutions (5)* and *(11)*, it is not possible to enlarge *Solutions (11)-(13)* with a further introduction of the terms $\mathbf{I}{\otimes}\mathbf{I}$ attached to a_1 ad $\bar{a}_1$. This is revealed by the consideration that the introduction in Table 1 of a row and column corresponding to new-inserted terms $(a_1)\,\mathbf{I}{\otimes}\mathbf{I}$, $(\bar{a}_1)\,\mathbf{I}{\otimes}\mathbf{I}$ would lead to two new tensorial terms outbalancing the number of the single extra coefficient that could be fixed (see also comments in the table caption).

5.2 Derivation based on Sherman-Morrison's formulas

Instead of using multiplication tables to work around the problem of making the product of candidate dual damage-effect tensors equal to the identity, an alternative approach is that of performing first the inversion by using Sherman-Morrison's formulas and then of checking if the obtained inverse does display the complementary structure, may be under specific conditions that need to be fulfilled. Sherman-Morrison's inversion formulas give the inverse of a non-singular fourth-order tensor modified by multiple rank-one updates.

Such formulas are listed in Appendix C for the cases of one and two rank-one updates, eqns (C.1), (C.2), respectively.

In exploring the damage-effect tensor (58a) of *Solution (11)* as a possible candidate, one may interpret it as a single rank-one update $\mathbb{A}_1{=}\mathbb{A}_0{+}a_1\,\mathbf{B}_1{\otimes}\mathbf{C}_1$ of Valanis-type damage-effect tensor $\mathbb{A}_0{=}a_6\,\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w}{+}a_3\,\mathbf{w}{\otimes}\mathbf{w}$, eqn (17a), *Solution (1)*. Indeed, this tensor can be adopted as 'pivoting' reference since it has a known convenient inverse with complementary structure, namely $\mathbb{A}_0^{-1}{=}1/a_6\,\bar{\mathbf{w}}\,\underline{\overline{\otimes}}\,\bar{\mathbf{w}}{-}a_3/(a_6(3a_3{+}a_6))\,\bar{\mathbf{w}}{\otimes}\bar{\mathbf{w}}$. If one then formalizes that eqn (58a) can be interpreted as a single rank-one update of $\mathbb{A}_0$ by taking $a_1{=}1$, $\mathbf{B}_1{=}\mathbf{I}$, $\mathbf{C}_1{=}a_{92}\mathbf{w}^2{+}a_{72}\mathbf{w}$, that is

$$\mathbb{A}_0 = a_6\,\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w} + a_3\,\mathbf{w}\otimes\mathbf{w}\,; \qquad \mathbb{A}_1 = \mathbb{A}_0 + \mathbf{I}\otimes\left(a_{92}\,\mathbf{w}^2 + a_{72}\,\mathbf{w}\right), \tag{129}$$

Sherman-Morrison's inversion formula (C.1) can be applied to get $\mathbb{A}_1^{-1}$. In doing so one gets first the second-order tensors $\mathbb{A}_0^{-1}:\mathbf{B}_1{=}1/a_6\,\bar{\mathbf{w}}^2{-}a_3/(a_6(3a_3{+}a_6))\,\mathrm{tr}\,\bar{\mathbf{w}}\,\bar{\mathbf{w}}$ and $\mathbf{C}_1:\mathbb{A}_0^{-1}{=}a_{92}/a_6\,\mathbf{I}{+}c/(a_6(3a_3{+}a_6))\,\bar{\mathbf{w}}$, where $c{=}a_6a_{72}{-}a_3a_{92}\mathrm{tr}\,\mathbf{w}$, and recognizes that a dual structure can be acquired by setting $c{=}0$, which leads precisely to constraint (59a), i.e. $a_6a_{72}{=}a_3a_{92}\mathrm{tr}\,\mathbf{w}$. Then, this constraint is imposed on the denominator factor D_1 in formula (C.1), to get $D_1{=}1{+}\mathbf{C}_1{:}\mathbb{A}_0^{-1}{:}\mathbf{B}_1{=}(3a_{92}{+}a_6)/a_6$, and the final expressions of the sought dual coefficients $\bar{a}_{91}$, $\bar{a}_{71}$ in eqn (58b) are identified, which are indeed those given in eqns (60c,d). Thus, *Solution (11)* is fully recovered within the Sherman-Morrison's approach.

A similar procedure could be applied as well to get 'reciprocal' *Solution (12)*, since this case may be seen as well as a single rank-one update of the same Valanis-type reference tensor $\mathbb{A}_0$, namely $\mathbb{A}_1{=}\mathbb{A}_0{+}(a_{92}\,\mathbf{I}{+}a_{82}\,\mathbf{w}){\otimes}\mathbf{w}^2$. Inversion formula (C.1) would then lead to the constraint (64a) and to the dual coefficients $\bar{a}_{91}$, $\bar{a}_{81}$ in eqns (65c,d).

An alternative of the procedure above could consider the damage-effect tensor (58a) as formed by two rank-one updates $\mathbb{A}_2{=}\mathbb{A}_0{+}a_1\,\mathbf{B}_1{\otimes}\mathbf{C}_1{+}a_2\,\mathbf{B}_2{\otimes}\mathbf{C}_2$ of the 'basic' damage-effect tensor $\mathbb{A}_0{=}a_6\,\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w}$, *Solution (0.2)*, which is characterized by the simpler complementary inverse $\mathbb{A}_0^{-1}{=}1/a_6\,\bar{\mathbf{w}}\,\underline{\overline{\otimes}}\,\bar{\mathbf{w}}$. Actually, this leads indeed to an independent derivation of more general *Solution (13)*, since in this way *Solutions (11)* and *(12)* can just be seen as the particular cases recovered for $a_{82}{=}0$ and $a_{72}{=}0$, respectively. As a matter of fact, one should notice that, since *Solutions (11)* and *(12)* work through constraints on the coefficients, it is not feasible to interpret directly *Solution (13)* just as a single rank-one update of the former cases. Then, the candidate damage-effect tensor (66a) is inspected for identifying the update factors. One way to do that leads to, same as before, $a_1{=}1$, $\mathbf{B}_1{=}\mathbf{I}$, $\mathbf{C}_1{=}a_{92}\mathbf{w}^2{+}a_{72}\mathbf{w}$, and to $a_2{=}1$, $\mathbf{B}_2{=}\mathbf{w}$, $\mathbf{C}_2{=}a_{82}\mathbf{w}^2{+}a_3\mathbf{w}$, i.e.:

$$\mathbb{A}_0 = a_6\,\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w}\,; \qquad \mathbb{A}_2 = \mathbb{A}_0 + \mathbf{I}\otimes\left(a_{92}\mathbf{w}^2 + a_{72}\mathbf{w}\right) + \mathbf{w}\otimes\left(a_{82}\mathbf{w}^2 + a_3\mathbf{w}\right). \tag{130}$$

To apply inversion formula (C.2), one gets first the second-order tensors $\mathbb{A}_0^{-1}{:}\mathbf{B}_1{=}1/a_6\,\bar{\mathbf{w}}^2$, $\mathbf{C}_1{:}\mathbb{A}_0^{-1}{=}a_{92}/a_6\,\mathbf{I}{+}a_{72}/a_6\,\bar{\mathbf{w}}$, $\mathbb{A}_0^{-1}{:}\mathbf{B}_2{=}1/a_6\,\bar{\mathbf{w}}$, $\mathbf{C}_2{:}\mathbb{A}_0^{-1}{=}a_{82}/a_6\,\mathbf{I}{+}a_3/a_6\,\bar{\mathbf{w}}$, and the scalar factors $\mathbf{C}_1{:}\mathbb{A}_0^{-1}{:}\mathbf{B}_1{=}(3a_{92}{+}a_{72}\mathrm{tr}\,\bar{\mathbf{w}})/a_6$, $\mathbf{C}_2{:}\mathbb{A}_0^{-1}{:}\mathbf{B}_2{=}(3a_3{+}a_{82}\mathrm{tr}\,\mathbf{w})/a_6$, $\mathbf{C}_1{:}\mathbb{A}_0^{-1}{:}\mathbf{B}_2{=}(3a_{72}{+}a_{92}\mathrm{tr}\,\mathbf{w})/a_6$, $\mathbf{C}_2{:}\mathbb{A}_0^{-1}{:}\mathbf{B}_1{=}(3a_{82}{+}a_3\mathrm{tr}\,\bar{\mathbf{w}})/a_6$. This leads to evaluate the denominator factor D_2 in formula (C.2) as $D_2{=}d_2/{a_6}^2$, where d_2 is the denominator factor already given in eqns (68a,c) and, finally, to recover the expressions of the dual coefficients $\bar{a}_3, \bar{a}_{91}, \bar{a}_{71}, \bar{a}_{81}$ given in eqns (67)-(68). An equivalent way to operate could consider the damage-effect tensor (66a) as obtained by the two updates $\mathbb{A}_2{=}\mathbb{A}_0{+}(a_{72}\mathbf{I}{+}a_3\mathbf{w}){\otimes}\mathbf{w}{+}(a_{92}\mathbf{I}{+}a_{82}\mathbf{w}){\otimes}\mathbf{w}^2$. Following the same steps as before, one would be led to the same relation $D_2{=}d_2/a_6{}^2$, with d_2 now given in eqns (68b,c), and to the same expressions of the coefficients. Notice that the inversion operation generates just the complementary terms, so that the solution succeeds without any constraint on the coefficients. Similar procedures could be applied as well to derive 'twin' *Solutions (14)-(16)*.

