A stochastic programming model for hedging options in a market with transaction costs

ICSP Bergamo, 2013

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2013-07-09
Outline

1. Introduction and overview of the hedging problem
2. Estimating option implied surfaces
3. PCA of local volatility surfaces
4. Scenario generation
5. SP model
6. Results
How should the option hedging problem be studied?

Important properties that have a large impact on optimal decisions:

- **Transactions** at market prices (bid/ask)
- **Significant** transaction fees
- Random variables **not normally distributed**
- Behavior of **option market prices**
- **SP** provides the **tool** to be able to solve this type of problem
- The optimization **model** should take these **aspects into account**
Stylized facts of options markets

- Need to **model uncertainty** in market **prices**
- **Stylized facts** of equity market returns
- **Uncertainty** in **option** prices: $C(S, K, T, r, \delta, \sigma)$
- One **volatility** per **option**
- Need to have knowledge of the **covariance structure** of the **option prices**
- **Careful** not to introduce **arbitrage** into the model
Overview of the hedging problem

Equity market prices
Q
Estimate surfaces
PCA

Option prices
Futures prices
Interest rates
Index levels
SP model
Option hedging
Monte Carlo simulation

Equity derivatives prices
Interest rates
Estimate models

Equity market prices
P
Estimate models
Model selection
Plots of option implied surfaces

Surfaces implied by option prices.

(a) Call price

(b) Implied volatility

(c) Local volatility

(d) RND
Methods for surface estimation described in the literature

- The **standard methods** described in the literature assumes some parametric form
- Some methods do **not guarantee arbitrage-free** surfaces
- Assuming a parametric form will typically give rise to **non-convex** optimization problems with **many local optima**
- \( \Rightarrow \) Estimated volatility surfaces can **change a lot** from one day to the next

**Consequences for the SP model:**

- The SP model will **identify arbitrage** and **exploit** it
- The SP model will be **confused** by all the **noise** and the large jumps
Estimation of the local volatility surface

- **Pricing** of derivatives via **finite difference** methods

- **Extended** the **methodology** in Andersen and Brotherton-Ratcliffe (1997) to a **non-uniform** grid

- The optimization problem is **not convex** but the optimization algorithm does **not** seem to get **stuck** in **local optima**
Discretized optimization problem (local volatility estimation)

\[
\begin{align*}
\min_{\hat{v}, q, \Gamma, z_e, z_b} & \quad h(\hat{v}) + \frac{1}{2} z_e^T E_e z_e + \frac{1}{2} z_b^T E_b z_b \\
\text{s. t.} & \quad q_{i+1}^T = q_i^T \Phi_i(\hat{v}, \Gamma), \quad i = 0, \ldots, M \\
& \quad g_e(q) + F_e z_e = x_e \\
& \quad x_l \leq g_b(q) + F_b z_b \leq x_u \\
& \quad q \geq 0 \\
& \quad \hat{v} \geq 0,
\end{align*}
\]

where \( h(\hat{v}) \) is the discretized version of

\[
h(v) = \frac{1}{2} \int_0^{\tau_n} \int_{x_1}^{x_N} \left( a''(\tau, x) \left( \frac{\partial^2 v}{\partial \tau^2} \right)^2 + b''(\tau, x) \left( \frac{\partial^2 v}{\partial x^2} \right)^2 \right) d\tau dx. \]
Empirical demonstration

Estimated surfaces from OMXS30 intraday data from 6 March 2013.

(a) Implied volatility

(b) Local volatility
Demonstration - film of estimated surfaces (2008)
A SP model for hedging options in a market with transaction costs

PCA of local volatilities - plots of eigen matrices

Eigen matrices of the first 4 principal components.

(a) PC1

(b) PC2

(c) PC3

(d) PC4
Model for the underlying asset returns

GARCH model **without jumps** in the index level (GARCH(1,1)) and GARCH-Poisson:

\[
\Delta Y_{t+1} = \mu \Delta t + \sigma_t \sqrt{\Delta t} z^1_t + \alpha \sigma_t \sqrt{N_t \Delta t} z^2_t, 
\]

\[
\sigma^2_t = \beta_0 + \beta_1 \sigma^2_{t-1} + \beta_2 \frac{(\Delta Y_{t+1} - \gamma \Delta t)^2}{\Delta t}, 
\]

\[
z^1_t, z^2_t \sim i.i.d. N(0, 1), N_t \sim i.i.d. Po(\lambda \Delta t). 
\]

Stochastic volatility model **without jumps** in the index level (log SVLE) and log SVPJLE:

\[
Y_t = Y_{t-1} + \mu \Delta t + (\exp(h_{t-1}) \Delta t) \frac{1}{2} \epsilon^Y_t + J_t, 
\]

\[
h_t = h_{t-1} + \kappa(\theta - h_{t-1}) \Delta t + \sigma_h \sqrt{\Delta t} \epsilon^V_t, 
\]

\[
\epsilon^Y_t, \epsilon^V_t \sim i.i.d. N(0, 1), \ E[\epsilon^Y_t \epsilon^V_t] = \rho, \ J_t = q_t \xi_t, 
\]

\[
q_t \sim i.i.d. Bernoulli(\lambda_y \Delta t), \ \xi_t \sim i.i.d. N(\mu_y, \sigma^2_y). 
\]
Evaluation of the models

QQ plots for the 4 models.

(a) GARCH(1,1)

(b) GARCH-Poisson

(c) log SVLE

(d) log SVPJLE
Scenario generation

Mean-reverting AR(1) model for the normalized principal components of the local volatility surface

\[ x_k(t) = \ln(U_t) V^{-1/2} b_k, \quad k = 1, \ldots, 10 \]  
\[ x_k(t + 1) = e^{-\lambda_k} x_k(t) + (1 - e^{-\lambda_k}) \bar{x}_k + \epsilon_k(t), \quad k = 1, \ldots, 10, \]  

where

\[ \epsilon_k(t) \sim N(0, \sigma_k). \]  

The model parameters are estimated with regression.

Scenarios for the GARCH Poisson model generated via the inverse transform method.
Hedging model

- Two stage SP model without recourse
- 100 scenarios for each trade date
- OMXS30 index options
- Hedging of portfolio of sold options with non-standard maturities
- Hedging possible with all standardized options and the index
- All transactions occur at the bid and ask prices
- Transaction fees of 0.5% for options and 0.05% for the index
- Option portfolio valued with estimated local volatility surface
The SP model that is solved for the hedging problem is given by:

\[
\begin{align*}
\min_{u_B, u_S, x, M} & \quad \frac{1}{N-1} \sum_{W_i < \mu_W} (W_i - \mu_W)^2 + \lambda (\tau_B^T u_B + \tau_S^T u_S) \\
\text{s. t.} & \quad W_i = Me^{r \Delta t_k} + P_i^T x + l_i, \quad i = 1, \ldots, N \\
& \quad x = x_l + u_B - u_S, \\
& \quad M = M_l - P_A^T u_B + P_B^T u_S, \\
& \quad \mu_W = \frac{1}{N} \sum_{k=1}^{N} W_k, \\
& \quad u_B, u_S \geq 0.
\end{align*}
\]
Hedging results

Propagation of the wealth of the option portfolio. Hedging performed during the period 18 June 2009 to 29 December 2009.

(a) Wealth of the portfolio

(b) Delta of the portfolio
Conclusions

- **Framework** that **realistically describes** the problem of **hedging** a portfolio of options

- **Realistic** description of the **dynamics** of market prices

- **Future improvements:**
  - **Increase** the **quality** of the estimated surfaces
  - **Choice** of **objective** function
Thank you for your attention!