Role of Stochastic Optimization in Revenue Management

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Revenue Management

- Revenue management involves making most use of **limited** inventories of **perishable** resources under **random** demand.

- Airlines form a typical application setting:
  - Resources correspond to capacities on flight legs.
  - Customers arrive randomly over time.

- Other application settings include hotels, car rentals, advertising, fashion retail.

- Primary **tradeoff** is whether to give resources to a currently available customer that is willing to pay a low price or keep the resources with the hope of a future customer that may be willing to pay a high price.
Revenue Management

Capacity Allocation

Dynamic Pricing

Assortment Offering

Decision Mechanism
Revenue Management

Static, Single Resource

Dynamic, Single Resource

Dynamic, Network of Resources

Modeling Detail
## Revenue Management

<table>
<thead>
<tr>
<th></th>
<th>Capacity Allocation</th>
<th>Dynamic Pricing</th>
<th>Assortment Offering</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Static Single Leg</strong></td>
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<td><strong>Dynamic Single Leg</strong></td>
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<td><strong>Dynamic Network</strong></td>
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• Requests arrive over time
• Make an acceptance or rejection decision for each request
• Accepted requests consume capacities on one or more flight legs
Deterministic Linear Programming Approximation

- \( \mathcal{L} \) : Set of flight legs in the airline network
- \( \mathcal{J} \) : Set of itineraries
- \( p_{jt} \) : Probability of getting a request for itinerary \( j \) at time \( t \)
- \( f_j \) : Revenue from itinerary \( j \)
- \( a_{ij} \) : 1 if itinerary \( j \) uses flight leg \( i \)
  
  0 otherwise
- \( c_i \) : Total available capacity on flight leg \( i \)
Deterministic Linear Programming Approximation

- Assume that the numbers of itinerary requests take on their expected values

\[
Z_{LP} = \max \sum_{j \in J} f_j w_j \\
\text{st} \sum_{j \in J} a_{ij} w_j \leq c_i \quad i \in \mathcal{L} \quad (\mu_i^*) \\
0 \leq w_j \leq \sum_{t \in T} p_{jt} \quad j \in \mathcal{I}
\]
Deterministic Linear Programming Approximation

- \( \{ \mu_i^* : i \in L \} \) are called bid prices
- They characterize the value of a seat
- Accept a request for itinerary \( j \) if

\[
f_j \geq \sum_{i \in L} a_{ij} \mu_i^*
\]

...subject to capacity availability
- As time progresses, it is customary to refresh the bid prices by resolving the linear program
- Optimal objective value of the linear program provides an upper bound on the performance of any nonancitipatory policy
Deterministic Linear Program

Decomposition by Displacement Adjustments

Decomposition by Fare Allocations
Dynamic Programming Formulation

- $\mathcal{L}$: Set of flight legs in the airline network
- $\mathcal{J}$: Set of itineraries
- $p_{jt}$: Probability of getting a request for itinerary $j$ at time $t$
- $f_j$: Revenue from itinerary $j$
- $a_{ij}$: $1$ if itinerary $j$ uses flight leg $i$
  $0$ otherwise
- $x_{it}$: Remaining capacity on flight leg $i$ at time $t$
- $u_{jt}$: $1$ if a request for itinerary $j$ is accepted at time $t$
  $0$ otherwise
Dynamic Programming Formulation

\[ V_t(x_t) = \max \sum_{j \in J} p_{jt} \left\{ f_j u_{jt} + V_{t+1}(x_t - u_{jt} \sum_{i \in L} a_{ij} e_i) \right\} \]

st  \( a_{ij} u_{jt} \leq x_{it} \)  \( i \in L, j \in J \)

\( u_{jt} \in \{0, 1\} \)  \( j \in J \)

- To obtain approximations to value functions, decompose the dynamic programming formulation by the flight legs
Decomposition by Fare Allocations
- $\lambda_{i,j}$: Fare allocation of itinerary $j$ over flight leg $i$

$$\sum_{i \in \mathcal{L}} \lambda_{i,j} = f_j$$
Decomposition by Fare Allocations

- Revenue allocations allow us to solve single-leg problems

\[ v_t^i(x_{it} | \lambda) = \max \sum_{j \in \mathcal{J}} p_{jt} \left\{ \lambda_{ij} u_{jt} + v_{t+1}^i(x_{it} - a_{ij} u_{jt} | \lambda) \right\} \]

\[ \text{st} \quad a_{ij} u_{jt} \leq x_{it} \quad j \in \mathcal{J} \]
\[ u_{jt} \in \{0, 1\} \quad j \in \mathcal{J} \]

