A stochastic programming perspective on contingent claims

Alan King

IBM Thomas J. Watson Research Center

kingaj@us.ibm.com

With Teemu Pennanen, Olga Streltchenko, Oleksandr Romanko and Ben de Prisco

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Outline

Pricing Contingent Claims by Arbitrage

Discussion of arbitrage theorem

Calibration

American Options

Summary
General Set Up

Discrete sample space
- Scenario tree with nodes \( n \in \mathcal{N}_t, t = 0, \ldots, T \)
- Parent nodes \( n^- \in \mathcal{N}_{t-1} \); child nodes \( n^+ \subset \mathcal{N}_{t+1} \)
- Vector price process \( \{Z_t\} \) with \( Z^0 := 1 \) (bond).

Options trading strategy
- Manufacture \( F_t = F(Z_t) \) by trading a portfolio \( \theta_t \)
- Solve the replication problem

\[
\begin{align*}
\min_{\theta} & \quad Z_0 \cdot \theta_0 \\
\text{st:} & \quad Z_t \cdot \theta_t + F_t \leq Z_t \cdot \theta_{t-1} \\
& \quad F_T \leq Z_T \cdot \theta_{T-1}.
\end{align*}
\]
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\begin{array}{ll}
\min_{\theta} & Z_0 \cdot \theta_0 \\
\text{st:} & Z_t \cdot \Delta \theta_t = -F_t \\
& Z_T \cdot \theta_T \geq 0,
\end{array}
\]

a more convenient notation based on “trades” \( \Delta \theta_t := \theta_t - \theta_{t-1} \).
Dual Problem

Duality for replication problem introduces dual multipliers $q_n, n \in \mathcal{N}_t$

- non-negative $q_n \geq 0, n \in \mathcal{N}_T$
- satisfy the martingale equality $q_n Z_n = \sum_{m \in n^+} q_m Z_m$
- sum to one $\sum_{n \in \mathcal{N}_T} q_n = 1$ (because $Z_0^n = 1$)
- maximize

$$\sum_{t=1}^{T} \sum_{n \in \mathcal{N}_t} q_n F_n := \langle q, F \rangle$$
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Let’s denote $\mathcal{M}$ to be the probability measures that make $\{Z_t\}$ into a martingale.
Duality

Theorem

The replication problem

\[ \begin{align*}
\min_{\theta} & \quad Z_0 \cdot \theta_0 \\
\text{st:} & \quad Z_t \cdot \Delta \theta_t = -F_t \\
& \quad Z_T \cdot \theta_T \geq 0
\end{align*} \]

has solutions iff there exist dual solutions \( q \) to

\[ \max_{q \in \mathcal{M}} \langle q, F \rangle \]

- Prove by linear programming duality.
The buyer’s problem

The problem we began with was how to produce, at minimal value today, the option cash flows $F_t$. This is the option seller’s problem.

The buyer wants to maximize value today from consuming future option cash flows:

$$\max_{(\theta)} \quad Z_0 \cdot \theta_0$$

subject to:

$$Z_t \cdot \theta_t \leq Z_t \cdot \theta_{t-1} + F_t$$

$$Z_T \cdot \theta_{T-1} \leq F_T.$$
Discussion of arbitrage theorem

Buyer and Seller

- Minimum price seller needs to replicate the option

  \[ F_a = \min_{\theta} Z_0 \cdot \theta_0 \quad \text{st:} \quad Z_t \cdot \Delta \theta_t = -F_n, \ Z_T \cdot \theta_T \geq 0 \]

  \[ \text{Dual: } = \max_{q} \langle q, F \rangle \quad \text{st:} \quad q \in \mathcal{M} \]

- Maximum price buyer wishes to bid

  \[ F_b = \max_{\theta} Z_0 \cdot \theta_0 \quad \text{st:} \quad Z_t \cdot \Delta \theta_t = F_t, \ Z_T \cdot \theta_T \leq 0 \]

  \[ \text{Dual: } = \min_{q} \langle q, F \rangle \quad \text{st:} \quad q \in \mathcal{M} \]

- \([F_b, F_a]\) is the (no-) Arbitrage Interval. Prices outside Arbitrage Interval create opportunities for traders.

- Note that \(F_b \leq F_a\). No trading!
No Trading?

➤ Buyers and sellers must have **different reasons** to trade
  - They must use different $\mathcal{M}$-measures to price it.
➤ Relevant differences between market actors:
  - **risk preferences** for gains / losses in trading
  - **existing positions** with risk that is to be hedged
➤ Transaction costs and access to market information tend, in fact, to **widen** the gap.
Stochastic Programming Perspective

- Risk preferences: utility

$$\sum_{n \in \mathcal{N}_T} p_n f(Z_n \cdot \theta_n) := E^p \left[ f(Z_T \cdot \theta_T) \right]$$

- with “real” probability distribution $p$.