In exploring now single rank-one updates of the identity term $\mathbb{A}_0{=}a_2\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}$ as candidate solutions, i.e. $\mathbb{A}_1{=}a_2\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}{+}a_x\,\mathbf{X}{\otimes}\mathbf{Y}$, Sherman-Morrison's inversion formula (C.1) immediately reveals that a complementary structure is achieved by taking $\mathbf{Y}{=}\bar{\mathbf{X}}{=}\mathbf{X}^{\text{-}1}$, i.e. that any tensor pair of the form

$$\mathbb{A}_1 = a_2\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I} + a_x\,\mathbf{X}\otimes\bar{\mathbf{X}}\;;\qquad \bar{\mathbb{A}}_1 = \frac{1}{a_2}\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I} + \bar{a}_x\,\mathbf{X}\otimes\bar{\mathbf{X}} \tag{131}$$

does form a complementary inverse pair, with $\bar{a}_x{=}{-}a_x/(a_2(3a_x{+}a_2))$. The same holds as well for the 'twin' forms $\mathbb{A}'_1{=}a_2\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}{+}a'_x\,\bar{\mathbf{X}}{\otimes}\mathbf{X}$, $\bar{\mathbb{A}}'_1{=}1/a_2\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}{+}\bar{a}'_x\,\bar{\mathbf{X}}{\otimes}\mathbf{X}$, where again $\bar{a}'_x{=}{-}a'_x/(a_2(3a'_x{+}a_2))$. Besides of course the isotropic case, *Solution (0.1)*, this leads to consider the possible 'twin' instances $\mathbb{A}{=}a_2\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}{+}\alpha\,\mathbf{w}{\otimes}\bar{\mathbf{w}}$, $\mathbb{A}'{=}a_2\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}{+}\alpha'\,\bar{\mathbf{w}}{\otimes}\mathbf{w}$ based on $\mathbf{X}{=}\mathbf{w}$, and $\mathbb{A}{=}a_2\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}{+}\beta\,\mathbf{w}^2{\otimes}\bar{\mathbf{w}}^2$, $\mathbb{A}'{=}a_2\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}{+}\beta'\,\bar{\mathbf{w}}^2{\otimes}\mathbf{w}^2$, based on $\mathbf{X}{=}\mathbf{w}^2$. Through the application of the Cayley-Hamilton theorem applied to $\bar{\mathbf{w}}$, these considerations direct the solution search towards non-symmetric *Solutions (17)-(20)*, which in turn conduct immediately to symmetric *Solutions (21)-(24)*.

Finally, the Sherman-Morrison's approach could be further applied in the present context to explore additional candidate solutions interpreted as subsequent rank-one updates of known ones. However, it should be said that this procedure can be handled conveniently by hand only until when the 'pivoting' damage-effect tensor used in the inversion operation possesses an inverse with conveniently simple form.

6 Conclusions

A set of dual orthotropic fourth-order damage-effect tensors possessing complementary structures has been derived. It includes *twenty-six* new solution instances, *fifteen* of which are *symmetric* (*Solutions (1)-(3), (5)-(10), (21)-(26)*), and *eleven non-symmetric* (*Solutions (4), (11)-(20)*). Other solution cases have not been reported in the paper since they are expressed by relations even more involved than some of those presented here. Most of the obtained solutions are particular cases of others and work *through constraints on the coefficients*. *Six* of the solutions presented here succeed in reaching the complementary structure *without constraints on the coefficients*: *three* are *symmetric* (*Solutions (1), (7), (9)*), and *three non-symmetric* (*Solutions (4), (13), (16)*). 'Shear-like' generators *Solutions (s.a), (s.b), (s.c)* (Section 4.3) have led respectively to: *eight Solutions (17)-(24)*, *fourteen Solutions (1), (4)-(16)*, *four Solutions (2)-(3), (25)-(26)*. All the solutions *without constraints on the coefficients* belong to 'shear-like' generator *Solution (s.b)*.

Although the complete enumeration of all convenient particular cases of the general solution of the problem at hand does not appear easy to be handled, the solution instances provided in this paper furnish new interesting orthotropic forms of damage-effect tensors that satisfy the duality requirement. The latter tensors lead to damaged compliance and stiffness embedding less restricted forms of orthotropic material symmetry than that of Valanis-type, with the simultaneous compromise, from an algebraic point of view, of keeping the formulation at a reasonable degree of complexity.

While the present study generally addresses the algebraic properties of complementary fourth-order tensor inverses, the ultimate convenience of using any of the damage-effect tensors solutions advanced here in the final development and implementation of a constitutive model of orthotropic elastic damage remains to be explored on analytical and numerical grounds and, obviously, and most important, validated on its physical significance.

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Appendix A: Representations of orthotropic tensors

Consider the orthotropic non-symmetric representation (6) of fourth-order damage-effect tensor $\mathbb{A}$ as an isotropic tensor-valued function of symmetric second-order damage tensor $\mathbf{w}$:

$$\begin{aligned}\mathbb{A} = a_1\,\mathbf{I}\otimes\mathbf{I} + a_2\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I} + a_3\,\mathbf{w}\otimes\mathbf{w} + a_4\,(\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{I} + \mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{w}) + a_5\,\mathbf{w}^2\otimes\mathbf{w}^2 + a_6\,\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w}\\ + a_{71}\,\mathbf{w}\otimes\mathbf{I} + a_{72}\,\mathbf{I}\otimes\mathbf{w} + a_{81}\,\mathbf{w}^2\otimes\mathbf{w} + a_{82}\,\mathbf{w}\otimes\mathbf{w}^2 + a_{91}\,\mathbf{w}^2\otimes\mathbf{I} + a_{92}\,\mathbf{I}\otimes\mathbf{w}^2 .\end{aligned} \tag{A.1}$$

Following Zysset and Curnier (1995), account for the spectral decomposition of tensor $\mathbf{w}$: $\mathbf{w}=w_1\mathbf{O}_1+w_2\mathbf{O}_2+w_3\mathbf{O}_3$; $\mathbf{O}_i=\mathbf{o}_i\otimes\mathbf{o}_i$, where w_i and $\mathbf{o}_i$, i=1-3, are the eigenvalues and unit orthonormal eigenvectors of $\mathbf{w}$. The principal directions defined by $\mathbf{o}_i$ play the role of orthotropic damage axes and $\mathbf{O}_i$ that of the relevant structural orthotropic second-order tensors. Notice that $\mathbf{O}_1+\mathbf{O}_2+\mathbf{O}_3=\mathbf{I}$. Also $\mathbf{O}_i\,\underline{\overline{\otimes}}\,\mathbf{O}_i=\mathbf{O}_i\otimes\mathbf{O}_i$, i=1-3. Fourth-order tensors $\mathbf{O}_i\otimes\mathbf{O}_i$, i=1-3, and idempotent tensors $(\mathbf{O}_i\,\underline{\overline{\otimes}}\,\mathbf{O}_j+\mathbf{O}_j\,\underline{\overline{\otimes}}\,\mathbf{O}_i)$, $i<j$=1-3, provide a decomposition of the symmetric identity (Walpole, 1984):

$$\mathbb{I}^s = \mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I} = \sum_{i=1}^{3}\mathbf{O}_i\otimes\mathbf{O}_i + \sum_{i<j=1}^{3}(\mathbf{O}_i\,\underline{\overline{\otimes}}\,\mathbf{O}_j + \mathbf{O}_j\,\underline{\overline{\otimes}}\,\mathbf{O}_i) . \tag{A.2}$$

Then, the damage-effect tensor (A.1) admits the following general orthotropic representation (see also Walpole, 1984):

$$\mathbb{A} = \sum_{i\neq j=1}^{3} l_{ij}\,\mathbf{O}_i\otimes\mathbf{O}_j + \sum_{i<j=1}^{3} 2m_{ij}\,(\mathbf{O}_i\,\underline{\overline{\otimes}}\,\mathbf{O}_j + \mathbf{O}_j\,\underline{\overline{\otimes}}\,\mathbf{O}_i) + \sum_{i=1}^{3}(l_{ii}+2m_{ii})\,\mathbf{O}_i\otimes\mathbf{O}_i , \tag{A.3}$$

where the nine 'Lamé's-like' parameters l_{ij}, i,j=1-3 and six 'shear-like' parameters m_{ij}, $i\leq j$=1-3 are given in terms of coefficients a_i, i=1-6; a_{i1}, a_{i2}, i=7-9, and principal values w_i, i=1-3, as follows:

$$\begin{aligned} l_{ij} &= a_1 + a_3\,w_i w_j + a_5\,w_i^2 w_j^2\\ &\quad + a_{71}\,w_i + a_{72}\,w_j + a_{81}\,w_i^2 w_j + a_{82}\,w_i w_j^2 + a_{91}\,w_i^2 + a_{92}\,w_j^2 , \quad i,j = 1\text{-}3 ;\\ 2m_{ij} &= a_2 + a_4(w_i + w_j) + a_6 w_i w_j , \quad i \leq j = 1\text{-}3 . \end{aligned} \tag{A.4}$$

Notice that the three parameters m_{11}, m_{22}, m_{33} are not independent from the other three 'shear-like' coefficients m_{12}, m_{23}, m_{13}, since they are related by:

$$\begin{aligned}(w_2-w_3)\,m_{11} &= (w_1-w_3)\,m_{12} - (w_1-w_2)\,m_{13} ,\\ (w_3-w_1)\,m_{22} &= (w_2-w_1)\,m_{23} - (w_2-w_3)\,m_{12} ,\\ (w_1-w_2)\,m_{33} &= (w_3-w_2)\,m_{13} - (w_3-w_1)\,m_{23} .\end{aligned} \tag{A.5}$$

So, the set of twelve independent parameters in eqn (A.3) may be identified by the nine l_{ij}, i,j=1-3, and the three m_{ij}, $i<j$=1-3.

