Revisiting the Superluminal Lorentz Transformations and Their Group-Theoretical Properties (*).

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Summary. – We briefly revisit the introduction of generalized Lorentz transformations (GLT) in four dimensions, by writing them in a new form which satisfies the requested group-theoretical properties. The new GLTs, in the particular case of boosts, reduce to the ones by Mignani and Recami (the latter, in fact, forming a group only in the case of collinear boosts). Some other aspects are briefly clarified.

Introduction. – It is well known that the extension of the principle of relativity to Superluminal inertial frames is straightforward only in the pseudo-Euclidean (« Minkowski ») space-times $M(n, n)$ having the same number n of space and time axes. In the two-dimensional case ($n = 1$) the Superluminal Lorentz transformations were first introduced—even if in a preliminary form (not yet satisfying the group-theoretical properties)—in ref. (1) and then independently rediscovered (2) and completed (3) in ref. (2,3). As to the six-dimensional case ($n = 3$), a preliminary suggestion first appeared in ref. (4), followed by various contributions (5).

When facing the problem of extending the Lorentz transformations to faster-than-light frames in four-dimensions, one meets the « no-go » theorem by GORINI *et al.* (6), that essentially stated *no* such extensions to exist when they are required to satisfy

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(1) L. PARKER: *Phys. Rev.*, **188**, 2287 (1969).

(2) V. S. OLKHOVSKY and E. RECAMI: *Visn. Kiivs. Univ. Ser. Fiz.*, **11**, 58 (1970); *About Lorentz transformations and tachyons*, preprint ITF/70 (Kiev, 1970); *Lett. Nuovo Cimento*, **1**, 165 (1971); E. RECAMI and R. MIGNANI: *Lett. Nuovo Cimento*, **4**, 144 (1972); R. MIGNANI, E. RECAMI and U. LOMBARDO: *Lett. Nuovo Cimento*, **4**, 624 (1972). See also M. PAVŠIČ: *The extended special theory of relativity*, preprint, University of Ljubljana (1971), unpublished; A. F. ANTIPPA: preprint, Université de Quebéc (1970), unpublished.

(3) R. MIGNANI and E. RECAMI: *Nuovo Cimento A*, **14**, 169 (1973). See also A. F. ANTIPPA: *Nuovo Cimento A*, **10**, 389 (1972).

(4) R. MIGNANI and E. RECAMI: *Lett. Nuovo Cimento*, **16**, 449 (1976).

(5) Cf. *e.g.* the bibliography written by V. F. PEREPELISTA: Reports ITEP-100 and ITEP-165, Inst. Theor. Exp. Phys. (Moscow, 1980).

(6) V. GORINI: *Commun. Math. Phys.*, **21**, 150 (1971), and references therein.

all the following properties: i) to refer to the 4-dimensional space-time M_4; ii) to be real; iii) to be linear; iv) to preserve space isotropy; v) to preserve the light-speed invariance; vi) to possess the prescribed group-theoretical properties. It appears then necessary to release one (at least) of the above requirements.

The first extension of Lorentz transformations in the 4-dimensional Minkowski space M_4 tried to stick to the requirement ii), cf. ref. [7]; with the interesting exception of the early paper by SOMIGLIANA [8].

According to us, however, it is essential to try to *keep* the characteristic postulates of special relativity (SR), *i.e.* the requirements iii), iv), v), vi), by releasing on the contrary the requirements i) or ii). This line of research has been *first* adopted in ref. [9,10], leading to the Generalized Lorentz transformations (GLT) by MIGNANI and RECAMI [10,11]. Such GLTs appeared till now, however, in a form suitable only for boosts: in the sense that they formed a group only in the case of collinear boosts. We want here to rederive them in a more general form, so to satisfy also the prescribed group-theoretical properties (requirement vi)).

The new form of SLTs. – The Superluminal Lorentz transformations (SLT) must transform—in the language of SR—timelike into spacelike quantities, and *vice-versa*. Therefore, we require the GLTs (both subluminal and Superluminal), $G_\mu{}^\nu$, connecting two generic inertial frames f, f'

$$\mathrm{d}x'_\mu = G_\mu{}^\nu \,\mathrm{d}x_\nu\,,$$

to have the following properties: 1) to transform inertial motion into inertial motion; 2) to form a (new) group $\mathbb{G}$; 3) to preserve space isotropy; 4) to leave the chronotopical quadratic form invariant, except for its sign (*): $\mathrm{d}x'_\mu \mathrm{d}x'^\mu = \pm\, \mathrm{d}x_\mu \mathrm{d}x^\mu$, $(u^2 \lessgtr c^2)$. The four-position x^μ is supposed of course to be a 4-vector even with respect to $\mathbb{G}$. In such an «extended relativity» the quadratic form $\mathrm{d}x_\mu \mathrm{d}x^\mu$ is a *scalar* under subluminal Lorentz transformation (LT) and a *pseudo-scalar* under SLTs.

