Time consistency of risk measures in markets with transaction costs

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joint work with Zach Feinstein (Princeton University)

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Multivariate risks $X \in L_d^p(\mathcal{F}_T)$,
1. Risk measures under transaction costs

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Collection of initial portfolio vectors that make \( X \) acceptable (w.r.t. some acceptance set \( A_t \subseteq L_d^p(\mathcal{F}_T) \))

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R_t(X) = \{ u \in L_d^p(\mathcal{F}_t) : X + u \in A_t \}.
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Usually initial capital just in a few ’eligible’ assets: subspace \( M_t \subseteq L^p_d(\mathcal{F}_t) \), typically \( \dim M_t = m << d \)

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\[ M_t \subseteq L^p_d(\mathcal{F}_t) \quad (M_t)_+ = M_t \cap L^p_d(\mathcal{F}_t)_+ \]

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**Conditional Set-Valued Risk Measure**

A set-valued function
\[ R_t : L^p_d(\mathcal{F}_T) \to \mathcal{P}((M_t)_+) = \{D \subseteq M_t : D = D + (M_t)_+\} \]
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- A conditional risk measure is **normalized** if for any \( X \in L^p_d(\mathcal{F}_T) \): \( R_t(X) + R_t(0) = R_t(X) \)
- **Dynamic risk measure:** sequence \( (R_t)_{t=0}^T \) of conditional risk measures

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Let $\mathcal{G}((M_t)_+) = \{D \subseteq M_t : D = \text{cl co}(D + (M_t)_+)\}$.

**Dual Representation, $1 \leq p \leq \infty$**

A function $R_t : L^p_d(\mathcal{F}_T) \to \mathcal{G}((M_t)_+)$ is a closed coherent conditional risk measure if and only if there is a nonempty set $\bar{\mathcal{W}}_t \subseteq \mathcal{W}_t$ such that

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- $Q$ vector probability measure with components $Q_i$ ($i=1,...,d$), $\frac{dQ_i}{dQ} \in L^q$ and $E^Q_t[X] = (E^Q_{t1}[X_1],...,E^Q_{td}[X_d])^T$. 

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- $w \in ((M_t)_+)^+$
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- \( w \in ((M_t)_+)^+ \)
- \( G_t(w) = \{ v \in L^p_d(\mathcal{F}_t) : E[w^Tv] \geq 0 \} \).
1. Risk measures under transaction costs

Proof of dual representation: Set-valued convex analysis.
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analog for convex set-valued risk measures $\triangleright$ Feinstein, Rudloff (2013)

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2. Time Consistency

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2. Time Consistency: Background

Time Consistency: scalar case
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2. Time Consistency: Set-Valued

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A dynamic set-valued risk measure \((R_t)^T_{t=0}\) is **time consistent** if for all \(t\), for all \(X, Y \in L^p_d(F_T)\) with

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In the scalar case \(R_t(X) = \{u \in L^p(\mathcal{F}_t): \rho_t(X) \leq u\}\) \((\rho_t)^T_{t=0}\) time consistent iff multi-portfolio time consistent.

In higher dimensions: multi-portfolio time consistency implies time consistency.
Multi-portfolio time consistency

For a normalized dynamic set-valued risk measure $(R_t)^T_{t=0}$ the following is equivalent
Multi-portfolio time consistency

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this is a set-valued Bellman’s principle!
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3. Examples

Examples
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(Kabanov 99, Schachermayer 04, Pennanen, Penner 08, ...)

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3.1 Superhedging

(proportional transaction costs at time $t$: closed convex cone $\mathbb{R}^d_+ \subseteq K_t(\omega) \subseteq \mathbb{R}^d$ (solvency cone), positions transferrable into nonnegative positions)
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- proportional transaction costs at time $t$: closed convex cone $\mathbb{R}^d_+ \subseteq K_t(\omega) \subseteq \mathbb{R}^d$ (solveny cone), positions transferrable into nonnegative positions

- $(V_t)_{t=0}^T$ self-financing portfolio process if

\[ V_t - V_{t-1} \in -K_t \quad P - a.s. \quad \forall t \in \{0, ..., T\} \quad (V_{-1} \equiv 0) \]
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- $L^p_d(\mathcal{F}_T)$-attainable claims (from zero cost at time $t$)

\[ C_{t,T} = \sum_{s=t}^{T} -L^p_d(\mathcal{F}_s; K_s) \]
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Set of superhedging portfolios for $X \in L^p_d(\mathcal{F}_T)$

$$SHP_t(X) := \{ u \in L^p_d(\mathcal{F}_t) : -X + u \in -C_{t,T} \}. $$
Under robust no arbitrage condition (NA$^r$):
\[ R_t(X) := SHP_t(-X) \]
is a closed market-compatible \textbf{coherent dynamic risk measure} on \( L^p_d(F_T) \) that is \textbf{multi-portfolio time consistent}.
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It follows

$$SHP_t(X) = \bigcup_{Z \in SHP_{t+1}(X)} SHP_t(Z) =: SHP_t(SHP_{t+1}(X)).$$
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This is equivalent to a sequence of linear vector optimization problems that can be solved by Benson’s algorithm for finite \(\Omega\).
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3.1 Superhedging