The twelve non-zero components of $\mathbb{A}$ in the principal axes of damage become:

$$\begin{aligned} A_{iiii} &= l_{ii} + 2m_{ii} = a_1 + a_2 + (2a_4 + a_{71} + a_{72})\,w_i + (a_3 + a_6 + a_{91} + a_{92})\,w_i^2\\ &\qquad + (a_{81} + a_{82})\,w_i^3 + a_5\,w_i^4 , \quad i = 1\text{-}3 .\\ A_{iijj} &= l_{ij} , \quad i \neq j = 1\text{-}3 ; \qquad A_{ijij} = m_{ij} , \quad i < j = 1\text{-}3 . \end{aligned} \tag{A.6}$$

These non-zero components may be depicted in the 6×6 matrix representation introduced and discussed by Walpole (Walpole, 1984, eqn (45)), in which diagonal entries are represented by the diagonal matrix diag$\{A_{1111}, A_{2222}, A_{3333},\ 2A_{1212}, 2A_{2323}, 2A_{1313}\}$ and the fourth-order symmetric identity $\mathbb{I}^s = \mathbf{I} \underline{\overline{\otimes}} \mathbf{I}$ maps to a 6×6 identity matrix:

$$[\mathbb{A}] = \begin{bmatrix} l_{11}+2m_{11} & l_{12} & l_{13} & & & \\ l_{21} & l_{22}+2m_{22} & l_{23} & & & \\ l_{31} & l_{32} & l_{33}+2m_{33} & & & \\ & & & 2m_{12} & & \\ & & & & 2m_{23} & \\ & & & & & 2m_{13} \end{bmatrix} = \begin{bmatrix} \underset{3\times 3}{\mathbf{L}} & \\ & \underset{3\times 3}{2\mathbf{M}} \end{bmatrix} . \quad \text{(A.7)}$$

This matrix may be seen as the composition of an upper-left 3×3 submatrix $\mathbf{l}$ with 'Lamé's-like' entries l_{ij}, i,j=1-3 (independent of coefficients a_2, a_4, a_6) and of a diagonal 6×6 matrix $2\mathbf{m}$=diag$\{2m_{11}, 2m_{22}, 2m_{33},\ 2m_{12}, 2m_{23}, 2m_{13}\}$ with 'shear-like' entries $2m_{ij}$, $i \leq j$=1-3 (dependent only on coefficients a_2, a_4, a_6). The three upper diagonal entries (A.6a,b) depend on all coefficients a_i, i=1-6; a_{i1}, a_{i2}, i=7-9. Alternatively, this originates the $\mathbf{L}$-2$\mathbf{M}$ decomposition in eqn (A.7b), where $\mathbf{L}$ is the upper-left 3×3 submatrix of $[\mathbb{A}]$ and 2$\mathbf{M}$=diag$\{2m_{12}, 2m_{23}, 2m_{13}\}$ the lower-right 3×3 diagonal submatrix of $[\mathbb{A}]$. The algebraic decomposition of fourth-order tensor $\mathbb{A}$ in terms of the symbolic $\mathbf{L}$-2$\mathbf{M}$ representation has been discussed by Walpole (1984).

When the eigenvalues w_1, w_2, w_3 are distinct, the nine entries l_{ij}, i,j=1-3, are in one-to-one correspondence with the nine coefficients $a_1, a_3, a_5,\ a_{71}, a_{72},\ a_{81}, a_{82},\ a_{91}, a_{92}$, eqn (A.4a), while the three coefficients $2m_{ij}$, $i<j$=1-3, are in one-to-one relation with the three coefficients a_2, a_4, a_6, eqn (A.4b). Posing for compactness

$$^{\mathrm{w}}d = (w_1 - w_2)\,(w_2 - w_3)\,(w_3 - w_1)\ , \tag{A.8}$$

the first set of nine relations inverses to (A.4a) in terms of l_{ij}, i,j=1-3, is given by:

$$\begin{aligned} a_1\,{}^{\mathrm{w}}d^2 &= w_2^2 w_3^2 (w_2 - w_3)^2 l_{11} + w_1^2 w_3^2 (w_1 - w_3)^2 l_{22} + w_1^2 w_2^2 (w_1 - w_2)^2 l_{33} \\ &+ w_1 w_2 w_3^2 (w_2 - w_3)(w_3 - w_1)(l_{12} + l_{21}) \\ &+ w_1 w_2^2 w_3 (w_2 - w_3)(w_1 - w_2)(l_{13} + l_{31}) \\ &+ w_1^2 w_2 w_3 (w_3 - w_1)(w_1 - w_2)(l_{23} + l_{32})\ ; \\ a_3\,{}^{\mathrm{w}}d^2 &= (w_2^2 - w_3^2)^2 l_{11} + (w_1^2 - w_3^2)^2 l_{22} + (w_1^2 - w_2^2)^2 l_{33} \\ &+ (w_2^2 - w_3^2)(w_3^2 - w_1^2)(l_{12} + l_{21}) \\ &+ (w_2^2 - w_3^2)(w_1^2 - w_2^2)(l_{13} + l_{31}) \\ &+ (w_3^2 - w_1^2)(w_1^2 - w_2^2)(l_{23} + l_{32})\ ; \\ a_5\,{}^{\mathrm{w}}d^2 &= (w_2 - w_3)^2 l_{11} + (w_1 - w_3)^2 l_{22} + (w_1 - w_2)^2 l_{33} \\ &+ (w_2 - w_3)(w_3 - w_1)(l_{12} + l_{21}) \\ &+ (w_2 - w_3)(w_1 - w_2)(l_{13} + l_{31}) \\ &+ (w_3 - w_1)(w_1 - w_2)(l_{23} + l_{32})\ ; \end{aligned} \tag{A.9}$$

$$
\begin{aligned}
-a_{71}{}^{\mathsf{w}}d^2 &= w_2w_3(w_2+w_3)(w_2-w_3)^2 l_{11} + w_1w_3(w_3+w_1)(w_3-w_1)^2 l_{22} \\
&+ w_1w_2(w_1+w_2)(w_1-w_2)^2 l_{33} \\
&+ w_1w_3(w_3-w_1)(w_2^2-w_3^2) l_{12} + w_2w_3(w_2-w_3)(w_3^2-w_1^2) l_{21} \\
&+ w_1w_2(w_1-w_2)(w_2^2-w_3^2) l_{13} + w_2w_3(w_2-w_3)(w_1^2-w_2^2) l_{31} \\
&+ w_1w_2(w_1-w_2)(w_3^2-w_1^2) l_{23} + w_1w_3(w_3-w_1)(w_1^2-w_2^2) l_{32}\ ; \\
-a_{72}{}^{\mathsf{w}}d^2 &= w_2w_3(w_2+w_3)(w_2-w_3)^2 l_{11} + w_1w_3(w_3+w_1)(w_3-w_1)^2 l_{22} \\
&+ w_1w_2(w_1+w_2)(w_1-w_2)^2 l_{33} \\
&+ w_2w_3(w_2-w_3)(w_3^2-w_1^2) l_{12} + w_1w_3(w_3-w_1)(w_2^2-w_3^2) l_{21} \\
&+ w_2w_3(w_2-w_3)(w_1^2-w_2^2) l_{13} + w_1w_2(w_1-w_2)(w_2^2-w_3^2) l_{31} \\
&+ w_1w_3(w_3-w_1)(w_1^2-w_2^2) l_{23} + w_1w_2(w_1-w_2)(w_3^2-w_1^2) l_{32}\ ;
\end{aligned}
\tag{A.10}
$$

$$
\begin{aligned}
-a_{81}{}^{\mathsf{w}}d^2 &= (w_2+w_3)(w_2-w_3)^2 l_{11} + (w_3+w_1)(w_3-w_1)^2 l_{22} \\
&+ (w_1+w_2)(w_1-w_2)^2 l_{33} \\
&+ (w_2-w_3)(w_3^2-w_1^2) l_{12} + (w_3-w_1)(w_2^2-w_3^2) l_{21} \\
&+ (w_2-w_3)(w_1^2-w_2^2) l_{13} + (w_1-w_2)(w_2^2-w_3^2) l_{31} \\
&+ (w_3-w_1)(w_1^2-w_2^2) l_{23} + (w_1-w_2)(w_3^2-w_1^2) l_{32}\ ; \\
-a_{82}{}^{\mathsf{w}}d^2 &= (w_2+w_3)(w_2-w_3)^2 l_{11} + (w_3+w_1)(w_3-w_1)^2 l_{22} \\
&+ (w_1+w_2)(w_1-w_2)^2 l_{33} \\
&+ (w_3-w_1)(w_2^2-w_3^2) l_{12} + (w_2-w_3)(w_3^2-w_1^2) l_{21} \\
&+ (w_1-w_2)(w_2^2-w_3^2) l_{13} + (w_2-w_3)(w_1^2-w_2^2) l_{31} \\
&+ (w_1-w_2)(w_3^2-w_1^2) l_{23} + (w_3-w_1)(w_1^2-w_2^2) l_{32}\ ;
\end{aligned}
\tag{A.11}
$$

$$
\begin{aligned}
a_{91}{}^{\mathsf{w}}d^2 &= w_2w_3(w_2-w_3)^2 l_{11} + w_3w_1(w_3-w_1)^2 l_{22} + w_1w_2(w_1-w_2)^2 l_{33} \\
&+ w_1w_3(w_2-w_3)(w_3-w_1) l_{12} + w_2w_3(w_2-w_3)(w_3-w_1) l_{21} \\
&+ w_1w_2(w_2-w_3)(w_1-w_2) l_{13} + w_2w_3(w_2-w_3)(w_1-w_2) l_{31} \\
&+ w_1w_2(w_3-w_1)(w_1-w_2) l_{23} + w_1w_3(w_3-w_1)(w_1-w_2) l_{32}\ ; \\
a_{92}{}^{\mathsf{w}}d^2 &= w_2w_3(w_2-w_3)^2 l_{11} + w_3w_1(w_3-w_1)^2 l_{22} + w_1w_2(w_1-w_2)^2 l_{33} \\
&+ w_2w_3(w_2-w_3)(w_3-w_1) l_{12} + w_1w_3(w_2-w_3)(w_3-w_1) l_{21} \\
&+ w_2w_3(w_2-w_3)(w_1-w_2) l_{13} + w_1w_2(w_2-w_3)(w_1-w_2) l_{31} \\
&+ w_1w_3(w_3-w_1)(w_1-w_2) l_{23} + w_1w_2(w_3-w_1)(w_1-w_2) l_{32}\ ,
\end{aligned}
\tag{A.12}
$$

while the second set of three relations inverses to (A.4b) in terms of m_{ij}, $i<j$=1-3, is given by:

$$
\begin{aligned}
-a_2{}^{\mathsf{w}}d &= w_3^2(w_1-w_2)2m_{12} + w_1^2(w_2-w_3)2m_{23} + w_2^2(w_3-w_1)2m_{13}\ , \\
a_4{}^{\mathsf{w}}d &= w_3(w_1-w_2)2m_{12} + w_1(w_2-w_3)2m_{23} + w_2(w_3-w_1)2m_{13}\ , \\
-a_6{}^{\mathsf{w}}d &= \quad (w_1-w_2)2m_{12} + \quad (w_2-w_3)2m_{23} + \quad (w_3-w_1)2m_{13}\ .
\end{aligned}
\tag{A.13}
$$