Stated differently, the GLTs, $G_\mu{}^\nu$, are unimodular, special and such that the LTs are *orthogonal* whilst the SLTs are *antiorthogonal*:

$$G^T G = +1\,, \qquad u^2 < c^2\,, \tag{1}$$

$$G^T G = -1\,, \qquad u^2 > c^2\,. \tag{2}$$

At this point let us explicitly recall that, in two different bases, the scalar products

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(7) V. S. OLKHOVSKY and E. RECAMI: in *Visn. Kiivs. Univ. Ser. Fiz.*, **11**, 58 (1970); preprint ITF/70, Ukrainian Academy of Science (Kiev, 1970); *Lett. Nuovo Cimento*, **1**, 165 (1971). Many other contributions, especially by A. F. ANTIPPA and A. E. EVERETT, followed along this line.

(8) C. SOMIGLIANA: *Rend. Accad. Naz. Lincei (Roma)*, **31**, 53 (1922).

(9) See *e.g.* E. RECAMI and R. MIGNANI: *Lett. Nuovo Cimento*, **4**, 144 (1972); R. MIGNANI, E. RECAMI and U. LOMBARDO: *Lett. Nuovo Cimento*, **4**, 624 (1972). See also, C. SOMIGLIANA: *Rend. Accad. Naz. Lincei (Roma)*, **31**, 53 (1922); M. PAVŠIČ: preprint, Ljubljana University (1971), unpublished.

(10) R. MIGNANI and E. RECAMI: *Nuovo Cimento A*, **14**, 169 (1973); E. RECAMI and R. MIGNANI: *Lett. Nuovo Cimento*, **8**, 110 (1972); *Riv. Nuovo Cimento*, **4**, 209, 398 (1974), and many subsequent papers. See also H. C. CORBEN: *Nuovo Cimento A*, **29**, 415 (1975).

(11) See also, *e.g.*, E. RECAMI (Editor): *Tachyons, Monopoles, and Related Topics* (Amsterdam, 1978); E. RECAMI: in *A. Einstein 1879-1979: Relativity, Quanta and Cosmology*, Chapt. 16, edited by F. DE FINIS and M. PANTALEO, Vol. **2** (New York, N. Y., 1979), p. 1021; P. CALDIROLA and E. RECAMI: in *Italian Studies in the Philosophy of Science*, edited by M. DELLA CHIARA (Boston, Mass., 1980), p. 249.

(*) To compare this fact with what ordinarily happens in general relativity, see the following.

define

$$x_\mu x^\mu \equiv x_\mu g^{\mu\nu} x_\nu ; \qquad x'_\mu x'^\mu \equiv x'_\mu g'^{\mu\nu} x'_\nu ,$$

and the *anti*orthogonal transformations operate in M_4 in such a way that ([12]) *either*

$$g'_{\mu\nu} = +\eta_{\mu\nu} \quad \text{and} \quad x_\mu x^\mu = -x'_\mu x'^\mu , \tag{3a}$$

or

$$g'_{\mu\nu} = -\eta_{\mu\nu} \quad \text{and} \quad x_\mu x^\mu = +x'_\mu x'^\mu . \tag{3b}$$

Notice that our antiorthogonal transformations do *not* belong to the ordinary group of transformations usually adopted in general relativity: see ref. ([13]), where this fact is clearly mentioned.

In other words, a SLT from a subluminal to a Superluminal frame, $s_0 \to S'$, can be identical with a suitable (ordinary) LT—let us can it the « dual » transformation—*except for the fact* that it must operate so as in eq. (3*a*), *i.e.* change timelike into spacelike vectors.

Alternatively, one may also say that a SLT is *identical* with its dual subluminal LT, *provided that* we impose the primed observer S' to use the opposite metric signature $g'_{\mu\nu} = -g_{\mu\nu}$, according to eq. (3*b*) (however, without changing the signs into the definitions of timelike and spacelike quantities) ([10,14]).

It follows that a generic SLT, corresponding to a Superluminal velocity $\boldsymbol{U}$, will be formally expressed by the *product* of the dual LT, corresponding to the subluminal velocity $\boldsymbol{u} \| \boldsymbol{U}$ with $u \equiv c^2/U$, *by* the metric $\mathscr{S} \equiv i\mathbf{1}$:

$$\begin{cases} \mathrm{SLT}(\boldsymbol{U}) = \pm \mathscr{S}[\mathrm{LT}(\boldsymbol{u})] & (4) \\ \mathscr{S} \equiv i\mathbf{1} , & (5) \end{cases} \qquad \begin{bmatrix} \boldsymbol{u} \| \boldsymbol{U}; & u \equiv c^2/U \\ u^2 < c^2; & U^2 > c^2 \end{bmatrix},$$

where $\mathscr{S} \equiv \mathscr{S}_4$ plays the rôle of the « transcendent SLT », since for $\boldsymbol{u} \to 0$ one gets $\mathrm{SLT}(\boldsymbol{U} \to \infty) = \pm i\mathbf{1}$.