- Existing positions: Income / outflow $L_n$

- Stochastic Program: (seller’s)

$$\min_{\theta} \quad Z_0 \cdot \theta_0 - E^p \left[ f(Z_T \cdot \theta_T) \right]$$

$$\text{st:} \quad Z_t \cdot \Delta \theta_t = L_t - F_t$$

This prices the option relative to an asset-liability management problem.
Risk Neutral

The stochastic linear program in the arbitrage pricing theorem has risk preferences $f()$

$$f(W) = \begin{cases} 
0 & \text{if } W \geq 0 \\
-\infty & \text{if } W < 0, 
\end{cases}$$

... the indicator function of the non-negative real line.

- Risks arising from “real” probabilities $p$ are neutralized by this $f()$.
- This version of incomplete markets pricing is called risk-neutral.
Incorporating Risk into Duality framework

Suppose we use a (non-decreasing, concave) utility function \( f() \) that is not so extreme as the indicator of the non-negative cone. The dual objective then contains a distance term between \( p \) and \( q \):

\[
\max_q \langle q, F \rangle + E^p[f^*(q/p)]
\]

\[
st: \quad q \in \mathcal{M}
\]

where \( f^* \) is the concave conjugate of \( f() \).

- \( f(w) = w - 1/4w^2 \) \( \rightarrow pf^*(q/p) = -(q - p)^2/p \) “goodness of fit”.
- \( f(w) = -e^w \) \( \rightarrow pf^*(q/p) = p \ln(p/q) \) “relative entropy”
In incomplete markets, there are many possible choices of martingale measure.

The calibration problem: given observed market prices, retrieve a pricing measure compatible with them.

- We add calibration constraints to the stochastic program.
- Investors may be buyers or sellers of particular claims depending on particular liabilities and goals.
- Calibration is sensitive to even the slightest “mispricing”.
- Incorporating risk may stabilize this problem.
Calibration in Incomplete markets

Require martingale $q$ to reproduce observed prices $C_0$ of options

$$C_0 \sim \langle q, C \rangle,$$

where $C_t$ is the contingent cash flow of the option $C$.

Apply as soft-constraints in the martingale (dual) problem, with weights $V$:

$$\max_{q, U^+, U^-} \langle q, F \rangle + V \cdot (U^+ + U^-)$$

subject to:

$$U^+ - U^- = C_0 - \langle q, C \rangle$$

$$U^+, U^- \geq 0$$

$$q \in \mathcal{M}$$

(2)
Private Valuation

The primal version of the calibration problem is

\[
\min_{\theta, \zeta} \quad \theta_0 \cdot Z_0 + \zeta_0 \cdot C_0 - E^P f(\theta_T \cdot Z_T) \\
\text{st:} \quad \Delta \theta_t \cdot Z_t = \zeta_0 \cdot C_t + L_t - F_t \\
\zeta_0 \in [-V, V]
\]  

(3)

- Positions in options calibrate the ALM to market prices \((Z_0, C_0)\).
- Incorporating investor policies on positions \((\theta, \zeta)\) will specialize the ALM.
- Provides a “private” valuation of \(F\).
Results are Fragile

We have tested this on stock market and treasury futures data.

- Incorporate publicly traded options on equities or futures.
- Use *volume traded* as the natural candidate for $V$.
- Mark-to-market each option as $F$, in turn.

Results are very sensitive to choice of volumes.

But really a pretty good fit given the strangeness of the market data!
American Options

American options can exercise at any time. Let \( e_t \in \{0, 1\} \) denote the exercise.

\[
\begin{align*}
\min_{\theta} \max_e & \quad Z_0 \cdot \theta_0 \\
\text{st:} & \quad Z_t \cdot \Delta \theta_t = -e_tF_t \\
& \quad Z_T \cdot \theta_T \geq 0, \\
& \quad e_t \in \{0, 1\} \\
& \quad \sum_t e_t = 1
\end{align*}
\]

Then the LP relaxation \( e_t \in [0, 1] \) at optimality has \( e_t \in \{0, 1\} \).

Proved by examining complementary slackness conditions. (Sketch of proof in Pennanen-King, complete proof in Pinar.)
Options pricing in incomplete markets can give practical guidance for the calibration of ALM problems to current markets.

Utility functions correspond to natural distance measures between the “real” and “implied” probability measures.

Risk-neutral utility is extremely sensitive to market mispricing.

Next step: investigate quadratic utility / goodness of fit, and real probability measure based on historical volatilities and covariances. Incorporate American option formalization into calibration.