Representations (A.1), (A.3) may be compared to the typical orthotropic representation of a non-symmetric fourth-order tensor in structural axes $\mathbf{o}_i$ in terms of twelve coefficients l_{ij}, $i{\neq}j$=1-3; l_i, m_i, i=1-3 (see, for symmetric tensors, Boehler, 1987, eqn (30),

p. 62, and Bigoni and Loret, 1999, eqns (A.1)-(A.2); Curnier et al., 1995, eqn (2.27)):

$$\mathbb{A} = \sum_{i\neq j=1}^{3} l_{ij}\, \mathbf{O}_i \otimes \mathbf{O}_j + \sum_{i=1}^{3} m_i\, (\mathbf{O}_i \,\underline{\overline{\otimes}}\, \mathbf{I} + \mathbf{I} \,\underline{\overline{\otimes}}\, \mathbf{O}_i) + l_i\, \mathbf{O}_i \otimes \mathbf{O}_i\ . \tag{A.14}$$

Comparison with eqns (A.1), (A.3) confirms the orthotropic character of the former representations and shows that the six coefficients l_{ij}, $i\neq j$=1-3 are given as before, eqn (A.4a), while parameters m_i are related to m_{ij}, $i<j$=1-3, by $2m_{ij}=m_i+m_j$, or conversely

$$\begin{aligned}
2m_1 &= 2(m_{12}+m_{13}-m_{23}) = a_2 + 2a_4 w_1 + a_6(w_1w_2 + w_1w_3 - w_2w_3)\ ,\\
2m_2 &= 2(m_{12}+m_{23}-m_{13}) = a_2 + 2a_4 w_2 + a_6(w_1w_2 + w_2w_3 - w_1w_3)\ ,\\
2m_3 &= 2(m_{13}+m_{23}-m_{12}) = a_2 + 2a_4 w_3 + a_6(w_1w_3 + w_2w_3 - w_1w_2)\ ,
\end{aligned} \tag{A.15}$$

and the three parameters l_i are linked to the three l_{ii}, i=1-3, as $l_i+2m_i=l_{ii}+2m_{ii}$, namely:

$$\begin{aligned}
l_1 &= l_{11} + 2(m_{11}-m_1) = a_1 + (a_{71}+a_{72})\, w_1 + (a_3+a_6+a_{91}+a_{92})\, w_1^2\\
&\qquad + (a_{81}+a_{82})\, w_1^3 + a_5\, w_1^4 - a_6\,(w_1w_2+w_1w_3-w_2w_3)\ ,\\
l_2 &= l_{22} + 2(m_{22}-m_2) = a_1 + (a_{71}+a_{72})\, w_2 + (a_3+a_6+a_{91}+a_{92})\, w_2^2\\
&\qquad + (a_{81}+a_{82})\, w_2^3 + a_5\, w_2^4 - a_6\,(w_1w_2+w_2w_3-w_1w_3)\ ,\\
l_3 &= l_{33} + 2(m_{33}-m_3) = a_1 + (a_{71}+a_{72})\, w_3 + (a_3+a_6+a_{91}+a_{92})\, w_3^2\\
&\qquad + (a_{81}+a_{82})\, w_3^3 + a_5\, w_3^4 - a_6\,(w_1w_3+w_2w_3-w_1w_2)\ .
\end{aligned} \tag{A.16}$$

Notice that both m_i and l_i depend on all principal values of $\mathbf{w}$, while m_{ii} and l_{ii} depend only on the direct eigenvalue w_i. Indeed $m_i\neq m_{ii}$ and $l_i\neq l_{ii}$ precisely differ accordingly: $l_i-l_{ii}=2(m_i-m_{ii})$.

Appendix B: Rivlin's tensorial identities

The following Rivlin's tensorial identities involving a *symmetric* second-order tensor $\mathbf{w}$ and the identity $\mathbf{I}$ in a three-dimensional inner product space are here collected:

$$\begin{aligned}
(\mathbf{w}^2 \,\underline{\overline{\otimes}}\, \mathbf{I} + \mathbf{I} \,\underline{\overline{\otimes}}\, \mathbf{w}^2) &= (\mathbf{w}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{w}^2) - (\mathbf{w} \,\underline{\overline{\otimes}}\, \mathbf{w} - \mathbf{w} \otimes \mathbf{w}) - {}^{\mathbf{w}}I_2\, (\mathbf{I} \,\underline{\overline{\otimes}}\, \mathbf{I} - \mathbf{I} \otimes \mathbf{I})\\
&\quad + {}^{\mathbf{w}}I_1 \Big((\mathbf{w} \,\underline{\overline{\otimes}}\, \mathbf{I} + \mathbf{I} \,\underline{\overline{\otimes}}\, \mathbf{w}) - (\mathbf{w} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{w}) \Big)\ ;
\end{aligned} \tag{B.1}$$

$$\begin{aligned}
(\mathbf{w}^2 \,\underline{\overline{\otimes}}\, \mathbf{w} + \mathbf{w} \,\underline{\overline{\otimes}}\, \mathbf{w}^2) &= (\mathbf{w}^2 \otimes \mathbf{w} + \mathbf{w} \otimes \mathbf{w}^2) + {}^{\mathbf{w}}I_1\, (\mathbf{w} \,\underline{\overline{\otimes}}\, \mathbf{w} - \mathbf{w} \otimes \mathbf{w})\\
&\quad - {}^{\mathbf{w}}I_3\, (\mathbf{I} \,\underline{\overline{\otimes}}\, \mathbf{I} - \mathbf{I} \otimes \mathbf{I})\ ;
\end{aligned} \tag{B.2}$$

$$\begin{aligned}
\mathbf{w}^2 \,\underline{\overline{\otimes}}\, \mathbf{w}^2 &= \mathbf{w}^2 \otimes \mathbf{w}^2 + {}^{\mathbf{w}}I_2\, (\mathbf{w} \,\underline{\overline{\otimes}}\, \mathbf{w} - \mathbf{w} \otimes \mathbf{w})\\
&\quad - {}^{\mathbf{w}}I_3 \Big((\mathbf{w} \,\underline{\overline{\otimes}}\, \mathbf{I} + \mathbf{I} \,\underline{\overline{\otimes}}\, \mathbf{w}) - (\mathbf{w} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{w}) \Big)\ ,
\end{aligned} \tag{B.3}$$

where ${}^{\mathbf{w}}I_1$, ${}^{\mathbf{w}}I_2$, ${}^{\mathbf{w}}I_3$ are the three principal invariants of $\mathbf{w}$, ${}^{\mathbf{w}}I_1=\mathrm{tr}\,\mathbf{w}$, ${}^{\mathbf{w}}I_2=(\mathrm{tr}^2\,\mathbf{w}-\mathrm{tr}\,\mathbf{w}^2)/2$, ${}^{\mathbf{w}}I_3=\det\mathbf{w}=\mathrm{tr}\,\mathbf{w}^3/3+\mathrm{tr}^3\,\mathbf{w}/6-\mathrm{tr}\,\mathbf{w}\ \mathrm{tr}\,\mathbf{w}^2/2$ entering the Cayley-Hamilton theorem applied to $\mathbf{w}$, namely $\mathbf{w}^3-{}^{\mathbf{w}}I_1\,\mathbf{w}^2+{}^{\mathbf{w}}I_2\,\mathbf{w}-{}^{\mathbf{w}}I_3\,\mathbf{I}=\mathbf{0}$.

Eqns (B.1), (B.2) and (B.3) can be extracted, respectively, from 3×3 matrix identities (4.22), (4.25) and (4.26) in Rivlin (1955), where eqn (4.22) is a generalization of the Cayley-Hamilton theorem and eqns (4.25), (4.26) are derived by repeated application of the same theorem. Eqns (B.1), (B.2) and (B.3) were listed by Rosati (2000) in similar form in his eqns (31), (30) and (34), respectively (here the symmetrized dyadic product is just used). Eqn (B.1) was also given by Lam and Zhang (1995) in their eqn (3.14).

Appendix C: Sherman and Morrison's formulas

The relations of the inverses of one and two rank-one updates of a given non-singular fourth-order tensor are here given. They can be extracted from the formula of Sherman and Morrison (1950), which provides the inverse of a square matrix modified just in one of its entries. Additional information on the spectral properties of multiple rank-one updates is available in Rizzi et al. (1996), Appendix A.

Given non-singular fourth-order tensor $\mathbb{A}_0$, second-order tensors $\mathbf{B}_1$, $\mathbf{C}_1$ and scalar a_1, the following relation holds:

$$\begin{aligned} &\mathbb{A}_1^{-1} = (\mathbb{A}_0 + a_1\, \mathbf{B}_1 \otimes \mathbf{C}_1)^{-1} = \mathbb{A}_0^{-1} - \frac{a_1}{D_1}\, \mathbb{A}_0^{-1} : \mathbf{B}_1 \otimes \mathbf{C}_1 : \mathbb{A}_0^{-1}\,, \\ &\text{provided that} \qquad D_1 = \frac{\det \mathbb{A}_1}{\det \mathbb{A}_0} = 1 + a_1\, \mathbf{C}_1 : \mathbb{A}_0^{-1} : \mathbf{B}_1 \neq 0\,. \end{aligned} \tag{C.1}$$