It is easy to recognize that the group-theoretical properties and space isotropy are preserved only by an operation $\mathscr{S}$ which be representable by a 4×4 matrix *symmetrical* with respect to all possible axis-permutations, so as it is in eq. (5). Notice that our present operation $\mathscr{S}$ does not affect the speed u (*i.e.*, does *not* operate any change $\beta \to 1/\beta$), differently from what stated in ref. ([11]).

If we call $\boldsymbol{Z}(4)$ the discrete group of the 4th roots of unity

$$\boldsymbol{Z}(4) \equiv \{\sqrt[4]{1}\} \equiv \{+1, -1, +i, -i\} , \tag{6}$$

then the new group $\mathbb{G}$ of GLTs (both subluminal and Superluminal) is

$$\mathbb{G} = \boldsymbol{Z}(4) \otimes \mathscr{L}_+^\uparrow , \tag{7}$$

where $\mathscr{L}_+^\uparrow$ is the ordinary (proper, orthochronous) Lorentz group.

([12]) G. D. MACCARRONE and E. RECAMI: preprint PP/693, Catania University (1982), to be published.
([13]) See *e.g.* C. MØLLER: *The Theory of Relativity* (Oxford, 1962), p. 234.
([14]) Cf. *e.g.* R. MIGNANI and E. RECAMI: *Lett. Nuovo Cimento*, **9**, 357 (1974). See also K. T. SHAH: *Lett. Nuovo Cimento*, **18**, 156 (1977).

If G is a GLT, then ([10]) also $-G$ belongs to $\mathbb{G}$:

$$G \in \mathbb{G} \Rightarrow -G \in \mathbb{G}\,, \qquad \forall G \in \mathbb{G}\;; \tag{8}$$

so that, when we confine ourselves to subluminal LTs, we are left with the group $[\mathbf{Z}(2) \equiv \{\sqrt[2]{1}\} \equiv \{+1, -1\}]$

$$\mathscr{L}_+ \equiv \mathscr{L}_+^\uparrow \cup \mathscr{L}_+^\downarrow = \mathbf{Z}(2) \otimes \mathscr{L}_+^\uparrow\,,$$

which is the *true* Lorentz group of special relativity, since it describes both particles and antiparticles within a purely relativistic scenario ([11,15]).

At last, let us emphasize the important symmetry property of the ordinary Lorentz boosts expressed (in natural units: $c = 1$) by the identities

$$\begin{cases} \dfrac{x - ut}{\sqrt{1-u^2}} \equiv -\dfrac{t - Ux}{\sqrt{U^2-1}}\,, \\[2mm] \dfrac{t - ux}{\sqrt{1-u^2}} \equiv -\dfrac{x - Ut}{\sqrt{U^2-1}}\,, \end{cases} \qquad [U \equiv 1/u]\,. \tag{9}$$

Reduction to Mignani and Recami's transformations. – In the particular case of boosts along x, eqs. (4) and (5) yield

$$\begin{cases} t' = \pm i\,\dfrac{t - ux}{\sqrt{1-u^2}} \equiv \mp i\,\dfrac{x - Ut}{\sqrt{U^2-1}}\,, \\[2mm] x' = \pm i\,\dfrac{x - ut}{\sqrt{1-u^2}} \equiv \mp i\,\dfrac{t - Ux}{\sqrt{U^2-1}}\,, \\[2mm] y' = \pm iy\,, \\ z' = \pm iz\,, \end{cases} \qquad \begin{bmatrix} \text{Superluminal case} \\ U \equiv c^2/u \\ u^2 < c^2;\; U^2 > c^2 \end{bmatrix}, \tag{10}$$

where we took advantage of eqs. (9). Let us observe that, under transformations (10), the $\mathbb{G}$–four-velocity $u_\mu \equiv dx^\mu/d\tau_0$ changes in such a way that $u'_\mu u'^\mu = -u_\mu u^\mu$, so that from $u^2 = +1$ if follows $u'^2 = -1$ (that is to say bradyonic speeds are transformed into tachyonic speeds, notwithstanding the appearance of eqs. (10): cf. ref. ([12])).