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recursion also in Roux, Zastawniak 13, but no connection to algorithms
Multiple correlated assets (basket options):
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Tree approximating \((d - 1)\)-dim Black-Scholes-Model by Korn, Müller (09)
3.1 Superhedging

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Tree approximating \((d - 1)\)-dim Black-Scholes-Model by Korn, Müller (09)

**Example:** Exchange Option, \(d = 3\) includes transaction costs for bond

<table>
<thead>
<tr>
<th>( r = 5%), ( \lambda = (1%, 2%, 4%)^T )</th>
<th>((13.341 \ 0.000 \ -7.760 \ 0.347 \ 0.498 \ 0.584 \ -0.446 \ -0.331 \ -0.260))</th>
</tr>
</thead>
<tbody>
<tr>
<td>vertex of (SHP_0(X))</td>
<td>(\pi_0^a(X)) (in bonds)</td>
</tr>
<tr>
<td></td>
<td>(\pi^a(X)) (in cash)</td>
</tr>
<tr>
<td>( r = 5%), ( \lambda = (0.2%, 0.4%, 0.1%)^T )</td>
<td>((12.403 \ 8.230 \ 0.000 \ -6.236 \ -4.237 \ 0.308 \ 0.353 \ 0.441 \ 0.507 \ 0.486 \ -0.433 \ -0.394 \ -0.317 \ -0.257 \ -0.276))</td>
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The set $SHP_t(X)$ can be equivalently described by a family of scalarizations

$$\{ \phi_v(X) = \text{ess}\inf_{u \in SHP_t(X)} v^T u; \quad v \in K_t^+ \}$$
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Historically, the superhedging price was studied in one currency Jouini, Kallal (95) representation for $d$ assets:

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where $Q$ is the set of all processes $(S_t)_{t=0}^T$ with $S_t \in K_t^+$ for all $t$ and their equivalent martingale measures $Q$. 
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It holds

$$\pi^a(X) = \operatorname{ess.inf}_{u \in SHP_0(X) \cap M} u^T v,$$

for $M = \mathbb{R}e^1$ and $v = e^1$. Thus, the scalar risk measure $\pi^a(X)$ involves the sets $SHP_t(X)$ for all $t$!!! The reason is time consistency!
3.2 AV@R

**Definition:** set-valued AV@R (static case): 

HAMEL, RUDLOFF, YANKOVA 12
Definition: set-valued AV@R (static case): Hamel, Rudloff, Yankova 12

Let $\alpha \in (0, 1]^d$ and $X \in L^1_d$.

$$AV@R_{\alpha}^{reg}(X) = \left\{ \text{diag}(\alpha)^{-1} \mathbb{E}[Z] - z : Z \in (L^1_d)_+, X + Z - z \mathbb{I} \in (L^1_d)_+, z \in \mathbb{R}^d \right\} \cap M.$$
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Remark: If $m = d = 1$: $AV@R^{reg}_\alpha(X) = AV@R^{sca}_\alpha(X) + \mathbb{R}_+$

with

$$AV@R^{sca}_\alpha(X) = \inf_{z \in \mathbb{R}} \left\{ \frac{1}{\alpha} E \left[ (-X + z \mathbb{I})^+ \right] - z \right\}$$

which is optimized certainty equivalent representation of the AV@R by Rockafellar and Uryasev ’00.
Good-deal bounds

The market extension $R^{mar}$ of a risk measure $R$ satisfies

$$R^{mar} (X) = \inf_{\mathcal{P}(M_+)} \{ R(X + Y) : Y \in C_{0,T} \}.$$ 

and is a again set-valued risk measure, corresponds to so called Good-deal price bounds.
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Let $\Omega$ be finite. Then, $AV@R_{\alpha}^{reg}(X)$ and $AV@R_{\alpha}^{mar}(X)$ can be calculated by solving a linear vector optimization problem (using Benson’s algorithm)
Example: $d = 12$ correlated assets, $m = 2$, one-period model, $X$ payoff of an outperformance option.

Figure: $AV@R_{\alpha}^{mar} (X)$ (left) and its geometric dual (right).
dynamic version \((AV@R_\alpha)_t\) is not multi-portfolio time-consistent, nor time consistent...
3.2 AV@R

- dynamic version $(AV@R_\alpha)_t$ is not multi-portfolio time-consistent, nor time consistent...
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4. Time Consistency: Stability

Dual representation of composed AV@R (here $M_t = L^p_d(F_t)$)

\[
\widetilde{AV@R}_t^\alpha (X) := \bigcap_{(Q,w) \in \widetilde{W}_t^\alpha} (E_t^Q[-X] + G_t(w)),
\]

where

\[
\widetilde{W}_t^\alpha = \{(Q, w) \in M_d^\mathbb{P} \times L^q_d(F_t) : \forall \tau \in \{t, \ldots, T-1\} : \mathbb{P}\left(\xi_{\tau+1}(Q^i) \leq (\alpha_i^\tau)^{-1} \text{ or } w_i = 0\right) = 1 \quad i = 1, \ldots, d\}.
\]


Thank you!