A further application of eqn (C.1), given additional second-order tensors $\mathbf{B}_2$, $\mathbf{C}_2$ and scalar a_2 yields the inverse of two rank-one updates of $\mathbb{A}_0$:

$$\begin{aligned} \mathbb{A}_2^{-1} &= (\mathbb{A}_0 + a_1\, \mathbf{B}_1 \otimes \mathbf{C}_1 + a_2\, \mathbf{B}_2 \otimes \mathbf{C}_2)^{-1} \\ &= \mathbb{A}_0^{-1} - \frac{a_1(1 + a_2\, \mathbf{C}_2 : \mathbb{A}_0^{-1} : \mathbf{B}_2)}{D_2}\, \mathbb{A}_0^{-1} : \mathbf{B}_1 \otimes \mathbf{C}_1 : \mathbb{A}_0^{-1} \\ &\quad - \frac{a_2(1 + a_1\, \mathbf{C}_1 : \mathbb{A}_0^{-1} : \mathbf{B}_1)}{D_2}\, \mathbb{A}_0^{-1} : \mathbf{B}_2 \otimes \mathbf{C}_2 : \mathbb{A}_0^{-1} \\ &\quad + \frac{a_1\, a_2(\mathbf{C}_1 : \mathbb{A}_0^{-1} : \mathbf{B}_2)}{D_2}\, \mathbb{A}_0^{-1} : \mathbf{B}_1 \otimes \mathbf{C}_2 : \mathbb{A}_0^{-1} \\ &\quad + \frac{a_1\, a_2(\mathbf{C}_2 : \mathbb{A}_0^{-1} : \mathbf{B}_1)}{D_2}\, \mathbb{A}_0^{-1} : \mathbf{B}_2 \otimes \mathbf{C}_1 : \mathbb{A}_0^{-1}\,, \\ \text{provided that} \qquad D_2 &= \frac{\det \mathbb{A}_2}{\det \mathbb{A}_0} = (1 + a_1\, \mathbf{C}_1 : \mathbb{A}_0^{-1} : \mathbf{B}_1)(1 + a_2\, \mathbf{C}_2 : \mathbb{A}_0^{-1} : \mathbf{B}_2) \\ &\qquad - a_1\, a_2\, (\mathbf{C}_1 : \mathbb{A}_0^{-1} : \mathbf{B}_2)(\mathbf{C}_2 : \mathbb{A}_0^{-1} : \mathbf{B}_1) \neq 0\,. \end{aligned} \tag{C.2}$$

Eqn (C.2) could also be rewritten in terms of only two rank-one updates of $\mathbb{A}_0^{-1}$ by collecting terms with either common factors $\mathbf{B}_1$, $\mathbf{B}_2$ or $\mathbf{C}_1$, $\mathbf{C}_2$.

Appendix D: Multiplication table of orthotropic tensor addends

	$(\bar{a}_3)$ $\bar{\mathbf{w}} \otimes \bar{\mathbf{w}}$	$(\bar{a}_6)$ $\bar{\mathbf{w}} \,\overline{\underline{\otimes}}\, \bar{\mathbf{w}}$	$(\bar{a}_{91})$ $\bar{\mathbf{w}}^2 \otimes \mathbf{I}$	$(\bar{a}_{71})$ $\bar{\mathbf{w}} \otimes \mathbf{I}$	$(\bar{a}_{81})$ $\bar{\mathbf{w}}^2 \otimes \bar{\mathbf{w}}$
(a_3) $\mathbf{w} \otimes \mathbf{w}$	3 $\mathbf{w} \otimes \bar{\mathbf{w}}$	$\mathbf{w} \otimes \bar{\mathbf{w}}$	$\mathrm{tr}\,\bar{\mathbf{w}}$ $\mathbf{w} \otimes \mathbf{I}$	3 $\mathbf{w} \otimes \mathbf{I}$	$\mathrm{tr}\,\bar{\mathbf{w}}$ $\mathbf{w} \otimes \bar{\mathbf{w}}$
(a_6) $\mathbf{w} \,\overline{\underline{\otimes}}\, \mathbf{w}$	$\mathbf{w} \otimes \bar{\mathbf{w}}$	$\mathbf{I} \,\overline{\underline{\otimes}}\, \mathbf{I}$	$\mathbf{I} \otimes \mathbf{I}$	$\mathbf{w} \otimes \mathbf{I}$	$\mathbf{I} \otimes \bar{\mathbf{w}}$
(a_{92}) $\mathbf{I} \otimes \mathbf{w}^2$	$\mathrm{tr}\,\mathbf{w}$ $\mathbf{I} \otimes \bar{\mathbf{w}}$	$\mathbf{I} \otimes \mathbf{I}$	3 $\mathbf{I} \otimes \mathbf{I}$	$\mathrm{tr}\,\mathbf{w}$ $\mathbf{I} \otimes \mathbf{I}$	3 $\mathbf{I} \otimes \bar{\mathbf{w}}$
(a_{72}) $\mathbf{I} \otimes \mathbf{w}$	3 $\mathbf{I} \otimes \bar{\mathbf{w}}$	$\mathbf{I} \otimes \bar{\mathbf{w}}$	$\mathrm{tr}\,\bar{\mathbf{w}}$ $\mathbf{I} \otimes \mathbf{I}$	3 $\mathbf{I} \otimes \mathbf{I}$	$\mathrm{tr}\,\bar{\mathbf{w}}$ $\mathbf{I} \otimes \bar{\mathbf{w}}$
(a_{82}) $\mathbf{w} \otimes \mathbf{w}^2$	$\mathrm{tr}\,\mathbf{w}$ $\mathbf{w} \otimes \bar{\mathbf{w}}$	$\mathbf{w} \otimes \mathbf{I}$	3 $\mathbf{w} \otimes \mathbf{I}$	$\mathrm{tr}\,\mathbf{w}$ $\mathbf{w} \otimes \mathbf{I}$	3 $\mathbf{w} \otimes \bar{\mathbf{w}}$

Table 1: Multiplication table of tensor terms belonging to the solution set. Notice the 'complementary symmetry' of the table entries with respect to the table diagonal. Also, note that the five tensor terms $\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I}$, $\mathbf{w}\otimes\bar{\mathbf{w}}$, $\mathbf{I}\otimes\mathbf{I}$, $\mathbf{w}\otimes\mathbf{I}$, $\mathbf{I}\otimes\bar{\mathbf{w}}$ filling the table entries are already generated through the first three coefficients a_3, a_6, a_{92} and $\bar{a}_3, \bar{a}_6, \bar{a}_{91}$. The three frames indicate the instances of the solution set that work without constraints on the coefficients and are based on the complementary pairs: ***(1)*** (a_3, a_6), $(\bar{a}_3, \bar{a}_6)$; ***(0.3)*** (a_6, a_{92}), $(\bar{a}_6, \bar{a}_{91})$; ***(13)*** $(a_3, a_6, a_{92}, a_{72}, a_{82})$, $(\bar{a}_3, \bar{a}_6, \bar{a}_{91}, \bar{a}_{71}, \bar{a}_{81})$. *Solutions (1)* and *(0.3)* are particular cases of *(13)*. Additional 'reciprocal' solutions ***(11)*** $(a_3, a_6, a_{92}, a_{72})$, $(\bar{a}_3, \bar{a}_6, \bar{a}_{91}, \bar{a}_{71})$ and ***(12)*** $(a_3, a_6, a_{92}, a_{82})$, $(\bar{a}_3, \bar{a}_6, \bar{a}_{91}, \bar{a}_{81})$ are also obtained as particular cases of *(13)* through the constraints on the coefficients arising respectively from the conditions a_{82}=0, $\bar{a}_{81}$=0 and a_{72}=0, $\bar{a}_{71}$=0. Four additional 'twin' solution cases can be obtained just by inverting the roles between coefficients with and without bars: ***(4)*** (a_6, a_{91}), $(\bar{a}_6, \bar{a}_{92})$; ***(14)*** $(a_3, a_6, a_{91}, a_{71})$, $(\bar{a}_3, \bar{a}_6, \bar{a}_{92}, \bar{a}_{72})$; ***(15)*** $(a_3, a_6, a_{91}, a_{81})$, $(\bar{a}_3, \bar{a}_6, \bar{a}_{92}, \bar{a}_{82})$; ***(16)*** $(a_3, a_6, a_{91}, a_{71}, a_{81})$, $(\bar{a}_3, \bar{a}_6, \bar{a}_{92}, \bar{a}_{72}, \bar{a}_{82})$. Insertion of a row and column corresponding to $(a_1)\,\mathbf{I}\otimes\mathbf{I}$ and $(\bar{a}_1)\,\mathbf{I}\otimes\mathbf{I}$ would add to the table entries the two new complementary terms $\mathbf{w}^2\otimes\mathbf{I}$ and $\mathbf{I}\otimes\bar{\mathbf{w}}^2$ in correspondence of row (a_6) and column $(\bar{a}_6)$, respectively, so that the solutions indicated here *cannot* be enlarged by adding the $\mathbf{I}\otimes\mathbf{I}$ terms corresponding to the coefficients a_1 and $\bar{a}_1$.

Appendix E: Factors of *Solution (25)*

The expressions of factors $\tilde{n}_3$, n_5, n_8 entering eqn (110) are reported below:

$$
\begin{aligned}
\tilde{n}_3 = {} & a_9^4\, {}^{\mathrm{w}}I_2{}^{4} \Big({}^{\mathrm{w}}I_1{}^{3}\, {}^{\mathrm{w}}I_2{}^{4} - 4\, {}^{\mathrm{w}}I_1{}^{4}\, {}^{\mathrm{w}}I_2{}^{2}\, {}^{\mathrm{w}}I_3 + 4\, {}^{\mathrm{w}}I_1{}^{5}\, {}^{\mathrm{w}}I_3{}^{2} + 9\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2{}^{2}\, {}^{\mathrm{w}}I_3{}^{2} \\
& + 6\, {}^{\mathrm{w}}I_1{}^{2}\, {}^{\mathrm{w}}I_3{}^{3} - 6\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3 \, ({}^{\mathrm{w}}I_2{}^{3} + 3\, {}^{\mathrm{w}}I_3{}^{2}) \Big) \\
& + a_6^2\, {}^{\mathrm{w}}I_3{}^{4} \Big[a_8^2\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2{}^{4} + 2\, a_8\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2{}^{2} \Big(a_5\, {}^{\mathrm{w}}I_2\, ({}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2 - 3\, {}^{\mathrm{w}}I_3) + 2\, a_6\, {}^{\mathrm{w}}I_3 \Big) \\
& + a_5\, a_6\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3\, (4\, {}^{\mathrm{w}}I_1{}^{2}\, {}^{\mathrm{w}}I_2 + 7\, {}^{\mathrm{w}}I_2{}^{2} - 24\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3) - 2\, a_6^2\, {}^{\mathrm{w}}I_3\, ({}^{\mathrm{w}}I_2{}^{2} - 4\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3) \\
& + a_5^2\, {}^{\mathrm{w}}I_2{}^{2}\, ({}^{\mathrm{w}}I_1{}^{3}\, {}^{\mathrm{w}}I_2{}^{2} - 6\, {}^{\mathrm{w}}I_1{}^{2}\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3 - 6\, {}^{\mathrm{w}}I_2{}^{2}\, {}^{\mathrm{w}}I_3 + 18\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3{}^{2}) \Big] \\
& + 2\, a_9^3\, {}^{\mathrm{w}}I_2{}^{3}\, {}^{\mathrm{w}}I_3 \Big[a_8\, {}^{\mathrm{w}}I_2 \Big(-2\, {}^{\mathrm{w}}I_1{}^{4}\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3 - 6\, {}^{\mathrm{w}}I_1{}^{2}\, {}^{\mathrm{w}}I_2{}^{2}\, {}^{\mathrm{w}}I_3 - 6\, {}^{\mathrm{w}}I_2{}^{3}\, {}^{\mathrm{w}}I_3 \\
& + 9\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3{}^{2} + {}^{\mathrm{w}}I_1{}^{3}\, ({}^{\mathrm{w}}I_2{}^{3} + 12\, {}^{\mathrm{w}}I_3{}^{2}) \Big) \\
& + a_6 \Big(2\, {}^{\mathrm{w}}I_1{}^{3}\, {}^{\mathrm{w}}I_2{}^{2}\, {}^{\mathrm{w}}I_3 + 6\, {}^{\mathrm{w}}I_2{}^{2}\, {}^{\mathrm{w}}I_3{}^{2} - {}^{\mathrm{w}}I_1{}^{2}\, ({}^{\mathrm{w}}I_2{}^{4} + 8\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3{}^{2}) + {}^{\mathrm{w}}I_1\, (4\, {}^{\mathrm{w}}I_2{}^{3}\, {}^{\mathrm{w}}I_3 + 6\, {}^{\mathrm{w}}I_3{}^{3}) \Big) \Big] \\
& + a_9^2\, {}^{\mathrm{w}}I_2{}^{2}\, {}^{\mathrm{w}}I_3{}^{2} \Big\{ a_8^2\, {}^{\mathrm{w}}I_2{}^{2}\, ({}^{\mathrm{w}}I_1{}^{3}\, {}^{\mathrm{w}}I_2{}^{2} - 12\, {}^{\mathrm{w}}I_1{}^{2}\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3 - 6\, {}^{\mathrm{w}}I_2{}^{2}\, {}^{\mathrm{w}}I_3 + 45\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3{}^{2}) \\
& + a_6^2 \Big(2\, {}^{\mathrm{w}}I_1{}^{2}\, {}^{\mathrm{w}}I_2{}^{2}\, {}^{\mathrm{w}}I_3 - 4\, {}^{\mathrm{w}}I_1{}^{3}\, {}^{\mathrm{w}}I_3{}^{2} + 6\, {}^{\mathrm{w}}I_3\, ({}^{\mathrm{w}}I_2{}^{3} + {}^{\mathrm{w}}I_3{}^{2}) + {}^{\mathrm{w}}I_1\, ({}^{\mathrm{w}}I_2{}^{4} - 20\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3{}^{2}) \Big) \\
& - a_6\, {}^{\mathrm{w}}I_2 \Big[4\, a_8\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3\, ({}^{\mathrm{w}}I_1{}^{2}\, {}^{\mathrm{w}}I_2 - 5\, {}^{\mathrm{w}}I_2{}^{2} + 3\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3) + a_5 \Big(4\, {}^{\mathrm{w}}I_1{}^{4}\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3 + 6\, {}^{\mathrm{w}}I_1{}^{2}\, {}^{\mathrm{w}}I_2{}^{2}\, {}^{\mathrm{w}}I_3 \\
& - 18\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3{}^{2} + 3\, {}^{\mathrm{w}}I_3\, (4\, {}^{\mathrm{w}}I_2{}^{3} + 3\, {}^{\mathrm{w}}I_3{}^{2}) - 2\, {}^{\mathrm{w}}I_1{}^{3}\, ({}^{\mathrm{w}}I_2{}^{3} + 6\, {}^{\mathrm{w}}I_3{}^{2}) \Big) \Big] \Big\} \\
& + a_7^2\, {}^{\mathrm{w}}I_2{}^{2} \Big\{ 2\, a_9^2\, {}^{\mathrm{w}}I_2{}^{2}\, (2\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2{}^{4} - 9\, {}^{\mathrm{w}}I_1{}^{2}\, {}^{\mathrm{w}}I_2{}^{2}\, {}^{\mathrm{w}}I_3 - {}^{\mathrm{w}}I_2{}^{3}\, {}^{\mathrm{w}}I_3 + 10\, {}^{\mathrm{w}}I_1{}^{3}\, {}^{\mathrm{w}}I_3{}^{2}) \\
& + {}^{\mathrm{w}}I_3{}^{2} \Big(a_5\, a_6\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3\, (8\, {}^{\mathrm{w}}I_1{}^{2}\, {}^{\mathrm{w}}I_2 + 7\, {}^{\mathrm{w}}I_2{}^{2} - 60\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3) - 2\, a_6^2\, {}^{\mathrm{w}}I_3\, ({}^{\mathrm{w}}I_2{}^{2} - 10\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3) \\
& + a_5^2\, {}^{\mathrm{w}}I_2{}^{2}\, ({}^{\mathrm{w}}I_1{}^{3}\, {}^{\mathrm{w}}I_2{}^{2} - 12\, {}^{\mathrm{w}}I_1{}^{2}\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3 - 6\, {}^{\mathrm{w}}I_2{}^{2}\, {}^{\mathrm{w}}I_3 + 45\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3{}^{2}) \Big) \\
& - 4\, a_9\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3 \Big[5\, a_6\, {}^{\mathrm{w}}I_3\, ({}^{\mathrm{w}}I_2{}^{2} - 2\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3) \\
& + a_5\, {}^{\mathrm{w}}I_2 \Big(-2\, {}^{\mathrm{w}}I_1{}^{2}\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3 - 8\, {}^{\mathrm{w}}I_2{}^{2}\, {}^{\mathrm{w}}I_3 + {}^{\mathrm{w}}I_1\, ({}^{\mathrm{w}}I_2{}^{3} + 15\, {}^{\mathrm{w}}I_3{}^{2}) \Big) \Big] \Big\} \\
& - 2\, a_6\, a_9\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3{}^{3} \Big\{ 2\, a_6^2\, {}^{\mathrm{w}}I_3\, (2\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2{}^{2} - 2\, {}^{\mathrm{w}}I_1{}^{2}\, {}^{\mathrm{w}}I_3 + {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3) \\
& - a_8\, {}^{\mathrm{w}}I_2{}^{2} \Big(a_8\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2\, ({}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2 - 6\, {}^{\mathrm{w}}I_3) + a_5\, ({}^{\mathrm{w}}I_1{}^{3}\, {}^{\mathrm{w}}I_2{}^{2} - 9\, {}^{\mathrm{w}}I_1{}^{2}\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3 - 6\, {}^{\mathrm{w}}I_2{}^{2}\, {}^{\mathrm{w}}I_3 + 27\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3{}^{2}) \Big) \\
& + a_6\, {}^{\mathrm{w}}I_2 \Big[a_5 \Big(-7\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2{}^{2}\, {}^{\mathrm{w}}I_3 - 3\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3{}^{2} + {}^{\mathrm{w}}I_1{}^{2}\, ({}^{\mathrm{w}}I_2{}^{3} + 6\, {}^{\mathrm{w}}I_3{}^{2}) \Big) \\
& + a_8 \Big(-2\, {}^{\mathrm{w}}I_1{}^{2}\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3 - 4\, {}^{\mathrm{w}}I_2{}^{2}\, {}^{\mathrm{w}}I_3 + {}^{\mathrm{w}}I_1\, ({}^{\mathrm{w}}I_2{}^{3} + 18\, {}^{\mathrm{w}}I_3{}^{2}) \Big) \Big] \Big\} \\
& - 2\, a_7\, {}^{\mathrm{w}}I_2 \Bigg(-2\, a_9^3\, {}^{\mathrm{w}}I_2{}^{3} \Big(-3\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3 + {}^{\mathrm{w}}I_1\, ({}^{\mathrm{w}}I_2{}^{2} - 2\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3) \Big) \Big({}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3 + {}^{\mathrm{w}}I_1\, ({}^{\mathrm{w}}I_2{}^{2} - 2\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3) \Big) \\
& + a_6\, {}^{\mathrm{w}}I_3{}^{3} \Big[2\, a_6\, {}^{\mathrm{w}}I_3\, (-a_6\, {}^{\mathrm{w}}I_2{}^{2} + 2\, a_8\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2{}^{2} + 6\, a_6\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3) \\
& + a_5^2\, {}^{\mathrm{w}}I_2{}^{2}\, ({}^{\mathrm{w}}I_1{}^{3}\, {}^{\mathrm{w}}I_2{}^{2} - 9\, {}^{\mathrm{w}}I_1{}^{2}\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3 - 6\, {}^{\mathrm{w}}I_2{}^{2}\, {}^{\mathrm{w}}I_3 + 27\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3{}^{2}) \\
& + a_5\, {}^{\mathrm{w}}I_2 \Big(a_8\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2{}^{2}\, ({}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2 - 6\, {}^{\mathrm{w}}I_3) + a_6\, {}^{\mathrm{w}}I_3\, (6\, {}^{\mathrm{w}}I_1{}^{2}\, {}^{\mathrm{w}}I_2 + 7\, {}^{\mathrm{w}}I_2{}^{2} - 36\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3) \Big) \Big] \\
& + a_9^2\, {}^{\mathrm{w}}I_2{}^{2}\, {}^{\mathrm{w}}I_3 \Big\{ 2\, a_6 \Big(-{}^{\mathrm{w}}I_1{}^{2}\, {}^{\mathrm{w}}I_2{}^{2}\, {}^{\mathrm{w}}I_3 + {}^{\mathrm{w}}I_2{}^{3}\, {}^{\mathrm{w}}I_3 - 2\, {}^{\mathrm{w}}I_1{}^{3}\, {}^{\mathrm{w}}I_3{}^{2} + {}^{\mathrm{w}}I_1\, ({}^{\mathrm{w}}I_2{}^{4} - 5\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3{}^{2}) \Big) \\
& + {}^{\mathrm{w}}I_2 \Big[a_5 \Big(-2\, {}^{\mathrm{w}}I_1{}^{4}\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3 - 6\, {}^{\mathrm{w}}I_1{}^{2}\, {}^{\mathrm{w}}I_2{}^{2}\, {}^{\mathrm{w}}I_3 - 6\, {}^{\mathrm{w}}I_2{}^{3}\, {}^{\mathrm{w}}I_3 + 9\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3{}^{2} + {}^{\mathrm{w}}I_1{}^{3}\, ({}^{\mathrm{w}}I_2{}^{3} + 12\, {}^{\mathrm{w}}I_3{}^{2}) \Big) \\
& + 2\, a_8\, {}^{\mathrm{w}}I_1 \Big(2\, {}^{\mathrm{w}}I_1{}^{2}\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3 + 8\, {}^{\mathrm{w}}I_2{}^{2}\, {}^{\mathrm{w}}I_3 - {}^{\mathrm{w}}I_1\, ({}^{\mathrm{w}}I_2{}^{3} + 15\, {}^{\mathrm{w}}I_3{}^{2}) \Big) \Big] \Big\} \\
& - a_9\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3{}^{2} \Big\{ 2\, a_6^2\, {}^{\mathrm{w}}I_3\, (6\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2{}^{2} - 8\, {}^{\mathrm{w}}I_1{}^{2}\, {}^{\mathrm{w}}I_3 + {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3) \\
& + a_5\, a_8\, {}^{\mathrm{w}}I_2{}^{2}\, (-{}^{\mathrm{w}}I_1{}^{3}\, {}^{\mathrm{w}}I_2{}^{2} + 12\, {}^{\mathrm{w}}I_1{}^{2}\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3 + 6\, {}^{\mathrm{w}}I_2{}^{2}\, {}^{\mathrm{w}}I_3 - 45\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_3{}^{2}) \\
& + a_6\, {}^{\mathrm{w}}I_2 \Big[a_5 \Big(-4\, {}^{\mathrm{w}}I_1{}^{3}\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3 - 20\, {}^{\mathrm{w}}I_1\, {}^{\mathrm{w}}I_2{}^{2}\, {}^{\mathrm{w}}I_3 - 3\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3{}^{2} + 3\, {}^{\mathrm{w}}I_1{}^{2}\, ({}^{\mathrm{w}}I_2{}^{3} + 8\, {}^{\mathrm{w}}I_3{}^{2}) \Big) \\
& + a_8 \Big(-8\, {}^{\mathrm{w}}I_1{}^{2}\, {}^{\mathrm{w}}I_2\, {}^{\mathrm{w}}I_3 - 4\, {}^{\mathrm{w}}I_2{}^{2}\, {}^{\mathrm{w}}I_3 + 2\, {}^{\mathrm{w}}I_1\, ({}^{\mathrm{w}}I_2{}^{3} + 15\, {}^{\mathrm{w}}I_3{}^{2}) \Big) \Big] \Big\} \Bigg),
\end{aligned}
\tag{E.1}
$$