We have to tackle the problem of physically interpreting our Superluminal boosts, eqs. (10). In a full paper to be published elsewhere ([12])—and in which recourse is temporarily made to an auxiliary six-dimensional space-time $M_6 \equiv M(3, 3)$—we show that one can deal with the first two equations (10) separately from the last two equations (10), and therefore split the reinterpretation procedure into *two steps*.

As to the two-dimensional space-time $M(1, 1) = (x, t)$, it can be regarded as a complex plane so that the imaginary unit $i \equiv \exp[\frac{1}{2}i\pi]$ operates there as a 90° (pseudo)rotation. The same can be said of course for the « two-dimensional » operation

$$\mathscr{S}_2 \equiv i\mathbb{1} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix},$$

([15]) E. RECAMI and W. A. RODRIGUES: *Found. Phys.*, **12**, 709 (1982).

which represents the transcendent SLT in $M(1, 1)$; actually, in the case of a transcendent boost along x, both observers s_0, S' agree about the axes x', t', x, t that: $t' \equiv x$; $x' \equiv t$, apart from their signs. This can be generalized [12]: for instance, the transcendent SLT in six dimensions ($\mathscr{S}_6 = i\mathbb{1}$, where $\mathbb{1}$ is now the 6-dimensional identity matrix) operates the «rotation» $\boldsymbol{x} \rightleftarrows \boldsymbol{t}$.

As a consequence, the first two eqsuations (10) can be reinterpreted, and eqs. (10) rewritten:

$$(11)\qquad \begin{cases} t' = \pm \dfrac{x-ut}{\sqrt{1-u^2}} \equiv \mp \dfrac{t-Ux}{\sqrt{U^2-1}}\,, \\ x' = \pm \dfrac{t-ux}{\sqrt{1-u^2}} \equiv \mp \dfrac{x-Ut}{\sqrt{U^2-1}}\,, \\ y' = \pm\, iy\,, \\ z' = \pm\, iz\,, \end{cases} \qquad \begin{bmatrix} \text{Superluminal case} \\ U \equiv c^2/u \\ u^2 < c^2;\ U^2 > c^2 \end{bmatrix},$$

where the imaginary units do remain in the transversal co-ordinates (even after the «excursion» in M_6) [12]. Let us observe that our (partially) reinterpreted SLTs, eqs. (11), yield, for the speed of the first frame s_0 with respect to the second frame S',

$$(12)\qquad \boldsymbol{x} \equiv 0 \Rightarrow \frac{x'}{t'} = \pm\left(-\frac{1}{u}\right) = \mp\, U\,, \qquad \begin{bmatrix} u^2 < c^2;\ U^2 > c^2 \\ U = c^2/u \end{bmatrix},$$

where u, U are the speeds (along the boost direction: $\boldsymbol{u} \| \boldsymbol{U}$) of the two *dual* frames s, S'; remember that the dual frames always possess infinite relative speed. This shows once more that eqs. (11) are actually associated to Superluminal relative motion. The problem of suitably defining the tachyon velocity will be considered elsewhere [12].

We have thus shown that our new SLTs, eqs. (10), *reduced to the «old» SLTs by* MIGNANI *and* RECAMI [10,11] in the case of Superluminal boosts. Actually, those SLTs formed a group only under collinear boosts.

Notice that our new SLTs together with the LTs form a group, namely the group $\mathbb{G}$ in eqs. (6)-(7), only in their original form (10); and *not* in their (partially) reinterpreted form (11). The reinterpretation (when necessary) is to be applied only *at the end* of any possible chain of GLTs. To act differently would mean [12] to use diverse metric signatures (in our sense) during the GLT-chain, and this is of course forbidden: cf. ref. [12].

The *second step* in the interpretation of the SLTs has to do with the the last two equations (11). But these equations, as mentioned above, coincide with Mignani and Recami's. We can therefore refer to the work already performed in this direction in recent times; and we confine ourselves to quoting ref. [16]. Further details will appear elsewhere [12].

Let us here mention, however, that we succeeded [12,16] in geometrico-physically interpreting—at least in some relevant cases—also the last two equations (11), thus identifying in a sense the space-time M_4 accessible to s_0 and the space-time M'_4 accessible to S'. For this reason we went on calling «transformations» (and not mappings) our SLTs, eqs. (11).

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(16) E. RECAMI and G. D. MACCARRONE: *Lett. Nuovo Cimento*, **28**, 151 (1980); P. CALDIROLA, D. G. MACCARRONE and E. RECAMI: *Lett. Nuovo Cimento*, **29**, 241 (1980); G. D. MACCARRONE and E. RECAMI: in *Proceedings Einstein Centenary Symposium* (Nagpur, 1981), p. 123; A. O. BARUT G. D. MACCARRONE and E. RECAMI: to be published.

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