- Single-leg problems provide an upper bound on exact value functions

\[ \sum_{i \in \mathcal{L}} v_t^i(x_{it} | \lambda) \geq V_t(x_t) \]

as long as

\[ \sum_{i \in \mathcal{L}} \lambda_{ij} = f_j \quad j \in \mathcal{J} \]
We obtain upper bounds on the optimal expected revenue

$$\sum_{i \in \mathcal{L}} v^i_t(x_{it} | \lambda) \geq V_t(x_t) \quad \sum_{i \in \mathcal{L}} v^i_1(c_i | \lambda) \geq V_1(c)$$

To obtain the tightest possible upper bound, we solve

$$\min_{\lambda} \sum_{i \in \mathcal{L}} v^i_1(c_i | \lambda) \quad \text{convex in } \lambda$$

subject to

$$\sum_{i \in \mathcal{L}} \lambda_{ij} = f_j \quad j \in \mathcal{J}$$

Tightest possible upper bound is better than the one from deterministic linear programming approximation

$$Z_{LP} \geq \sum_{i \in \mathcal{L}} v^i_1(c_i | \lambda^*) \geq V_1(c)$$
We can obtain a good revenue allocation from deterministic linear programming approximation

\[
\begin{align*}
\text{max} & \quad \sum_{j \in J} f_j w_j \\
\text{st} & \quad \sum_{j \in J} a_{ij} w_j \leq c_i \quad i \in \mathcal{L} \\
& \quad 0 \leq w_j \leq \sum_{t \in T} p_{jt} \quad j \in J
\end{align*}
\]
We can obtain a good revenue allocation from deterministic linear programming approximation

\[
\begin{align*}
\text{max} \quad & \sum_{j \in \mathcal{J}} f_j w_{\psi j} \\
\text{st} \quad & \sum_{j \in \mathcal{J}} a_{ij} w_{ij} \leq c_i \quad i \in \mathcal{L} \\
& 0 \leq w_{ij} \leq \sum_{t \in \mathcal{T}} p_{jt} \quad i \in \mathcal{L}, \; j \in \mathcal{J} \\
& w_{\psi j} - w_{ij} = 0 \quad j \in \mathcal{J}, \; i \in \mathcal{L}
\end{align*}
\]
We can obtain a good revenue allocation from deterministic linear programming approximation

\[
\max \sum_{j \in J} f_j w_{\psi j}
\]

\[
\text{st} \quad \sum_{j \in J} a_{i j} w_{i j} \leq c_i \quad i \in \mathcal{L}
\]

\[
0 \leq w_{i j} \leq \sum_{t \in T} p_{j t} \quad i \in \mathcal{L}, \ j \in \mathcal{J}
\]

\[
w_{\psi j} - w_{i j} = 0 \quad j \in \mathcal{J}, \ i \in \mathcal{L} \quad (\beta_{i j}^*)
\]

\[
\sum_{i \in \mathcal{L}} \beta_{i j}^* = f_j \quad j \in \mathcal{J}
\]
Making Decisions

- If we have the exact value functions, then we can make the decisions at time $t$ by solving

$$V_t(x_t) = \max \sum_{j \in \mathcal{J}} p_{jt} \left\{ f_j u_j t + V_{t+1}(x_t - u_j t \sum_{i \in \mathcal{L}} a_{ij} e_i) \right\}$$

subject to

$$a_{ij} u_j t \leq x_{it} \quad i \in \mathcal{L}, \quad j \in \mathcal{J}$$

$$u_j t \in \{0, 1\} \quad j \in \mathcal{J}$$

- It is optimal to accept a request for itinerary $j$ at time $t$ when there is enough capacity and

$$f_j + V_{t+1}(x_t - \sum_{i \in \mathcal{L}} a_{ij} e_i) \geq V_{t+1}(x_t)$$
If we have the exact value functions, then we can make the decisions at time $t$ by solving

$$V_t(x_t) = \max \sum_{j \in \mathcal{J}} p_{jt} \left\{ f_j u_{jt} + V_{t+1}(x_t - u_{jt} \sum_{i \in \mathcal{L}} a_{ij} e_i) \right\}$$

\[
\text{st} \quad a_{ij} u_{jt} \leq x_{it} \quad i \in \mathcal{L}, \ j \in \mathcal{J} \\
u_{jt} \in \{0, 1\} \quad j \in \mathcal{J}
\]