$$
\begin{aligned}
n_5 = {} & 4\,a_6\,a_9\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3{}^3 \Big[a_8\,{}^{\mathrm{w}}\!I_2{}^2 \Big(a_8\,{}^{\mathrm{w}}\!I_1{}^2 + a_5\,({}^{\mathrm{w}}\!I_1{}^3 - 18\,{}^{\mathrm{w}}\!I_3) \Big) - a_6\,{}^{\mathrm{w}}\!I_2 \Big(a_8\,({}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2 - 12\,{}^{\mathrm{w}}\!I_3) \\
& + a_5\,{}^{\mathrm{w}}\!I_1\,({}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2 - 5\,{}^{\mathrm{w}}\!I_3) \Big) - 2\,a_6^2\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_3 \Big] \\
& + 2\,a_6^2\,{}^{\mathrm{w}}\!I_3{}^4 \Big(a_8^2\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2{}^2 + 2\,a_5 a_8\,{}^{\mathrm{w}}\!I_1{}^2\,{}^{\mathrm{w}}\!I_2{}^2 + a_5^2\,{}^{\mathrm{w}}\!I_2{}^2\,({}^{\mathrm{w}}\!I_1{}^3 - 12\,{}^{\mathrm{w}}\!I_3) - 4\,a_6^2\,{}^{\mathrm{w}}\!I_3 + 14\,a_5 a_6\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3 \Big) \\
& + 2\,a_9^4\,{}^{\mathrm{w}}\!I_2{}^4\,({}^{\mathrm{w}}\!I_1{}^3\,{}^{\mathrm{w}}\!I_2{}^2 - {}^{\mathrm{w}}\!I_1{}^4\,{}^{\mathrm{w}}\!I_3 - 5\,{}^{\mathrm{w}}\!I_1{}^2\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3 - 3\,{}^{\mathrm{w}}\!I_2{}^2\,{}^{\mathrm{w}}\!I_3) \\
& - 4\,a_9^3\,{}^{\mathrm{w}}\!I_2{}^3\,{}^{\mathrm{w}}\!I_3 \Big(a_6\,{}^{\mathrm{w}}\!I_1\,({}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2{}^2 - {}^{\mathrm{w}}\!I_1{}^2\,{}^{\mathrm{w}}\!I_3 - 4\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3) + a_8\,{}^{\mathrm{w}}\!I_2\,(-{}^{\mathrm{w}}\!I_1{}^3\,{}^{\mathrm{w}}\!I_2 + 6\,{}^{\mathrm{w}}\!I_1{}^2\,{}^{\mathrm{w}}\!I_3 + 9\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3) \Big) \\
& + a_7^2\,{}^{\mathrm{w}}\!I_2{}^2 \Big[-4\,a_9\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3 \Big(2\,a_5\,{}^{\mathrm{w}}\!I_2\,({}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2 - 9\,{}^{\mathrm{w}}\!I_3) + 9\,a_6\,{}^{\mathrm{w}}\!I_3 \Big) + 2\,a_9^2\,{}^{\mathrm{w}}\!I_2{}^2\,(4\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2{}^2 \\
& - 9\,{}^{\mathrm{w}}\!I_1{}^2\,{}^{\mathrm{w}}\!I_3 - 9\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3) + {}^{\mathrm{w}}\!I_3{}^2 \Big(2\,a_5^2\,{}^{\mathrm{w}}\!I_2{}^2\,({}^{\mathrm{w}}\!I_1{}^3 - 27\,{}^{\mathrm{w}}\!I_3) - 18\,a_6^2\,{}^{\mathrm{w}}\!I_3 + 63\,a_5\,a_6\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3 \Big) \Big] \\
& - 2\,a_7\,{}^{\mathrm{w}}\!I_2 \Big[2\,a_9^3\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2{}^3\,(-2\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2{}^2 + 3\,{}^{\mathrm{w}}\!I_1{}^2\,{}^{\mathrm{w}}\!I_3 + 9\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3) + 2\,a_9^2\,{}^{\mathrm{w}}\!I_2{}^3\,{}^{\mathrm{w}}\!I_3\,(2\,a_6\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2 \\
& - 2\,a_8\,{}^{\mathrm{w}}\!I_1{}^2\,{}^{\mathrm{w}}\!I_2 + a_5\,{}^{\mathrm{w}}\!I_1{}^3\,{}^{\mathrm{w}}\!I_2 + 18\,a_8\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_3 - 6\,a_5\,{}^{\mathrm{w}}\!I_1{}^2\,{}^{\mathrm{w}}\!I_3 - 9\,a_5\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3) \\
& + 2\,a_6\,{}^{\mathrm{w}}\!I_3{}^3 \Big(a_5^2\,{}^{\mathrm{w}}\!I_2{}^2\,({}^{\mathrm{w}}\!I_1{}^3 - 18\,{}^{\mathrm{w}}\!I_3) - 6\,a_6^2\,{}^{\mathrm{w}}\!I_3 + a_5\,{}^{\mathrm{w}}\!I_2\,(a_8\,{}^{\mathrm{w}}\!I_1{}^2\,{}^{\mathrm{w}}\!I_2 + 21\,a_6\,{}^{\mathrm{w}}\!I_3) \Big) \\
& + a_9\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3{}^2 \Big(2\,a_5\,a_8\,{}^{\mathrm{w}}\!I_2{}^2\,({}^{\mathrm{w}}\!I_1{}^3 - 27\,{}^{\mathrm{w}}\!I_3) - 18\,a_6^2\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_3 \\
& + a_6\,{}^{\mathrm{w}}\!I_2\,(-4\,a_8\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2 - 6\,a_5\,{}^{\mathrm{w}}\!I_1{}^2\,{}^{\mathrm{w}}\!I_2 + 36\,a_8\,{}^{\mathrm{w}}\!I_3 + 39\,a_5\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_3) \Big) \Big] \\
& + a_9^2\,{}^{\mathrm{w}}\!I_2{}^2\,{}^{\mathrm{w}}\!I_3{}^2 \Big[2\,a_8^2\,{}^{\mathrm{w}}\!I_2{}^2\,({}^{\mathrm{w}}\!I_1{}^3 - 27\,{}^{\mathrm{w}}\!I_3) + 2\,a_6^2\,({}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2{}^2 + 3\,{}^{\mathrm{w}}\!I_1{}^2\,{}^{\mathrm{w}}\!I_3 + 4\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3) \\
& + a_6\,{}^{\mathrm{w}}\!I_2 \Big(24\,a_8\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_3 + a_5\,(4\,{}^{\mathrm{w}}\!I_1{}^3\,{}^{\mathrm{w}}\!I_2 - 17\,{}^{\mathrm{w}}\!I_1{}^2\,{}^{\mathrm{w}}\!I_3 - 24\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3) \Big) \Big],
\end{aligned}
\tag{E.2}
$$