It is optimal to accept a request for itinerary $j$ at time $t$ when there is enough capacity and

$$f_j \geq V_{t+1}(x_t) - V_{t+1}(x_t - \sum_{i \in \mathcal{L}} a_{ij} e_i)$$
We accept a request for itinerary $j$ at time $t$ when there is enough capacity and

$$f_j \geq \sum_{i \in \mathcal{L}} v^i_{t+1}(x_{it} \mid \lambda) - \sum_{i \in \mathcal{L}} v^i_{t+1}(x_{it} - a_{ij} \mid \lambda)$$

$$f_j \geq \sum_{i \in \mathcal{L}} a_{ij} \left[ v^i_{t+1}(x_{it} \mid \lambda) - v^i_{t+1}(x_{it} - 1 \mid \lambda) \right]$$

bid price of flight leg $i$
Numerical Performance

- An airline network with one hub serving multiple spokes

- There is a high-fare and a low-fare itinerary connecting each origin-destination pair

- Compare deterministic linear programming approximation (DLP), decomposition with fare allocations from the deterministic linear program (DFA-DLP) and decomposition with best fare allocations (DFA-OPT)
Comparing Expected Revenues

% gap with DLP

DFA-DLP
DFA-OPT

low load  medium load  high load  low load  medium load  high load

DFA-DLP
DFA-OPT

4 spokes  8 spokes
Deterministic Linear Program

Decomposition by Displacement Adjustments

Decomposition by Fare Allocations
Decomposition by Displacement Adjustments

- Start from the deterministic linear program to decompose the dynamic programming formulation

\[ Z_{LP} = \max \sum_{j \in J} f_j w_j \]

\[ \text{st } \sum_{j \in J} a_{ij} w_j \leq c_i \quad i \in \mathcal{L} \quad (\mu_i^*) \]

\[ 0 \leq w_j \leq \sum_{t \in T} p_{jt} \quad j \in J \]
Decomposition by Displacement Adjustments

- Start from the deterministic linear program to decompose the dynamic programming formulation

\[
Z_{LP} = \max \quad \sum_{j \in J} \left[ f_j - \sum_{k \in L \setminus \{i\}} a_{kj} \mu_k^* \right] w_j + \sum_{k \in L \setminus \{i\}} \mu_k^* c_k \\
\text{st} \quad \sum_{j \in J} a_{ij} w_j \leq c_i \\
0 \leq w_j \leq \sum_{t \in T} p_{jt} \quad j \in J
\]
We can solve the dynamic programming formulation of the single-leg problem over flight leg \( i \)

\[
v_t^i(x_{it}) = \max \sum_{j \in J} p_{jt} \left\{ \left[ f_j - \sum_{k \in \mathcal{L}\backslash\{i\}} a_{kj} \mu_k^* \right] u_{jt} + v_{t+1}^i(x_{it} - a_{ij} u_{jt}) \right\}
\]

\[
\text{st} \quad a_{ij} u_{jt} \leq x_{it} \quad j \in J
\]

\[
u_{jt} \in \{0, 1\} \quad j \in J
\]

Single-leg problem provides an upper bound on exact value functions

\[
v_t^i(x_{it}) + \sum_{k \in \mathcal{L}\backslash\{i\}} \mu_k^* x_{kt} \geq V_t(x_t)
\]
Upper bounds are tighter than the one from the deterministic linear programming approximation

\[
Z_{LP} \geq v^i_1(c_i) + \sum_{k \in \mathcal{L} \setminus \{i\}} \mu^*_k c_k \geq V_1(c)
\]

Apply the same idea for each flight leg i and approximate the value function \(V_t(x_t)\) by

\[
\sum_{i \in \mathcal{L}} v^i_t(x_{it})
\]
• An airline network with one hub serving multiple spokes

• There is a high-fare and a low-fare itinerary connecting each origin-destination pair

• Compare deterministic linear programming approximation (DLP), decomposition with fare allocations from the deterministic linear program (DFA-DLP), decomposition with best fare allocations (DFA-OPT) and decomposition by displacement adjustments (DDA)
Comparing Upper Bounds

- **DFA-DLP**
- **DFA-OPT**
- **DDA**

% gap with DLP

<table>
<thead>
<tr>
<th>low load</th>
<th>medium load</th>
<th>high load</th>
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<tr>
<td>4 spokes</td>
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- 4 spokes
- 8 spokes
Comparing Expected Revenues

DFA-DLP
DFA-OPT
DDA

% gap with DLP

low load
medium load
high load
low load
medium load
high load

4 spokes
8 spokes
Directions for Improvement

• We can obtain better policies by using **value function approximations that are non-separable** by flight legs
• It is important to incorporate additional **constraints on various performance measures**, such as occupancy
• Maximizing expected revenue is reasonable when dealing with large airlines, but **incorporating risk** becomes important for smaller businesses
• Number of **dimensions of the state vector** becomes even larger when we explicitly deal with overbooking and reservation fees
• Deterministic problem becomes large when we deal with **dynamic pricing** and **assortment planning**
# Partial List of References