$$
\begin{aligned}
n_8 = {} & a_6^2\,{}^{\mathrm{w}}\!I_3{}^4 \Big(a_8^2\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2{}^2 + 2a_5\,a_8\,{}^{\mathrm{w}}\!I_1{}^2\,{}^{\mathrm{w}}\!I_2{}^2 + a_5^2\,{}^{\mathrm{w}}\!I_2{}^2({}^{\mathrm{w}}\!I_1{}^3 - 12\,{}^{\mathrm{w}}\!I_3) - 4a_6^2\,{}^{\mathrm{w}}\!I_3 + 14a_5 a_6\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3 \Big) \\
& + a_9^4\,{}^{\mathrm{w}}\!I_2{}^3 \Big({}^{\mathrm{w}}\!I_2{}^3\,({}^{\mathrm{w}}\!I_1{}^3 - 6\,{}^{\mathrm{w}}\!I_3) - {}^{\mathrm{w}}\!I_1{}^2\,{}^{\mathrm{w}}\!I_2{}^2\,{}^{\mathrm{w}}\!I_3 + 2\,{}^{\mathrm{w}}\!I_1{}^3\,{}^{\mathrm{w}}\!I_3{}^2 - 2\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3\,({}^{\mathrm{w}}\!I_1{}^3 + 3\,{}^{\mathrm{w}}\!I_3) \Big) \\
& - a_9^3\,{}^{\mathrm{w}}\!I_2{}^3\,{}^{\mathrm{w}}\!I_3 \Big(2\,a_8\,(-{}^{\mathrm{w}}\!I_1{}^3\,{}^{\mathrm{w}}\!I_2{}^2 + {}^{\mathrm{w}}\!I_1{}^4\,{}^{\mathrm{w}}\!I_3 + 2\,{}^{\mathrm{w}}\!I_1{}^2\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3 + 12\,{}^{\mathrm{w}}\!I_2{}^2\,{}^{\mathrm{w}}\!I_3) \\
& - a_6\,(-2\,{}^{\mathrm{w}}\!I_1{}^2\,{}^{\mathrm{w}}\!I_2{}^2 + 2\,{}^{\mathrm{w}}\!I_1{}^3\,{}^{\mathrm{w}}\!I_3 + 7\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3 + 12\,{}^{\mathrm{w}}\!I_3{}^2) \Big) \\
& + a_7^2\,{}^{\mathrm{w}}\!I_2{}^2 \Big[2\,a_9^2\,{}^{\mathrm{w}}\!I_2{}^2\,(2\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2{}^2 - 5\,{}^{\mathrm{w}}\!I_1{}^2\,{}^{\mathrm{w}}\!I_3 - 3\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3) \\
& + {}^{\mathrm{w}}\!I_3{}^2 \Big(-6\,a_6^2\,{}^{\mathrm{w}}\!I_3 + a_5\,a_6\,(2\,{}^{\mathrm{w}}\!I_1{}^2 + 21\,{}^{\mathrm{w}}\!I_2)\,{}^{\mathrm{w}}\!I_3 + a_5^2\,{}^{\mathrm{w}}\!I_2\,({}^{\mathrm{w}}\!I_1{}^3\,{}^{\mathrm{w}}\!I_2 - 3\,{}^{\mathrm{w}}\!I_1{}^2\,{}^{\mathrm{w}}\!I_3 - 18\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3) \Big) \\
& + 2\,a_9\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3 \Big(-8\,a_6\,{}^{\mathrm{w}}\!I_3 + a_5\,(-2\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2{}^2 + {}^{\mathrm{w}}\!I_1{}^2\,{}^{\mathrm{w}}\!I_3 + 15\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3) \Big) \Big] \\
& + a_6\,a_9\,{}^{\mathrm{w}}\!I_3{}^3 \Big[-2\,a_6^2\,{}^{\mathrm{w}}\!I_3\,(3\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2 + 2\,{}^{\mathrm{w}}\!I_3) + a_8\,{}^{\mathrm{w}}\!I_2{}^2 \Big(a_8\,{}^{\mathrm{w}}\!I_1\,(2\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2 - 3\,{}^{\mathrm{w}}\!I_3) \\
& + a_5\,(2\,{}^{\mathrm{w}}\!I_1{}^3\,{}^{\mathrm{w}}\!I_2 - 3\,{}^{\mathrm{w}}\!I_1{}^2\,{}^{\mathrm{w}}\!I_3 - 30\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3) \Big) + a_6\,{}^{\mathrm{w}}\!I_2 \Big(-2\,a_8\,({}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2{}^2 + {}^{\mathrm{w}}\!I_1{}^2\,{}^{\mathrm{w}}\!I_3 - 10\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3) \\
& + a_5\,(-2\,{}^{\mathrm{w}}\!I_1{}^2\,{}^{\mathrm{w}}\!I_2{}^2 - 2\,{}^{\mathrm{w}}\!I_1{}^3\,{}^{\mathrm{w}}\!I_3 + 13\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3 + 6\,{}^{\mathrm{w}}\!I_3{}^2) \Big) \Big] \\
& + a_9^2\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3{}^2 \Big[a_8^2\,{}^{\mathrm{w}}\!I_2{}^2\,({}^{\mathrm{w}}\!I_1{}^3\,{}^{\mathrm{w}}\!I_2 - 3\,{}^{\mathrm{w}}\!I_1{}^2\,{}^{\mathrm{w}}\!I_3 - 18\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3) \\
& + a_6^2 \Big(2\,{}^{\mathrm{w}}\!I_1{}^2\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3 + 8\,{}^{\mathrm{w}}\!I_2{}^2\,{}^{\mathrm{w}}\!I_3 + {}^{\mathrm{w}}\!I_1\,({}^{\mathrm{w}}\!I_2{}^3 - 6\,{}^{\mathrm{w}}\!I_3{}^2) \Big) - a_6\,{}^{\mathrm{w}}\!I_2 \Big(2\,a_8\,{}^{\mathrm{w}}\!I_1\,(2\,{}^{\mathrm{w}}\!I_1{}^2 - 11\,{}^{\mathrm{w}}\!I_2)\,{}^{\mathrm{w}}\!I_3 \\
& + a_5\,(-2\,{}^{\mathrm{w}}\!I_1{}^3\,{}^{\mathrm{w}}\!I_2{}^2 + 2\,{}^{\mathrm{w}}\!I_1{}^4\,{}^{\mathrm{w}}\!I_3 + {}^{\mathrm{w}}\!I_1{}^2\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3 + 18\,{}^{\mathrm{w}}\!I_2{}^2\,{}^{\mathrm{w}}\!I_3 + 3\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_3{}^2) \Big) \Big] \\
& - a_7\,{}^{\mathrm{w}}\!I_2 \Big\{ a_6\,{}^{\mathrm{w}}\!I_3{}^3 \Big[2\,a_6\,(-5\,a_6 + a_8\,{}^{\mathrm{w}}\!I_1)\,{}^{\mathrm{w}}\!I_3 + a_5^2\,{}^{\mathrm{w}}\!I_2\,(2\,{}^{\mathrm{w}}\!I_1{}^3\,{}^{\mathrm{w}}\!I_2 - 3\,{}^{\mathrm{w}}\!I_1{}^2\,{}^{\mathrm{w}}\!I_3 - 30\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3) \\
& + a_5 \Big(a_8\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2\,(2\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2 - 3\,{}^{\mathrm{w}}\!I_3) + a_6\,(2\,{}^{\mathrm{w}}\!I_1{}^2 + 35\,{}^{\mathrm{w}}\!I_2)\,{}^{\mathrm{w}}\!I_3 \Big) \Big] \\
& + 2\,a_9^3\,{}^{\mathrm{w}}\!I_2{}^2 \Big(4\,{}^{\mathrm{w}}\!I_1{}^3\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3 + 6\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2{}^2\,{}^{\mathrm{w}}\!I_3 + 9\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3{}^2 - {}^{\mathrm{w}}\!I_1{}^2\,(2\,{}^{\mathrm{w}}\!I_2{}^3 + 3\,{}^{\mathrm{w}}\!I_3{}^2) \Big) \\
& - 2\,a_9^2\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3 \Big[a_6\,(-2\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2{}^3 + {}^{\mathrm{w}}\!I_1{}^2\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3 - 3\,{}^{\mathrm{w}}\!I_2{}^2\,{}^{\mathrm{w}}\!I_3 + 6\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_3{}^2) + {}^{\mathrm{w}}\!I_2 \Big(a_8\,{}^{\mathrm{w}}\!I_1\,(2\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2{}^2 \\
& - {}^{\mathrm{w}}\!I_1{}^2\,{}^{\mathrm{w}}\!I_3 - 15\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3) + a_5\,(-{}^{\mathrm{w}}\!I_1{}^3\,{}^{\mathrm{w}}\!I_2{}^2 + {}^{\mathrm{w}}\!I_1{}^4\,{}^{\mathrm{w}}\!I_3 + 2\,{}^{\mathrm{w}}\!I_1{}^2\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3 + 12\,{}^{\mathrm{w}}\!I_2{}^2\,{}^{\mathrm{w}}\!I_3) \Big) \Big] \\
& + a_9\,{}^{\mathrm{w}}\!I_3{}^2 \Big[-2\,a_6^2\,{}^{\mathrm{w}}\!I_3\,(10\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2 + 3\,{}^{\mathrm{w}}\!I_3) + 2\,a_5\,a_8\,{}^{\mathrm{w}}\!I_2{}^2\,({}^{\mathrm{w}}\!I_1{}^3\,{}^{\mathrm{w}}\!I_2 - 3\,{}^{\mathrm{w}}\!I_1{}^2\,{}^{\mathrm{w}}\!I_3 - 18\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3) \\
& + a_6\,{}^{\mathrm{w}}\!I_2 \Big(4\,a_8\,(-{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2{}^2 + {}^{\mathrm{w}}\!I_1{}^2\,{}^{\mathrm{w}}\!I_3 + 6\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3) + a_5\,(-6\,{}^{\mathrm{w}}\!I_1{}^2\,{}^{\mathrm{w}}\!I_2{}^2 + 40\,{}^{\mathrm{w}}\!I_1\,{}^{\mathrm{w}}\!I_2\,{}^{\mathrm{w}}\!I_3 + 9\,{}^{\mathrm{w}}\!I_3{}^2) \Big) \Big] \Big\}.
\end{aligned}
\tag{E.3}
$$